Linear Systems and Feedback Control II

Course Feedback Control and Real-time SystemsI HECS3: Performance and quantitative properties

High-confidence Embedded and Cyber-Physical Systems
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Plan

In this course we learn how to design a controller

- We have considered closed-loop **stability**
- Now we consider **time response specifications**
  1. By pole placement
  2. In a more systematic manner

Poles and Time Response

2nd order system transfer function: $G(s) = \frac{{\omega_n}^2}{s^2+2\xi\,\omega_n+{\omega_n}^2}$

The poles of the system: $P_{1,2} = \sigma \pm j\omega_d = -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2}$

ξ is Damping Ratio

ω_n is the Natural frequency

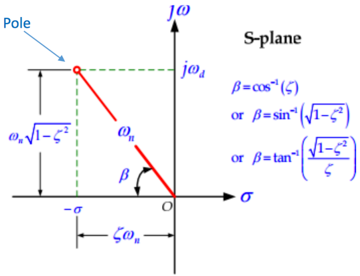
Rise Time: $T_r = \frac{\pi-\beta}{\omega_d}$

Settling Time: $T_s = \frac{4}{\xi\,\omega_n}$

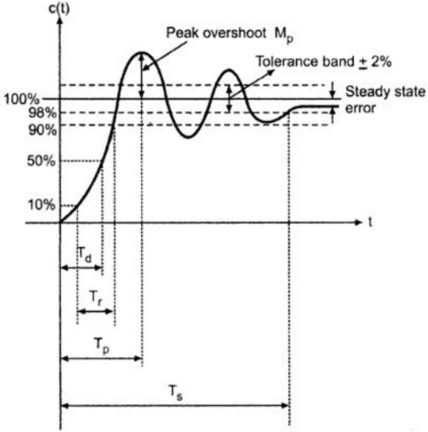
Delay Time: $T_d = \frac{1+0.7\,\xi}{\omega_n}$

Peak Time: $T_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\xi^2}}$

Maximum Overshoot: $M_p = e^{-\pi\xi/\sqrt{1-\xi^2}} \times 100\%$



Poles and Time Response



Poles and Time Response

- Delay Time (T_d): is the time required for the response to reach 50 percent of the final value.
- Rise Time (T_r): is the time required for the response to rise from 0 to 90 percent of the final value.
- Settling Time (T_s): is the time required for the response to reach and stay within a specified tolerance band (2 percent or 5 percent) of its final value.
- Peak Time (T_p): is the time required for the underdamped step response to reach the peak of time response.

Computing Time Response - Example 1

- Using Matlab
- Define a system by its transfer function

  $$\frac{s^2 + 5s + 5}{s^4 + 1.65s^3 + 5s^2 + 6.5s + 2}$$ by the following command:

  ```
  sys = tf([1 5 5],[1 1.65 5 6.5 2]);
  ```

- Compute its step response by:

  ```
  step(sys)
  ```

Computing Time Response - Example 2

- Define a system by its state space representation:

  $a = [-0.5572 - 0.7814; 0.78140];$
  $b = [1 - 1; 02];$
  $c = [1.96916.4493];$
  sys = ss(A,B,C,0);

- Compute its impulse response (that is, the response to the Dirac delta input, the Laplace transform of which is 1) by:

  impulse(sys)

Problem Setting

Given a linear system in a state space representation

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad\qquad x(0) = x_0$$
$$y(t) = Cx(t) + Du(t)$$

- $x(t) \in \mathbb{R}^n$ is the system state (vector of state variables), n: order of the state space representation
- $u(t) \in \mathbb{R}^m$ the control input
- $y(t) \in \mathbb{R}^p$ the measured output
- A, B, C and D are real-valued matrices
- x_0 is the initial state

Pole Placement Problem

Does there exist a state feedback control law

$$u(t) = -Fx(t)$$

such that the closed-loop poles are in predefined locations (denoted γ_i , $i = 1, \ldots, n$) in the complex plane?

Controllability

There **exists** a state feedback control $u(t) = -Fx(t)$ such that the poles of the closed-loop system are γ_i , $i = 1, \ldots, n$ if and only if the pair (A, B) is **controllable**.

Controllability matrix $\mathcal{C}(A, B) = [B \;\; AB \;\; A^2B \ldots A^{n-1}B]$ (where $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$)

- The pair (A, B) is controllable if and only if the controllability matrix $\mathcal{C}(A, B)$ is **full rank**
- For the case $m = 1$, the pair (A, B) is controllable if the square controllability matrix is **nonsingular**, that is $det(\mathcal{C}(A, B)) \neq 0$

Controllable Canonical Form

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \vdots & & 0 & 1 \\ -a_0 & -a_1 & \dots & \dots & -a_{n-1} \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix} \text{ and}$$

$$C = \begin{bmatrix} c_0 & c_1 & \dots & c_{n-1} \end{bmatrix}.$$

Let $F = \begin{bmatrix} f_1 & f_2 & \dots & f_n \end{bmatrix}$

Then

$$A - BF = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \vdots & & 0 & 1 \\ -a_0 - f_1 & -a_1 - f_2 & \dots & \dots & -a_{n-1} - f_n \end{bmatrix}$$

Pole Placement

The desired closed-loop polynomial
$(s-\gamma_1)(s-\gamma_2)\ldots(s-\gamma_n)$ can be developed as:

$$(s-\gamma_1)(s-\gamma_2)\ldots(s-\gamma_n) = s^n + \alpha_{n-1}s^{n-1} + \ldots + \alpha s + \alpha_0$$

Threfore $f_i = -a_{i-1} + \alpha_{i-1}$, $i = 1, \ldots, n$ ensures that the poles of $(A - BF)$ are $\{\gamma_i\}$, $i = 1, \ldots, n$

Procedure for General State Space Representation (1)

For a general state space representation, use a **change of basis** to put the system in the canonical form $\Rightarrow$ simplify the computation of the state feedback control gain F

In Matlab, use $F = acker(A, B, P)$, or a newer version $F = place(A, B, P)$, where P is the set of desired closed-loop poles

Procedure for General State Space Representation (2)

Procedure for the general case:

1. Check controllability of (A, B)
2. Calculate $\mathscr{C} = [B, AB, \dots, A^{n-1}B]$.

   Note $\mathscr{C}^{-1} = \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}$. Define $T = \begin{bmatrix} q_n \\ q_n A \\ \vdots \\ q_n A^{n-1} \end{bmatrix}^{-1}$
3. Note $\bar{A} = T^{-1}AT$ and $\bar{B} = T^{-1}B$ (which are under the controllable canonical form)
4. Choose the desired closed-loop poles and define the desired closed-loop characteristic polynomial:
   $s^n + \alpha_{n-1}s^{n-1} + \ldots + \alpha_1 s + \alpha_0$
5. Calculate the state feedback $u = -\bar{F}\bar{x}$ with:

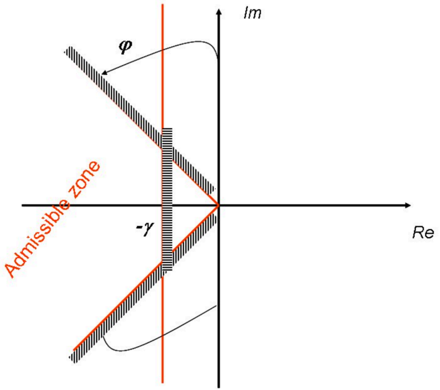
$$\bar{f}_i = -a_{i-1} + \alpha_{i-1}, i = 1, .., n$$

6. Calculate (for the original system):

$$u = -Fx, \text{ with } F = \bar{F}T^{-1}$$

Closed-Loop Poles as Specifications

The required closed-loop performances should be chosen in the following zone



which ensures a damping greater than $\xi = \sin\phi$.
$-\gamma$ implies that the real part of the CL poles are sufficiently negatives.

Closed-Loop Poles as Specifications

Some useful rules for selection the desired pole/zero locations (for a second order system):

- Rise time : $t_r \simeq \frac{1.8}{\omega_n}$
- Seetling time : $t_s \simeq \frac{4.6}{\xi \omega_n}$
- Overshoot $M_p = \exp(-\pi\xi / sqrt(1 - \xi^2))$:
  $\xi = 0.3 \Leftrightarrow M_p = 35\%$,
  $\xi = 0.5 \Leftrightarrow M_p = 16\%$,
  $\xi = 0.7 \Leftrightarrow M_p = 5\%$.