Multicuts in Series-Parallel Graphs and Box-TDIness

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The concept of total dual integrality dates back to the works of Edmonds, Giles and Pulleyblank in the late 70's of the previous century. This concept is strongly connected to min-max relations in combinatorial optimization problems. In this work we show a characterization of series-parallel graphs in terms of box-TDIness of the k-edge-connected spanning subgraph polyhedron.

Throughout the paper, all entries will be rational. Let us be given a linear system, and its linear programming duality equation:

$$\min\{cx : Ax \geq b\} = \max\{yb : yA = c,\, y \geq \mathbf{0}\} \tag{1}$$

The system $Ax \geq b$ is *totally dual integral* (*TDI*) if, for each integer vector c for which the minimum in (1) is finite, there exists an integer optimal solution for the maximum in (1). General properties of TDI systems can be found in [9, Chap. 22]. Every polyhedron P can be described by a TDI system $Ax \geq b$. When the polyhedron is integer, A and b can be chosen integer. The integrality of the TDI system is desirable because, then, (1) is a min-max relations between combinatorial objects.

We are interested in the stronger property of box-TDIness. A system $Ax \geq b$ is called *box-TDI* if the system $Ax \geq b, \ell \leq x \leq u$ is TDI for all rational vectors ℓ and u. A polyhedron that can be described by box-TDI system is called a *box-TDI polyhedron*. This definition is motivated the fact that any TDI system describing a box-TDI polyhedron is box-TDI [4]. The past few years, box-TDIness has received a renewed interest: new tools for recognising box-TDI systems and polyhedra are presented in [1] and [3]. Moreover, new box-TDI polyhedra have been discovered [2, 5, 7].

In this paper, we are interested in box-TDI systems and polyhedra associated with multicuts in series-parallel graphs. We provide systems having only integer coefficients for the flow cone and the k-edge-connected spanning subgraph polyhedron with k even. We prove that these systems are box-TDI if and only if the graph is series-parallel. We also prove that, when k is odd, the k-edge-connected spanning subgraph polyhedron is a box-TDI polyhedron.

Definitions. Throughout the paper, $G = (V, E)$ denotes a connected undirected graph, possibly with parallel edges and loops. A graph G is *series-parallel* if it does not contain K_4 as a minor [8]. For $S \subseteq V$, the set $\delta(S)$ is the set of edges of G with one endpoint in S and the other in $V \setminus S$. Such a set $\delta(S)$ is called a *cut*. Given a partition $S_1, S_2, \ldots, S_t$ of V, the *multicut* $\delta(S_1, \ldots, S_t)$ is the set of edges of G having their endpoints in two distinct S_i. Given a multicut $M = \delta(S_1, \ldots, S_t)$, we define $d_M = t$. A *flow* of G is obtained by signing the edges of a circuit of G, all but one receiving $+1$, the remaining one being assigned -1.

An integer box-TDI system for the polar of the cut cone. The *cut cone* of G is the cone generated by the incidence vectors of the cuts of G. In [5], series-parallel graphs are characterized in terms of the box-TDIness of the standard system describing the cut cone. The *flow cone* of a graph is the cone generated by the vectors χ^e for all $e \in E$ and the incidence vectors of the flows of G. In graphs with no K_5-minor, the cut cone and the flow cone are polar of one another, see e.g. [10]. In [3], the authors provide a system describing the flow cone which is box-TDI if and only if the graph is series-parallel. Unfortunately, this system has half-integer coefficients, and they raise the question of finding an integer box-TDI system describing this cone. Here, we provide an answer.

Theorem 1. *The system $x(M) \geq \mathbf{0}$, for all multicuts M of G is box-TDI if and only if the graph G is series-parallel.*

Box-TDIness for the k-edge-connected spanning subgraph polyhedron. Given $k \geq 2$, a graph is *k-edge-connected* if it remains connected after the deletion of any set of $k-1$ edges. The *k-edge-connected spanning subgraph polyhedron* of a graph G is the convex hull of the points $x \in \mathbb{Z}^E$ such that replacing each edge e by x_e copies yields a k-edge-connected spanning subgraph of G. In [6], Didi Biha and Mahjoub provide a linear description of this polyhedron for series-parallel graphs. In [2], Chen, Ding, and Zhang provide a box-TDI system describing the 2-edge-connected polyhedron of a series-parallel graph. More precisely, they prove that the system $\frac{1}{2}x(C) \geq 1,$ for all cuts C of $G, x \geq \mathbf{0}$ is box-TDI if and only if G is series-parallel. Their result implies that the k-edge-connected spanning subgraph polyhedron is box-TDI whenever k is even. However, there remains to find a box-TDI system with integer coefficients. We provide the following one.

Theorem 2. *Let $G = (V, E)$ be a graph and k even. The system*

$$x(M) \geq \frac{k}{2} d_M, \text{ for all multicuts } M \text{ of } G, x \geq \mathbf{0}$$

is box-TDI if and only if the graph G is series-parallel.

Perspectives. When the graph is series-parallel and k is odd, thanks to Didi Biha and Mahjoub [6], we know a linear description for k-edge-connected spanning subgraph polyhedron. Yet, neither a TDI nor a box-TDI system describing it is known. Our next goal is to provide an integer box-TDI system for this case. We already proved that the polyhedron is box-TDI, using a characterization of [5] for odd k.

Theorem 3. *Let $G = (V, E)$ be a graph and $k \geq 2$. The k-edge-connected spanning subgraph polyhedron of G is box-TDI if and only if G is series-parallel.*

Combining Theorem 3 with Cook's result [4], if one finds a TDI system describing this polyhedron for odd $k \geq 2$, then one gets a box-TDI system. This is our next goal, our starting point being the description of Didi Biha and Mahjoub in [6].

References

[1] X. Chen, G. Ding, and W. Zang. A Characterization of Box-Mengerian Matroid Ports *Mathematics of Operations Research*, 33(2):497 – 512, 2008.

[2] X. Chen, G. Ding, and W. Zang. The Box-TDI System associated with 2-edge Connected Spanning Subgraphs *Discrete Appl. Math.*, 157(1):118 – 125, 2009.

[3] P. Chervet, R. Grappe, L. Robert Characterizations of Box-Totally Dual Integral Polyhedra *Submitted*, 2016.

[4] W. Cook. On Box Totally Dual Integral Polyhedra. *Math. Program.*, 34(1):48–61, 1986.

[5] D. Cornaz, R. Grappe, and M. Lacroix. Trader Multiflow and Box-TDI Polyhedra. *Submitted*, 2016.

[6] M. Didi Biha, A.R. Mahjoub k-edge Connected Polyhedra on Series-parallel Graphs *Operations research letters*, 19(2):71 – 78, 1996.

[7] G. Ding, L. Tan, and W. Zang. When is the matching polytope box-totally dual integral? *Tech. Rep.*, 2015.

[8] R.J. Duffin. Topology of series-parallel networks *Journal of Mathematical Analysis and Applications*, 10(2):303 – 318, 1965.

[9] A. Schrijver. *Theory of linear and integer programming.* Wiley-Interscience series in discrete mathematics and optimization. Wiley, 1999.

[10] A. Schrijver. *Combinatorial optimization: polyhedra and efficiency.* Algorithms and combinatorics. Springer-Verlag, Berlin, Heidelberg, New York, N.Y., et al., 2003.