# LinearSystems and Feedback Control

Course Feedback Control and Real-time SystemsI

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# What to Expect in this Lecture

the most well-understood **class** of dynamical systems

- standard terminology
- types of behavior
- how to control the behavior

# Autonomous Linear Dynamical Systems

Linear ordinary differential equation (ODE)

$$\dot{x} = Ax$$

with **initial state** $x(0) = x_{\text{ini}}$.

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \qquad A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix}$$

# Example: Coasting Car

Assumptions:

- idealized car with no rolling resistance
- no brakes, no engine acceleration
- only wind resistance at slow speed

variable: horizontal position $x_1$

drag force: $F_{\mathrm{drag}} = -\beta \dot{x}_1$ with drag coefficient $\beta \geq 0$

Newton's equation of motion (mass $m$):

$$m\ddot{x}_1 = -\beta \dot{x}_1$$

# Example: Coasting Car

$$m\ddot{x}_1 = -\beta\dot{x}_1$$

solving for $\dot{x}_1$: $\ddot{x}_1 = -\frac{\beta}{m}\dot{x}_1$

adding auxiliary variable $x_2 := \dot{x}_1$ to get 1st order ODE:

$$\dot{x}_2 = -\frac{\beta}{m}x_2$$

vector-matrix form:

$$\begin{bmatrix}\dot{x}_1 \\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1 \\ 0 & -\frac{\beta}{m}\end{bmatrix}\begin{bmatrix}x_1 \\ x_2\end{bmatrix}$$

# Solution (scalar form)

Linear ordinary differential equation (scalar form)

$$\dot{x} = ax$$

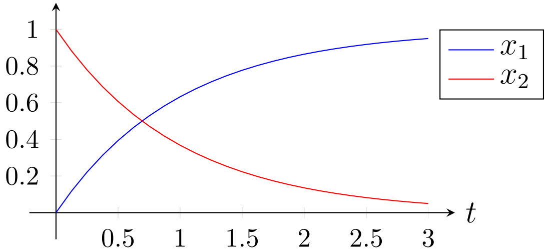
with **initial state** $x(0) = x_{\text{ini}}$ has the solution

$$x(t) = e^{at} x_{\text{ini}}.$$

Proof:

$$\dot{x}(t) = \tfrac{d}{dt}\big(e^{at}\big)x_{\text{ini}} = \big(ae^{at}\big)x_{\text{ini}} = a\big(e^{at}x_{\text{ini}}\big) = ax(t).$$

# Example: Coasting Car




solve $x_2(t)$ alone:

$$\dot{x}_2 = -\tfrac{\beta}{m} x_2 \quad \Rightarrow \quad x_2(t) = e^{-\frac{\beta}{m}t} x_{2,\text{ini}}.$$

solve $x_1(t)$ by integrating $x_2(t)$:

$$\dot{x}_1 = x_2 \quad \Rightarrow \quad x_1(t) = \frac{m x_{2,\text{ini}}}{\beta}\left(1 - e^{-\frac{\beta}{m}t}\right) + x_{1,\text{ini}}.$$

# Solution (general form)

Linear ordinary differential equation (ODE)

$$\dot{x} = Ax$$

with **initial state** $x(0) = x_{\text{ini}}$ has the solution

$$x(t) = e^{At} x_{\text{ini}},$$

$$e^{At} = \sum_{k=0}^{\infty} A^k \frac{t^k}{k!} = I + At + A^2 \frac{t^2}{2} + \cdot$$

# Overview

Autonomous Linear Dynamical Systems

Stability

Some Linear Algebra

Linear Time-Invariant (LTI) Systems

Feedback Control

# Stability: General notion

The definition is not limited to linear systems

An equilibrium state is a state $x_e$ such that in the absence of input we have: $x(0) = x_e \rightarrow \forall t > 0 \;\; x(t) = x_e$.

For system $\dot{x}(t) = f(x(t), u(t))$, equilibrium states are solutions to the equation $0 = f(x, 0)$.

An equilibrium state $x_e$ is

- **stable** if

$$\forall \epsilon > 0 \, \exists \delta > 0 : \|x(0) - x_e\| < \delta \rightarrow \|x(t) - x_e\| < \epsilon.$$

- **asymptotically stable** if

$$\forall \delta > 0 : \|x(0) - x_e\| < \delta \rightarrow \lim_{t \rightarrow \infty} x(t) = x_e.$$

# Stability

A linear system

$$\dot{x} = Ax$$

can have

- One unique equilibrium $x = 0$ if $A$ is invertible
- An infinite number of equilibria if $A$ is not invertible

Every linear system falls into one of the following categories:

- for $t \to \infty$, $\|x(t)\| \to \infty$: **unstable**
- for $t \to \infty$, $\|x(t)\| \to 0$: **asymptotically stable**
- otherwise, $x(t)$ periodic: **marginally stable**

$$\|x\| = \sqrt{x_1^2 + \cdots x_n^2}$$

# Example: Accelerating Car

assumption: no friction, no drag

acceleration $x_3 = \text{const}$

$$\ddot{x}_1 = x_3$$

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = x_3, \quad \dot{x}_3 = 0$$

vector-matrix form:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

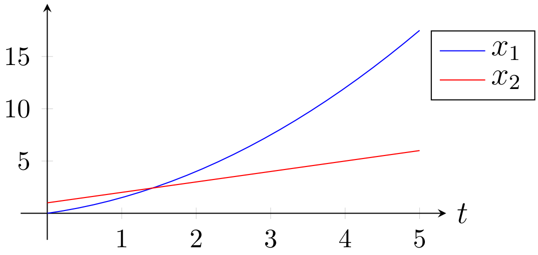
# Example: Accelerating Car

$$e^{At} = \sum_{k=0}^{\infty} A^k \frac{t^k}{k!} = I + At + A^2 \frac{t^2}{2} + \cdot$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, A^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A^3 = A^4 = \cdots = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} t + \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \frac{t^2}{2} = \begin{bmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$$

# Example: Accelerating Car




$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = e^{At} \begin{bmatrix} x_{1,\mathrm{ini}} \\ x_{2,\mathrm{ini}} \\ x_{3,\mathrm{ini}} \end{bmatrix} = \begin{bmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{1,\mathrm{ini}} \\ x_{2,\mathrm{ini}} \\ x_{3,\mathrm{ini}} \end{bmatrix} = \begin{bmatrix} x_{1,\mathrm{ini}} + t x_{2,\mathrm{ini}} + \frac{t^2}{2} x_{3,\mathrm{ini}} \\ x_{2,\mathrm{ini}} + t x_{3,\mathrm{ini}} \\ x_{3,\mathrm{ini}} \end{bmatrix}$$

diverging to infinity: **unstable**

# Example: Ideal Pendulum

Variables:

- horizontal position $x_1$, horizontal velocity $x_2$
- pendulum angle $\phi$ (only used in modeling phase)

Parameters:

- gravitational constant $g \approx 10$, length of pendulum $L$
- pendulum mass $m$ (only used in modeling phase)

Newton's equation of motion (assuming $\phi$ close to $0$):

$$m\ddot{x}_1 = -mg \sin\phi \cos\phi = -m\frac{g}{L}x_1 \underbrace{\cos\phi}_{\approx 1}$$

# Example: Pendulum

Newton's equation of motion (assuming $\phi$ close to $0$):

$$m\ddot{x}_1 = -m\tfrac{g}{L}x_1$$

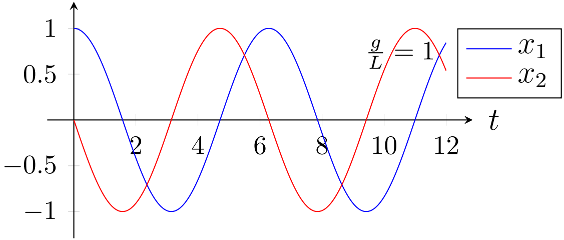
solving for $\dot{x}_1$:

$$\ddot{x}_1 = -\tfrac{g}{L}x_1$$

adding auxiliary variable $x_2 = \dot{x}_1$:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\tfrac{g}{L} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

# Example: Ideal Pendulum




$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = e^{At} \begin{bmatrix} x_{1,\text{ini}} \\ x_{2,\text{ini}} \end{bmatrix} = \begin{bmatrix} \cos(\omega t) & \frac{1}{\omega}\sin(\omega t) \\ -\omega\sin(\omega t) & \cos(\omega t) \end{bmatrix} \begin{bmatrix} x_{1,\text{ini}} \\ x_{2,\text{ini}} \end{bmatrix},$$

$$\omega = \sqrt{\frac{g}{L}}$$

perfect oscillation: **marginally stable**

# Example: Pendulum with Air Friction

drag force: $F_{\text{drag}} = -\beta\dot{x}$, with drag coefficient $\beta \geq 0$

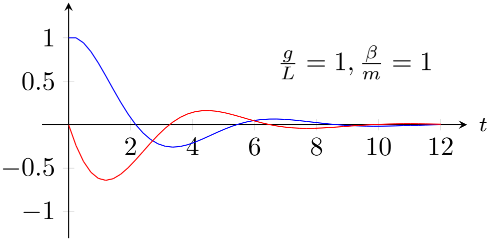
Newton's equation of motion (assuming $\phi$ close to $0$):

$$m\ddot{x}_1 = -m\tfrac{g}{L}x_1 - \beta\dot{x}_1$$

adding auxiliary variable $x_2 = \dot{x}_1$:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{L} & -\frac{\beta}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

# Example: Pendulum with Air Friction




$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = e^{At} \begin{bmatrix} x_{1,\text{ini}} \\ x_{2,\text{ini}} \end{bmatrix} = \begin{bmatrix} (\ldots) & (\ldots) \\ (\ldots) & (\ldots) \end{bmatrix} \begin{bmatrix} x_{1,\text{ini}} \\ x_{2,\text{ini}} \end{bmatrix}$$

the $(\ldots)$ are long terms made up of $e^{at}$, $\cos(\omega t)$, $\sin(\omega t)$

converging to equilibrium: **asymptotically stable**

# Solution (general from)

The elements of $e^{At}$ are sums of terms

$$c_{i,1} e^{a_i t} t^{p_i} \big( c_{i,2} \cos(b_i t) + c_{i,3} \sin(b_i t) \big)$$

$\lambda_i = a_i \pm \mathrm{i} b_i$ is the $i$-th **Eigenvalue** of $A$

$p_i \in 0, \ldots, m_i - 1$, where $m_i$ is the multiplicity of $\lambda_i$

- any $a_i > 0$: $||x(t)|| \to \infty$, unstable
- any $a_i = 0$ and $m_i > 1$: $||x(t)|| \to \infty$, unstable
- all $a_i = 0$ and $m_i = 1$: $x(t)$ periodic, marginally stable
- all $a_i < 0$: $||x(t)|| \to 0$, asymptotically stable

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# Determinant of a Matrix

## matrix determinant

$$|M| = \sum_{(i_1,\ldots,i_n)} \pm a_{i_1} a_{i_2} \cdot a_{i_n}$$

where $(i_1, \ldots, i_n)$ are permutations of $1, \cdots, n$ and $\pm$ is $+$ $(-)$ for even (odd) permutations

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

- $|M| = |M^T|$
- if $M$ (upper or lower) triangular, $|M| = m_{11} m_{22} \cdots m_{nn}$

# Eigenvalues of a Matrix

characteristic polynomial of $A$:

$$|A - \lambda I| = 0$$

characteristic polynomial has $n$ roots, called **Eigenvalues**

- EV of $A$ and of $A^T$ are identical
- if $A$ is (upper or lower) triangular, its EV are $a_{11}, \ldots, a_{nn}$

# Exercise: Stability

Compute the Eigenvalues to assess the stability of

1. coasting car
$$\begin{bmatrix} 0 & 1 \\ 0 & -\frac{\beta}{m} \end{bmatrix}$$
2. accelerating car
$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$
3. ideal pendulum
$$\begin{bmatrix} 0 & 1 \\ -\frac{g}{L} & 0 \end{bmatrix}$$
4. pendulum with air frinction
$$\begin{bmatrix} 0 & 1 \\ -\frac{g}{L} & -\frac{\beta}{m} \end{bmatrix}$$

# Exercise: Stability

What happens to an accelerating car with air friction?

$$m\ddot{x}_1 = -\beta\dot{x}_1 + x_3$$

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# Linear Time-Invariant (LTI) Systems

Linear system with inputs $u = \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}$, outputs $y = \begin{bmatrix} y_1 \\ \vdots \\ y_l \end{bmatrix}$

$$\dot{x} = \underbrace{A}_{n \times n} x + \underbrace{B}_{n \times m} u$$
$$y = \underbrace{C}_{l \times n} x$$

typically $m \ll n$, $l \ll n$

# Solution

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

solution from $x(0) = x_{\text{ini}}$ for given input function $u(t)$:

$$x(t) = e^{At}x_{\text{ini}} + e^{At}\int_{\tau=0}^{t} e^{-A\tau}Bu(\tau)d\tau$$
$$y(t) = Cx(t)$$