Ellipsoidal Toolbox

TCC Workshop

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Outline

- Problem setting and basic definitions
- Overview of existing methods and tools
- Ellipsoidal approach
- Systems with disturbances
- Hybrid systems
- Summary and outlook

System Equations

The controlled system:

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{t}, \boldsymbol{x}, \boldsymbol{u}), \quad \boldsymbol{t} \geq \boldsymbol{t_0}$$

state variable: $\boldsymbol{x} \in \mathbb{R}^{\mathrm{n}}$

Control:

- Open-loop: $\boldsymbol{u}(\boldsymbol{t}) \in \mathcal{P}(\boldsymbol{t}),\ \boldsymbol{t} \geq \boldsymbol{t_0}$
- Closed-loop: $\boldsymbol{u}(\boldsymbol{t}, \boldsymbol{x}) \in \mathcal{P}(\boldsymbol{t})\ (\boldsymbol{u}(\boldsymbol{t}, \boldsymbol{x}) \in \mathcal{P}(\boldsymbol{t}, \boldsymbol{x})),\ \boldsymbol{t} \geq \boldsymbol{t_0}$
- $\mathcal{P}(\boldsymbol{t})$ compact subset of $\mathbb{R}^{\mathrm{m}}$

Reachability (definitions)

- Reach set $X(\boldsymbol{t},\boldsymbol{t_0},X^{\mathbf{0}})$ at $\boldsymbol{t} > \boldsymbol{t_0}$ from $\{\boldsymbol{t_0},X^{\mathbf{0}}\}$:

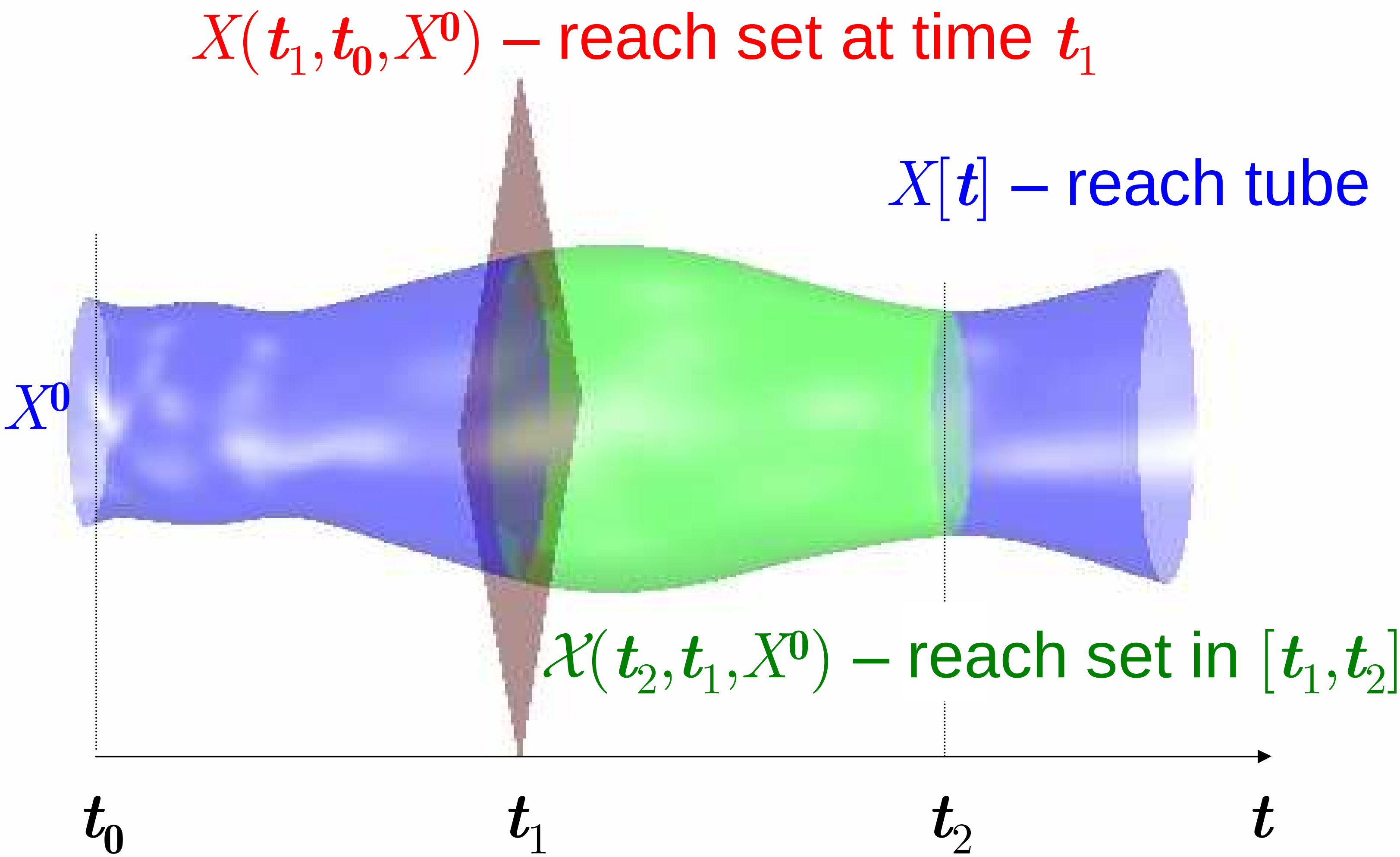
$$X(t,t_0,X^0) = \bigcup_{u(\cdot)\in\mathcal{P}(\cdot),x^0\in X^0} \{x(t,t_0,x^0|u(\cdot))\}$$

- Reach tube: map $\boldsymbol{t} \to X[\boldsymbol{t}] = X(\boldsymbol{t},\boldsymbol{t_0},X^{\mathbf{0}})$

- Reach set at some time within $[\boldsymbol{t}_1,\boldsymbol{t}_2]$:

$$\mathcal{X}(t_2,t_1,X^0) = \bigcup_{t_1\le\tau\le t_2} X(\tau,t_0,X^0)$$

Reachability (illustrations)



Reachability (properties)

- The reach sets are the same for open-loop and closed-loop controls

- Reach set $X(t, t_0, X^0)$ satisfies the semigroup property:

$$X(t, t_0, X^0) = X(t, \tau, X(\tau, t_0, X^0))$$

Also true for the reach tube $X[t]$

Backward Reach Set

Given:

- Target set Y^1
- Terminating time t_1

Backward reach set $Y(t, t_1, Y^1)$ at time t – set of all states y for each of which there exists control $u(\tau),\ t_0 \leq \tau < t$, such that $y(t) = y$ and $y(t_1) \in Y^1$

Linear Systems

- Continuous-time:

$$\dot{\boldsymbol{x}}(\boldsymbol{t}) = A(\boldsymbol{t})\boldsymbol{x}(\boldsymbol{t}) + B(\boldsymbol{t})\boldsymbol{u}(\boldsymbol{t})$$

- Discrete-time:

$$\boldsymbol{x}(\boldsymbol{t}+1) = A(\boldsymbol{t})\boldsymbol{x}(\boldsymbol{t}) + B(\boldsymbol{t})\boldsymbol{u}(\boldsymbol{t})$$

$$\boldsymbol{x}(\boldsymbol{t}_{\mathbf{0}}) \in X^{\mathbf{0}},\ \boldsymbol{u}(\boldsymbol{t}) \in \mathcal{P}(\boldsymbol{t})$$

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Algorithmic Methods

Polytopes (*MPT*)	linear systems exact reach set	ETHZ
Zonotopes (*MATISSE*)	linear systems external apprx.	UPenn/ Verimag
Hyperrectangles (*d/dt*)	linear systems external apprx.	Verimag
Oriented Rectangles (*CheckMate*)	autonomous systems external apprx.	CMU

Analytic Methods

Quantifier Elimination (*Requiem*)	linear nilpotent systems exact reach set	UPenn
Parallelotopes	linear systems external/internal apprx.	IMM
Level Sets (*Level Set Toolbox*)	any systems exact reach set	UBC
Barrier Certificates	polynomial systems no reach set computation	Caltech

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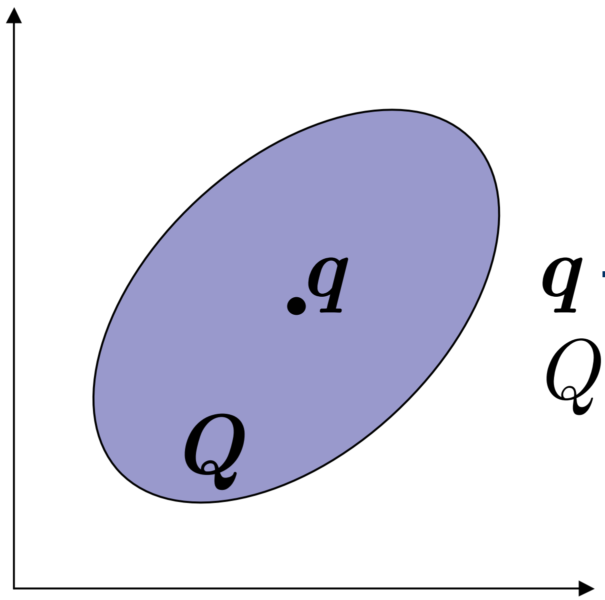
Ellipsoid

$$\mathcal{E}(\boldsymbol{q}, Q) = \{\boldsymbol{x} \in \mathbb{R}^{\mathrm{n}} \mid \langle(\boldsymbol{x} - \boldsymbol{q}),\ Q^{-1}(\boldsymbol{x} - \boldsymbol{q})\rangle \le 1\}$$

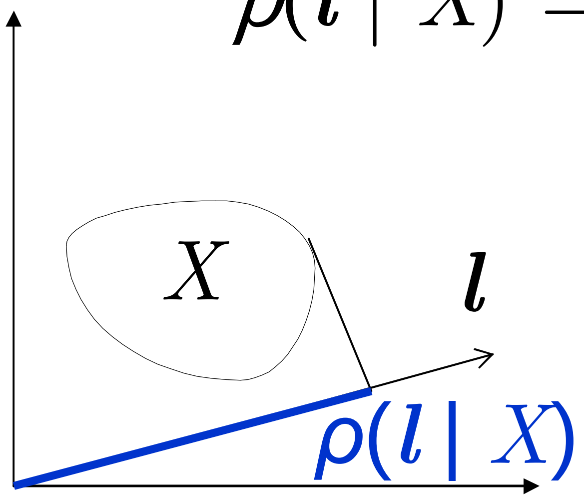


$\boldsymbol{q}$ – center of ellipsoid

Q – shape matrix ($Q = Q^{\mathrm{T}} > 0$)

Support Function

$$\rho(\boldsymbol{l} \mid X) = \mathbf{sup}\ \{\langle \boldsymbol{l},\ \boldsymbol{x}\rangle \mid \boldsymbol{x} \in X\}$$



Support function of ellipsoid:

$$\rho(\boldsymbol{l} \mid \mathcal{E}(\boldsymbol{q}, Q)) = \langle \boldsymbol{l},\ \boldsymbol{q}\rangle + \langle \boldsymbol{l},\ Q\boldsymbol{l}\rangle^{1/2}$$

Linear Systems

- Continuous-time:

$$\dot{\boldsymbol{x}}(\boldsymbol{t}) = A(\boldsymbol{t})\boldsymbol{x}(\boldsymbol{t}) + B(\boldsymbol{t})\boldsymbol{u}(\boldsymbol{t})$$

- Discrete-time:

$$\boldsymbol{x}(\boldsymbol{t}+1) = A(\boldsymbol{t})\boldsymbol{x}(\boldsymbol{t}) + B(\boldsymbol{t})\boldsymbol{u}(\boldsymbol{t})$$

$$\boldsymbol{x}(\boldsymbol{t}_{\mathbf{0}}) \in \boxed{\mathcal{E}(\boldsymbol{x}_{\mathbf{0}}, X_{\mathbf{0}})},\ \boldsymbol{u}(\boldsymbol{t}) \in \boxed{\mathcal{E}(\boldsymbol{p}(\boldsymbol{t}), P(\boldsymbol{t}))}$$

Reach Set of Linear System

Symmetric convex compact set in $\mathbb{R}^n$ evolving in time

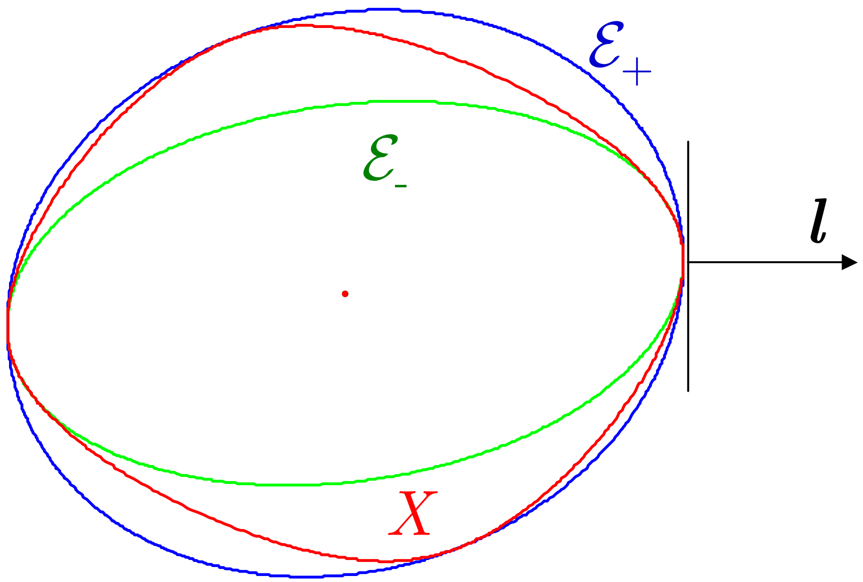
Tight Approximations

- External ellipsoidal approximation $\mathcal{E}_+$ of symmetric convex set X is tight if
  - $X \subseteq \mathcal{E}_+$
  - There exists $\boldsymbol{l}$ such that $\rho(\pm\boldsymbol{l} \mid \mathcal{E}_+) = \rho(\pm\boldsymbol{l} \mid X)$
- Internal ellipsoidal approximation $\mathcal{E}_-$ of symmetric convex set X is tight if
  - $\mathcal{E}_- \subseteq X$
  - There exists $\boldsymbol{l}$ such that $\rho(\pm\boldsymbol{l} \mid \mathcal{E}_-) = \rho(\pm\boldsymbol{l} \mid X)$

Reach Set Approximation

For any $\boldsymbol{l}$ there exist $\mathcal{E}_+$ and $\mathcal{E}_-$:

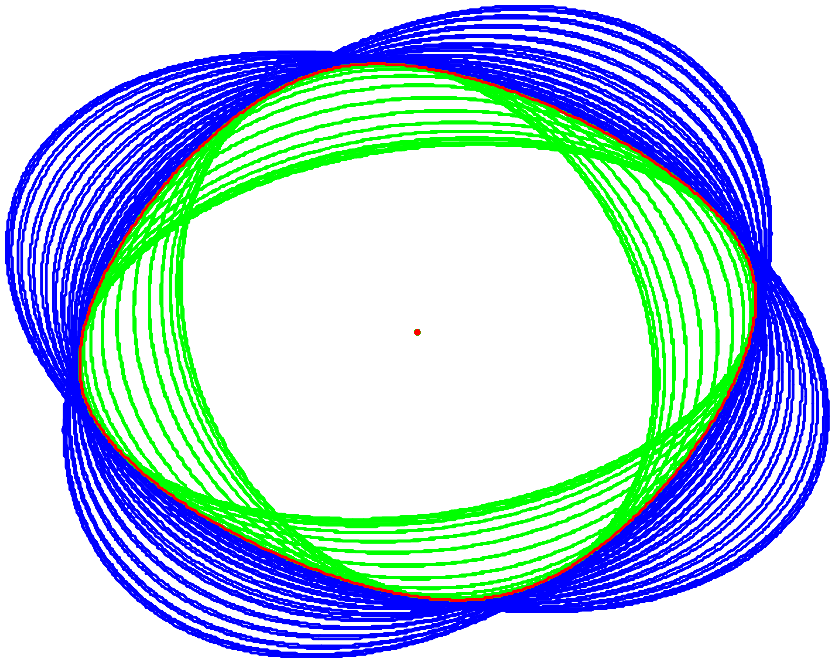
- $\mathcal{E}_- \subseteq X \subseteq \mathcal{E}_+$
- $\rho(\pm\boldsymbol{l} \mid \mathcal{E}_-) = \rho(\pm\boldsymbol{l} \mid X) = \rho(\pm\boldsymbol{l} \mid \mathcal{E}_+)$



Reach Set Approximation

For any $\boldsymbol{l}$ there exist $\mathcal{E}_+$ and $\mathcal{E}_-$:

- $\mathcal{E}_- \subseteq X \subseteq \mathcal{E}_+$
- $\rho(\pm\boldsymbol{l} \mid \mathcal{E}_-) = \rho(\pm\boldsymbol{l} \mid X) = \rho(\pm\boldsymbol{l} \mid \mathcal{E}_+)$

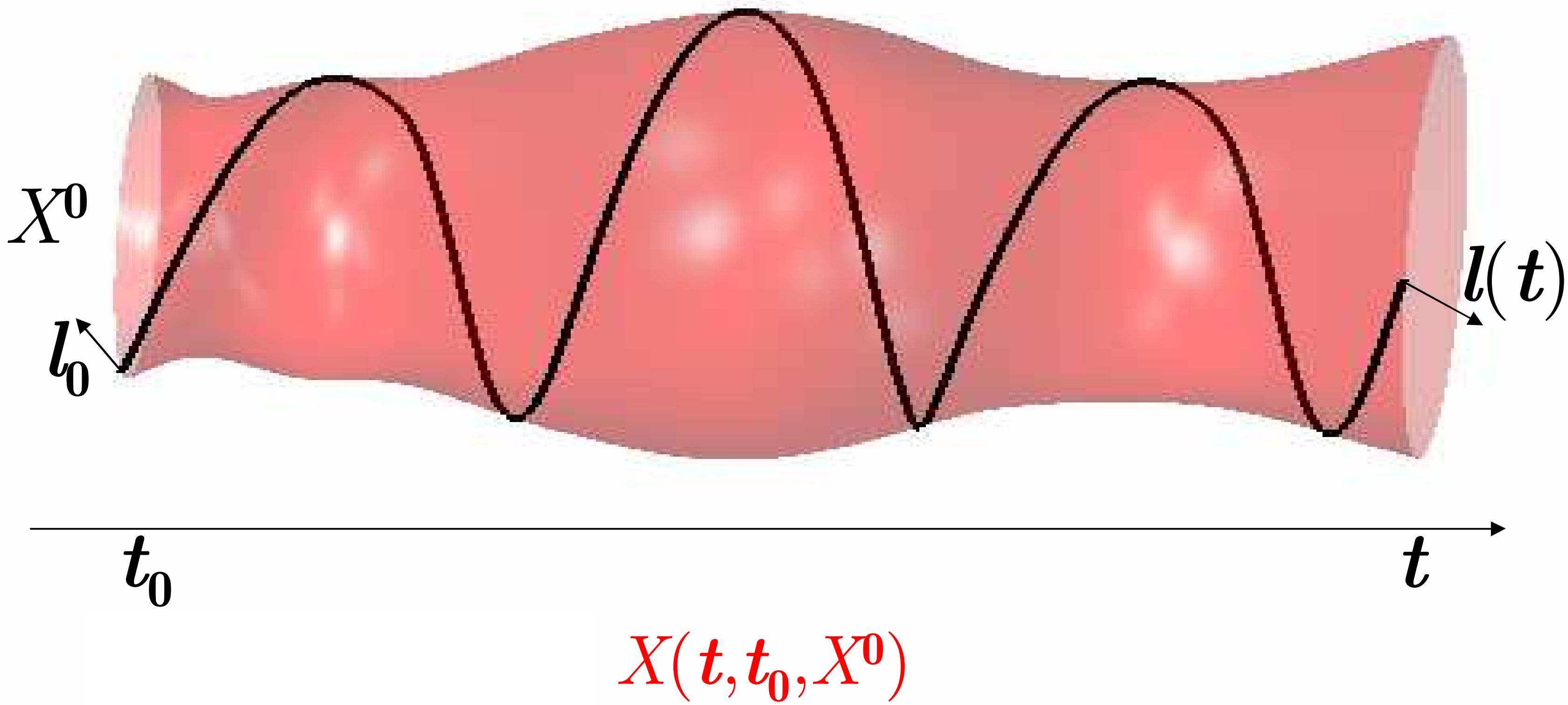


$$\bigcup_{\boldsymbol{l}} \mathcal{E}_- = X = \bigcap_{\boldsymbol{l}} \mathcal{E}_+$$

Good Curves (concept)

$$\dot{l}(t) = -A^{\mathrm{T}}(t)\,l(t),\ l(t_0) = l_0$$

<u>good curve</u>

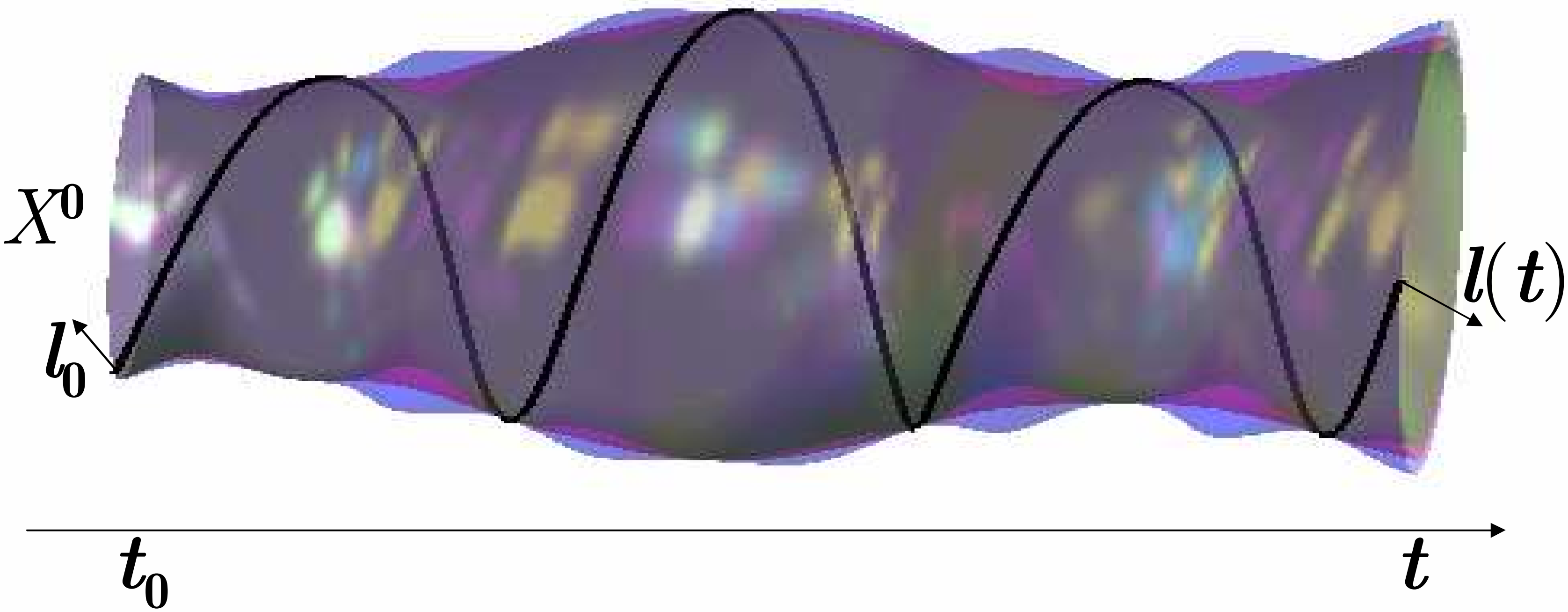


GC

Good Curves (concept)

$$\dot{l}(t) = -A^{\mathrm{T}}(t)\, l(t),\; l(t_0) = l_0$$

good curve

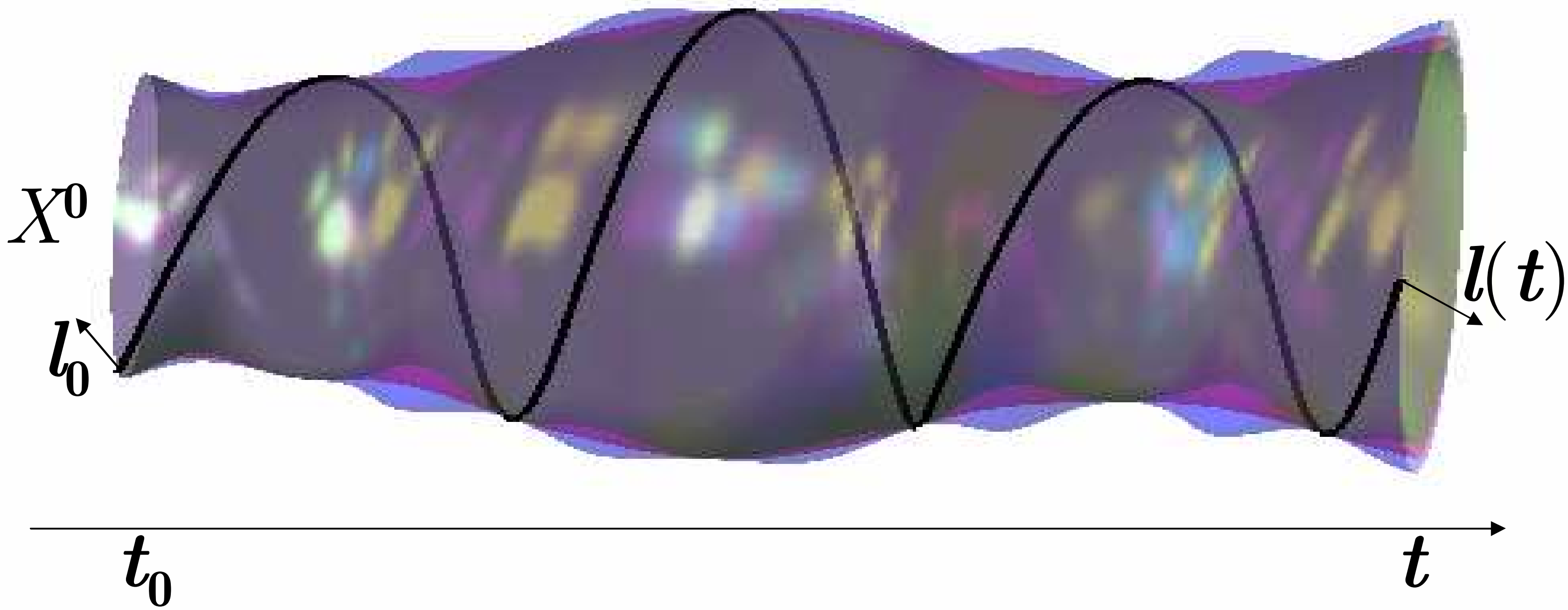


$$\mathcal{E}(x_c(t), X_l^-(t)) \subseteq X(t, t_0, X^0) \subseteq \mathcal{E}(x_c(t), X_l^+(t))$$

Good Curves (concept)

$$\dot{l}(t) = -A^{\mathrm{T}}(t)\,l(t),\; l(t_0) = l_0$$

good curve



$$\rho(l(t)|\ \mathcal{E}(x_c(t), X_l^-(t))) = \rho(l(t)|X(t, t_0, X^0)) = \rho(l(t)|\ \mathcal{E}(x_c(t), X_l^+(t)))$$

Good Curves (summary)

If $\boldsymbol{l}(\boldsymbol{t})$ satisfies $\dot{\boldsymbol{l}}(\boldsymbol{t}) = -A^{\mathrm{T}}(\boldsymbol{t})\boldsymbol{l}(\boldsymbol{t}),\ \boldsymbol{l}(\boldsymbol{t}_0) = \boldsymbol{l}_0$, then

- $\mathcal{E}(\boldsymbol{x}_c(\boldsymbol{t}), X_l^-(\boldsymbol{t})) \subseteq X(\boldsymbol{t}, \boldsymbol{t}_0, X^0) \subseteq \mathcal{E}(\boldsymbol{x}_c(\boldsymbol{t}), X_l^+(\boldsymbol{t}))$
- $\rho(\boldsymbol{l}(\boldsymbol{t}) | \mathcal{E}(\boldsymbol{x}_c(\boldsymbol{t}), X_l^-(\boldsymbol{t}))) = \rho(\boldsymbol{l}(\boldsymbol{t}) | X(\boldsymbol{t}, \boldsymbol{t}_0, X^0)) = \rho(\boldsymbol{l}(\boldsymbol{t}) | \mathcal{E}(\boldsymbol{x}_c(\boldsymbol{t}), X_l^+(\boldsymbol{t})))$

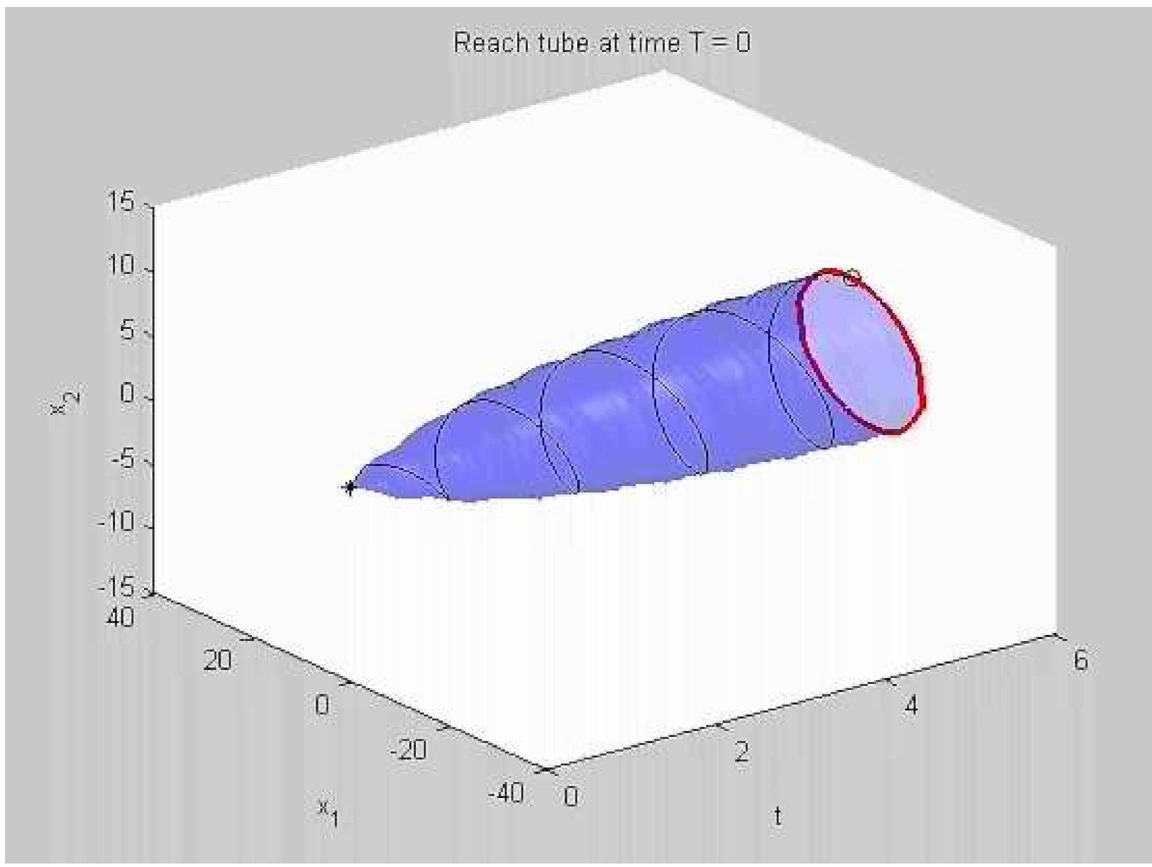
where $\dot{\boldsymbol{x}}_c(\boldsymbol{t}) = A(\boldsymbol{t})\boldsymbol{x}_c(\boldsymbol{t}) + B(\boldsymbol{t})\boldsymbol{p}(\boldsymbol{t}),\ \boldsymbol{x}_c(\boldsymbol{t}_0) = \boldsymbol{x}_0$,
and the shape matrices $X_l^+(\boldsymbol{t})$, $X_l^-(\boldsymbol{t})$ are governed by single ODEs

- *On ellipsoidal techniques for reachability analysis*
  by A.B.Kurzhanski, P.Varaiya (2000)

Steering the system
to a given target point at given time

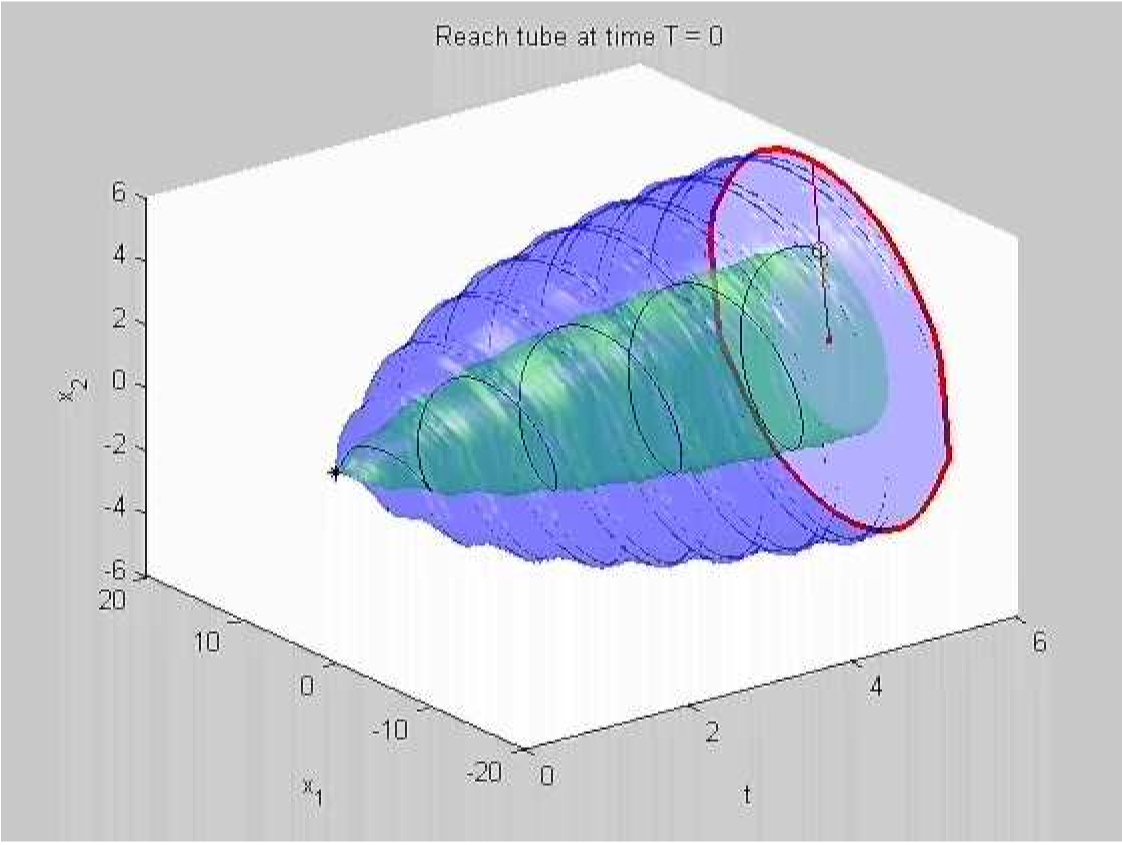
Good Curves (control)

$$u_l(t) = p(t) + \frac{P(t)B^T(t)\Phi(t_0,t)l_0}{\langle l_0, \Phi(t_0,t)B(t)P(t)B^T(t)\Phi(t_0,t)l_0\rangle^{1/2}}$$



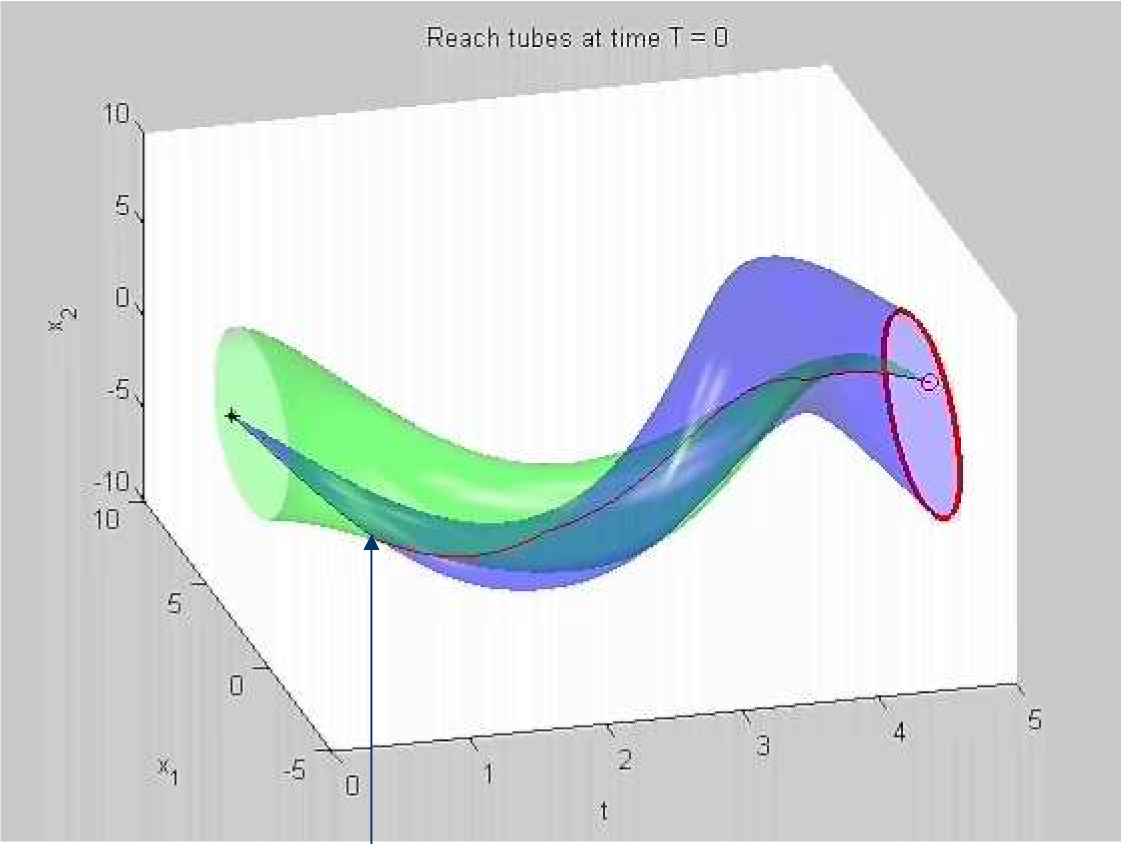
Reaching Internal Point

Scale the set of controls: $\mathcal{E}(\boldsymbol{p}(\boldsymbol{t}), \boldsymbol{\mu}^2 P(\boldsymbol{t})),\ |\boldsymbol{\mu}| \leq 1$



Reaching Internal Point

Colliding forward and backward reach tubes



switch good curves

Ellipsoidal Toolbox©

www.eecs.berkeley.edu/~akurzhan/ellipsoids

Ellipsoidal Toolbox

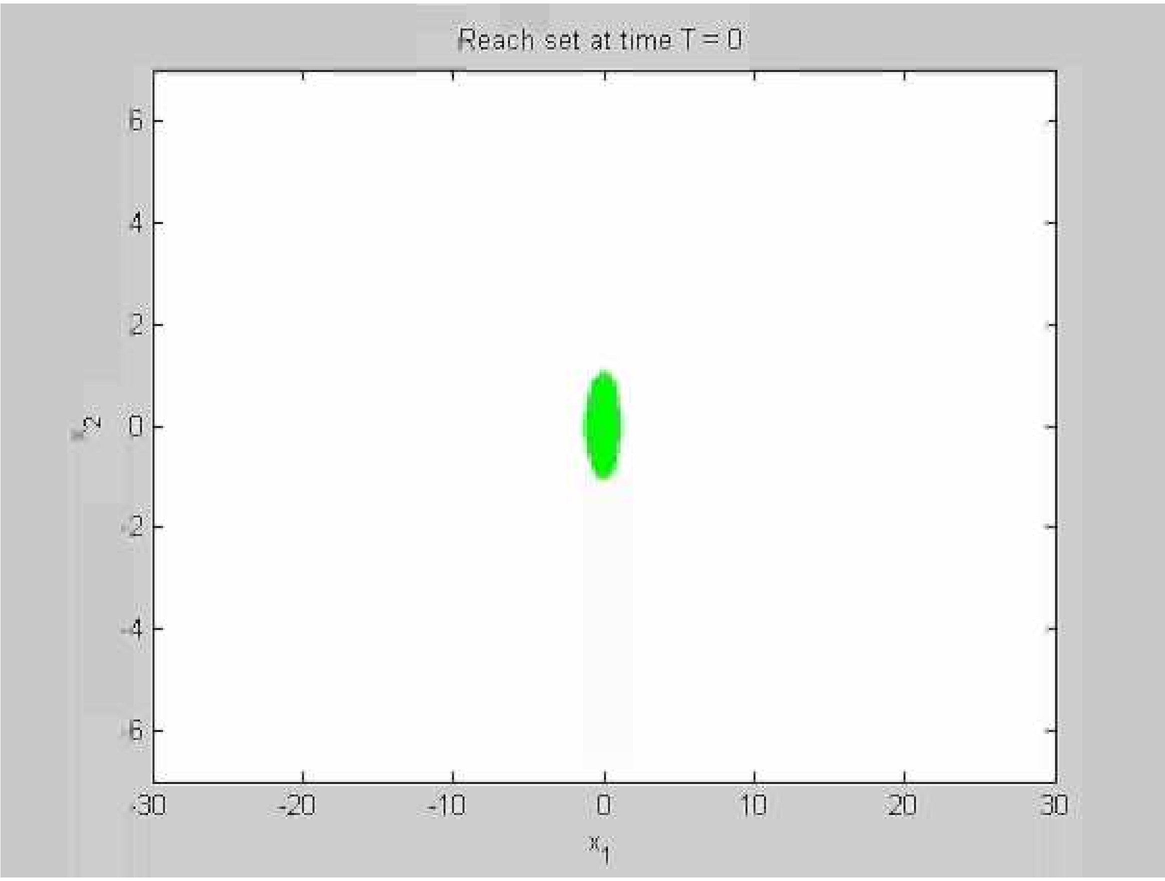
- Ellipsoidal calculus
  - Geometric sums and differences
  - Intersections with ellipsoids, hyperplanes, polyhedra
- Reachability analysis
  - Continuous- and discrete-time linear systems
  - Forward and backward reach sets
- Visualization (2D and 3D)
  - Plotting of ellipsoids, hyperplanes, reach sets
  - Projections

MATLAB Types

$\mathcal{ET}$ implements classes:

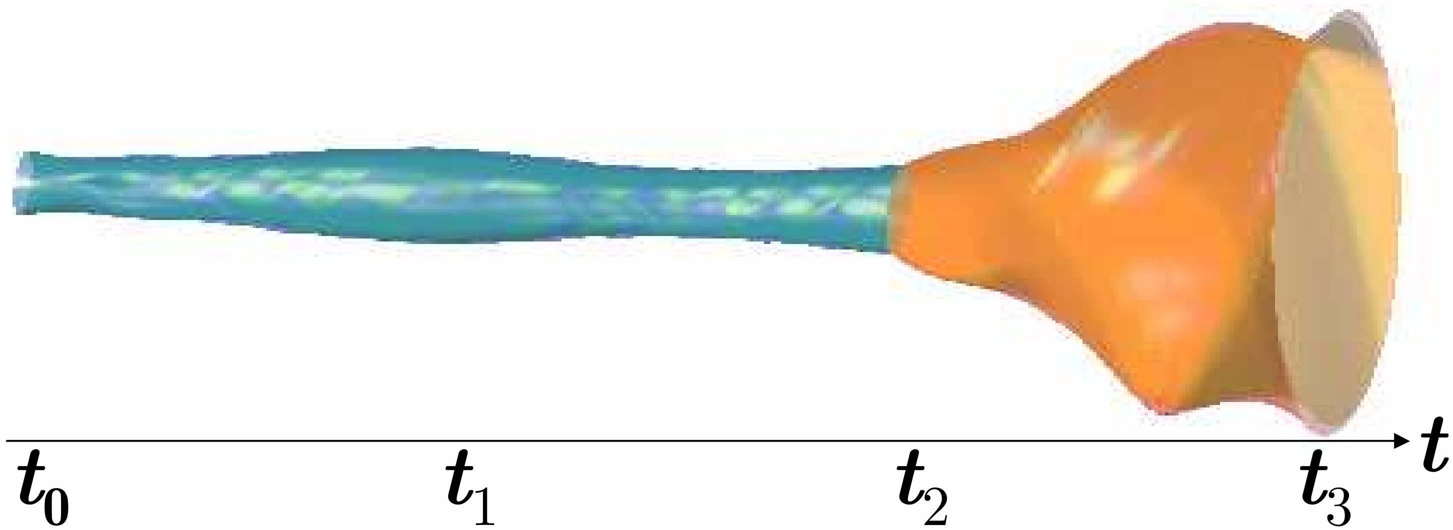
- **`ellipsoid`**
- **`hyperplane`**
- **`linsys`**
- **`reach`**

Approximation Refinement



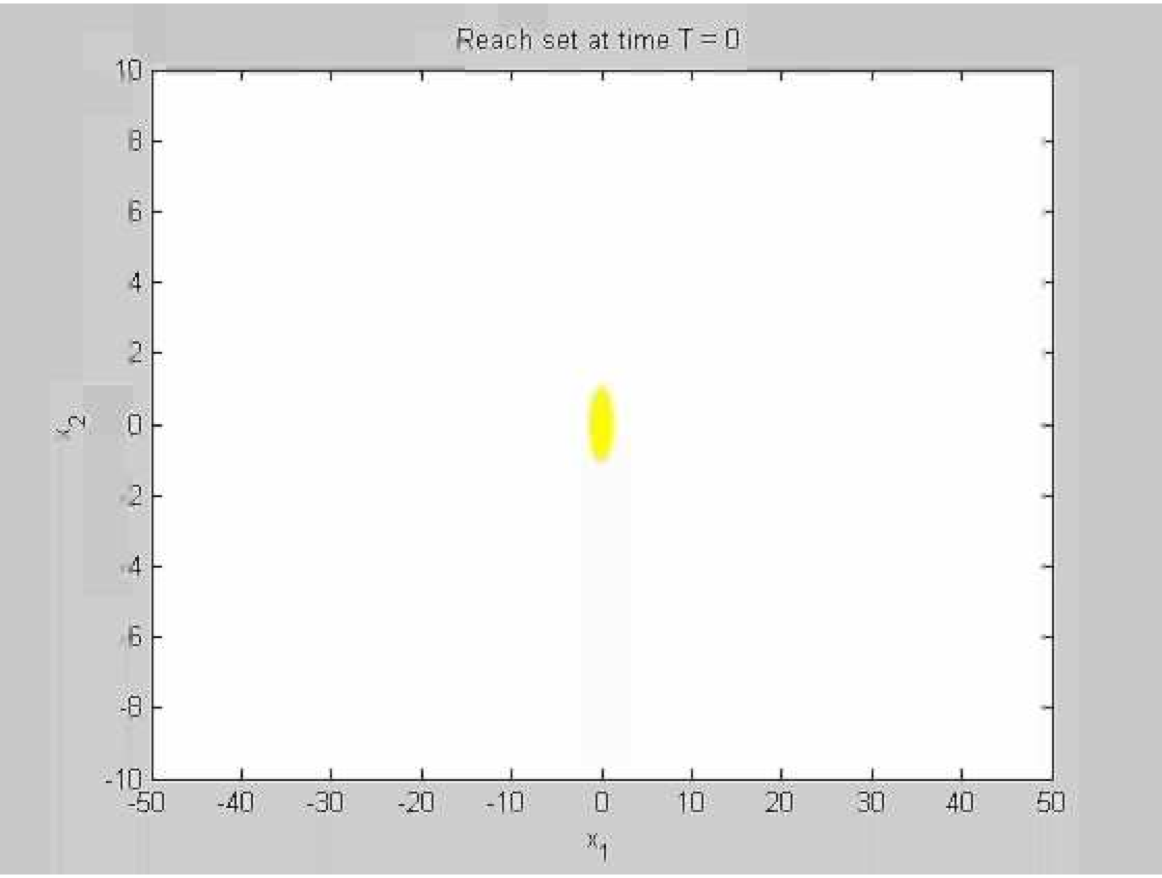
$\boldsymbol{ET}$ function: **`refine`**

Semigroup Property



ET function: **`evolve`**

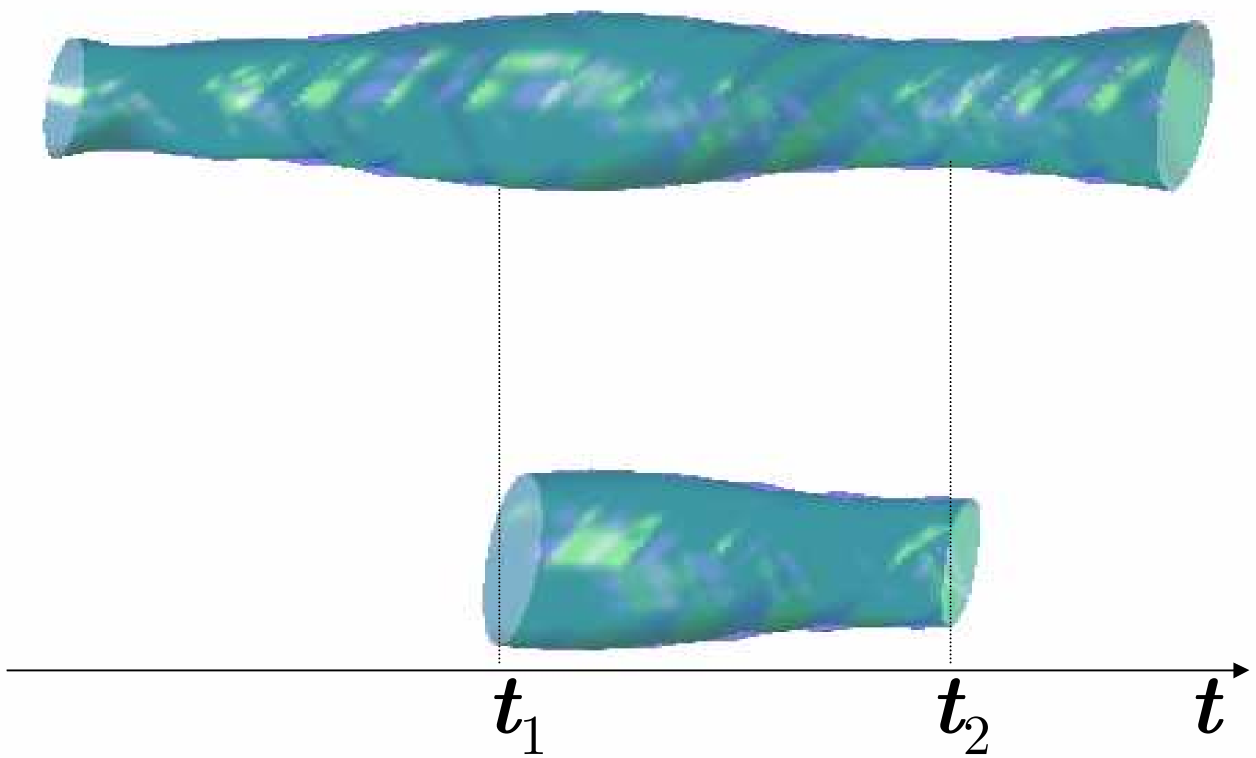
Switched System



$\mathcal{ET}$ function: **`evolve`**

3D

Cutting the Reach Tube



$\boldsymbol{ET}$ function: **`cut`**

Verification

- Check if reach set external (internal) approximation intersects with given object: **`ellipsoid`**, **`hyperplane`**, **`polytope`**

ET function: **`intersect`**

Discrete-Time Systems

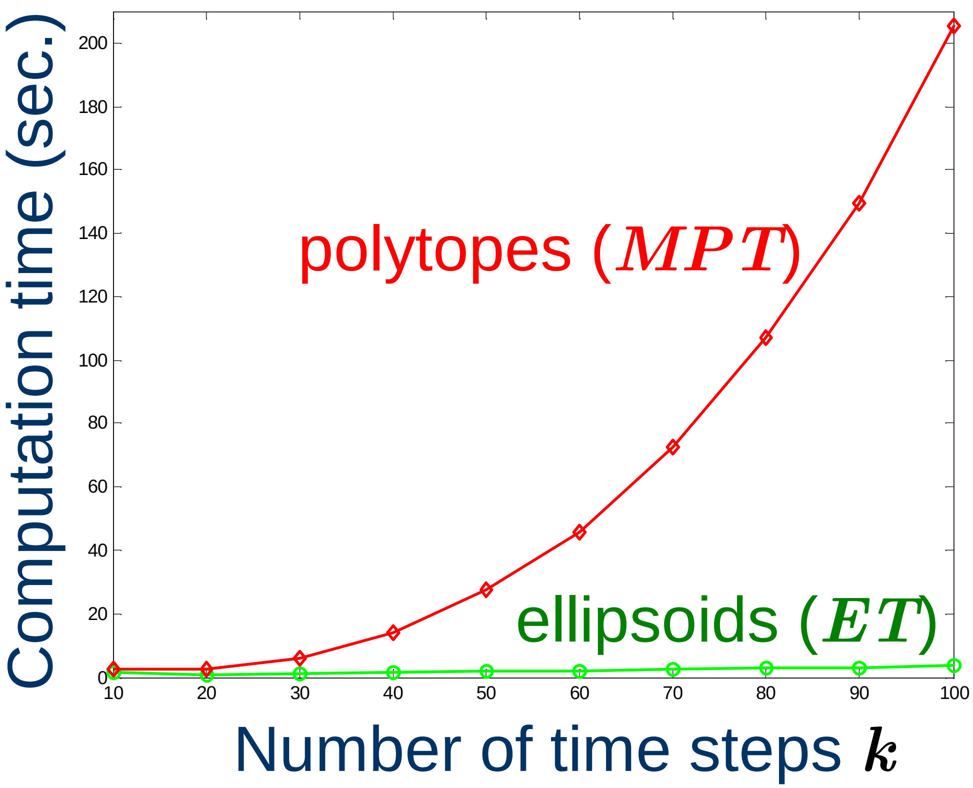
$$\boldsymbol{x}[\boldsymbol{k}+1] = A[\boldsymbol{k}]\boldsymbol{x}[\boldsymbol{k}] + B[\boldsymbol{k}]\boldsymbol{u}[\boldsymbol{k}]$$

$$\boldsymbol{x}[\boldsymbol{k_0}] \in \mathcal{E}(\boldsymbol{x_0}, X_{\boldsymbol{0}}),\ \boldsymbol{u}[\boldsymbol{k}] \in \mathcal{E}(\boldsymbol{p}[\boldsymbol{k}], P[\boldsymbol{k}])$$

Same ellipsoidal theory applies
with some adjustments

Ellipsoids vs Polytopes

$$x[k+1] = \begin{bmatrix} \cos(1) & -\sin(1) \\ \sin(1) & \cos(1) \end{bmatrix} x[k] + u[k], \; x[0] \in X^0, \; u[k] \in \mathcal{P}$$



Complexity:

$$\boldsymbol{k}(L(8n^3 + 4n^2 + 2n) + 2n^2)$$

number of directions l_0

state space dimension

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Linear System with Disturbance

- System equation:

$$\dot{\boldsymbol{x}}(\boldsymbol{t}) = A(\boldsymbol{t})\boldsymbol{x}(\boldsymbol{t}) + B(\boldsymbol{t})\boldsymbol{u}(\boldsymbol{t},\boldsymbol{x}(\boldsymbol{t})) + G(\boldsymbol{t})\boldsymbol{v}(\boldsymbol{t})$$

- Initial state: $\boldsymbol{x}(\boldsymbol{t}_0) \in X^0 = \mathcal{E}(\boldsymbol{x}_0, X_0)$
- Control
  - Open-loop: $\boldsymbol{u}(\boldsymbol{t}) \in \mathcal{P}(\boldsymbol{t}) = \mathcal{E}(\boldsymbol{p}(\boldsymbol{t}), P(\boldsymbol{t}))$
  - Closed-loop: $\boldsymbol{u}(\boldsymbol{t},\boldsymbol{x}(\boldsymbol{t})) \in \mathcal{P}(\boldsymbol{t}) = \mathcal{E}(\boldsymbol{p}(\boldsymbol{t}), P(\boldsymbol{t}))$

- Disturbance: $\boldsymbol{v}(\boldsymbol{t}) \in \mathcal{Q}(\boldsymbol{t}) = \mathcal{E}(\boldsymbol{q}(\boldsymbol{t}), Q(\boldsymbol{t}))$

Reach Sets

Open-loop reach set (OLRS) and closed-loop reach set (CLRS) of a system with disturbance are different

OLRS of MAXMIN Type

- Given initial set X^0, time $t > t_0$, $X^-(t, t_0, X^0)$ is the set of all x, such that for any $v(\tau) \in \mathcal{Q}(\tau)$ there exists $x^0 \in X^0$ and $u(\tau) \in \mathcal{P}(\tau)$, $t_0 \leq \tau < t$, which steer the system from $x(t_0) = x^0$ to $x(t) = x$

- $X^-(t, t_0, X^0)$ is subzero level set of

$$V^-(t, x) = \max_v \min_u \{\mathbf{dist}(x(t_0), X^0) \mid x(t) = x\}$$

OLRS of MINMAX Type

- Given initial set $X^{\mathbf{0}}$, time $t > t_0$, $X^+(t,t_0,X^{\mathbf{0}})$ is the set of all x, for which there exists $u(\tau) \in \mathcal{P}(\tau)$, that for all $v(\tau) \in \mathcal{Q}(\tau)$ assigns $x^0 \in X^{\mathbf{0}}$ such that trajectory $x(\tau)$, $t_0 \le \tau < t$, leads from $x(t_0) = x^0$ to $x(t) = x$

- $X^+(t,t_0,X^{\mathbf{0}})$ is subzero level set of

$$V^+(t,x) = \mathbf{min}_u \mathbf{max}_v \{\mathbf{dist}(x(t_0), X^{\mathbf{0}}) \mid x(t) = x\}$$

OLRS Properties

- MAXMIN reach set:

$$X^-(t, t_0, X^0) =$$
$$\left(\Phi(t, t_0)X^0 \oplus \int_{t_0}^{t} \Phi(t, \tau)B(\tau)\mathcal{P}(\tau)d\tau\right) \dot{-} \int_{t_0}^{t} \Phi(t, \tau)(-G(\tau))\mathcal{Q}(\tau)d\tau$$

geometric difference

- MINMAX reach set:

$$X^+(t, t_0, X^0) =$$
$$\left(\Phi(t, t_0)X^0 \dot{-} \int_{t_0}^{t} \Phi(t, \tau)(-G(\tau))\mathcal{Q}(\tau)d\tau\right) \oplus \int_{t_0}^{t} \Phi(t, \tau)B(\tau)\mathcal{P}(\tau)d\tau$$

- $X^+(\boldsymbol{t}, \boldsymbol{t_0}, X^{\boldsymbol{0}}) \subseteq X^-(\boldsymbol{t}, \boldsymbol{t_0}, X^{\boldsymbol{0}})$

Sequential MAXMIN

- Correction at t_1: $[t_0,\ t] = [t_0,\ t_1] \cup [t_1,\ t]$

$$X^-_1(t, t_0, X^0) = X^-(t, t_1, X^-(t_1, t_0, X^0))$$

$$X_1^-(t, t_0, X^0) \subseteq X^-(t, t_0, X^0)$$

- k corrections: $t_0 \leq t_1 \leq \ldots \leq t_k \leq t$

$$X^-_k(t, t_0, X^0) = X^-(t, t_k, X^-_{k\text{-}1}(t_1, t_0, X^0))$$

$$X_k^-(t, t_0, X^0) \subseteq \ldots \subseteq X_1^-(t, t_0, X^0) \subseteq X^-(t, t_0, X^0)$$

Sequential MINMAX

- Correction at $\boldsymbol{t}_1$: $[\boldsymbol{t}_0,\ \boldsymbol{t}] = [\boldsymbol{t}_0,\ \boldsymbol{t}_1] \cup [\boldsymbol{t}_1,\ \boldsymbol{t}]$

$$X^{+}{}_{1}(\boldsymbol{t},\boldsymbol{t}_0,X^{\mathbf{0}}) = X^{+}(\boldsymbol{t},\boldsymbol{t}_1,X^{+}(\boldsymbol{t}_1,\boldsymbol{t}_0,X^{\mathbf{0}}))$$

$$X^{+}(\boldsymbol{t},\boldsymbol{t}_0,X^{\mathbf{0}}) \subseteq X_1{}^{+}(\boldsymbol{t},\boldsymbol{t}_0,X^{\mathbf{0}})$$

- $\boldsymbol{k}$ corrections: $\boldsymbol{t}_0 \le \boldsymbol{t}_1 \le \ldots \le \boldsymbol{t}_{\boldsymbol{k}} \le \boldsymbol{t}$

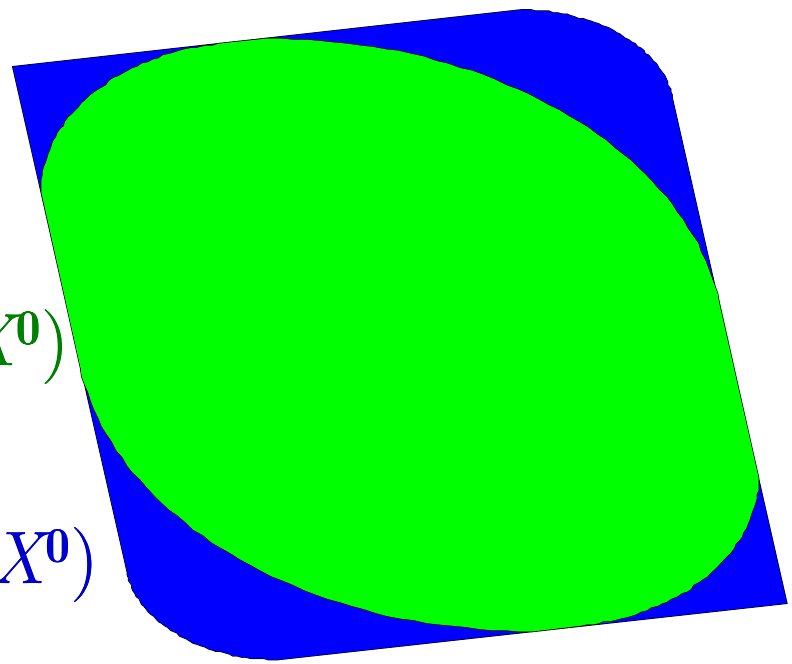
$$X^{+}{}_{\boldsymbol{k}}(\boldsymbol{t},\boldsymbol{t}_0,X^{\mathbf{0}}) = X^{+}(\boldsymbol{t},\boldsymbol{t}_{\boldsymbol{k}},X^{+}{}_{\boldsymbol{k}\text{-}1}(\boldsymbol{t}_1,\boldsymbol{t}_0,X^{\mathbf{0}}))$$

$$X^{+}(\boldsymbol{t},\boldsymbol{t}_0,X^{\mathbf{0}}) \subseteq X_1{}^{+}(\boldsymbol{t},\boldsymbol{t}_0,X^{\mathbf{0}}) \subseteq \ldots \subseteq X_{\boldsymbol{k}}{}^{+}(\boldsymbol{t},\boldsymbol{t}_0,X^{\mathbf{0}})$$

Piecewise Open-Loop

$$\boxed{X^{+}(t,t_0,X^0) \subseteq X^{+}{}_{k}(t,t_0,X^0)} \subseteq \boxed{X^{-}{}_{k}(t,t_0,X^0) \subseteq X^{-}(t,t_0,X^0)}$$

MINMAX MAXMIN



$X^{+}{}_{k}(t,t_0,X^0)$

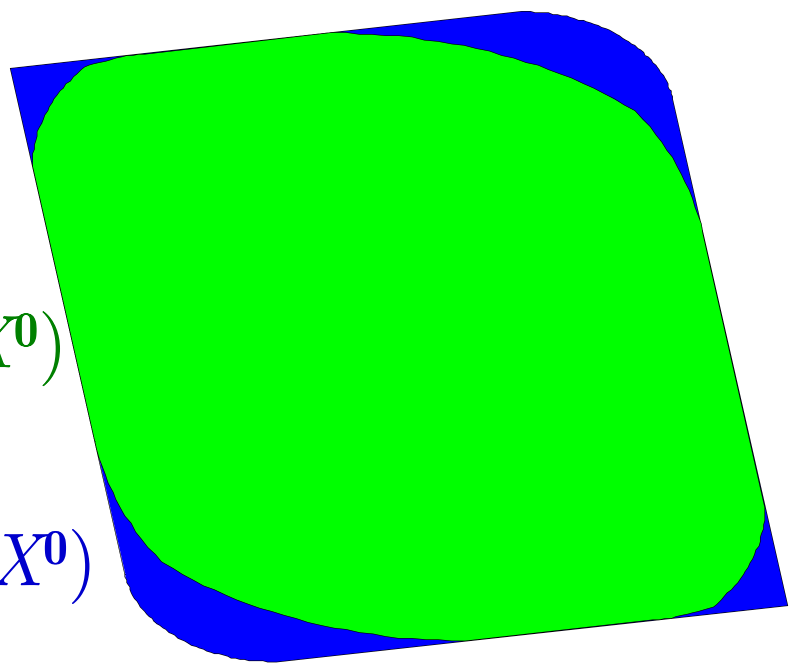
$X^{-}{}_{k}(t,t_0,X^0)$

$k = 0$

Piecewise Open-Loop

$$X^{+}(t,t_0,X^0) \subseteq X^{+}_{k}(t,t_0,X^0) \subseteq X^{-}_{k}(t,t_0,X^0) \subseteq X^{-}(t,t_0,X^0)$$

MINMAX MAXMIN

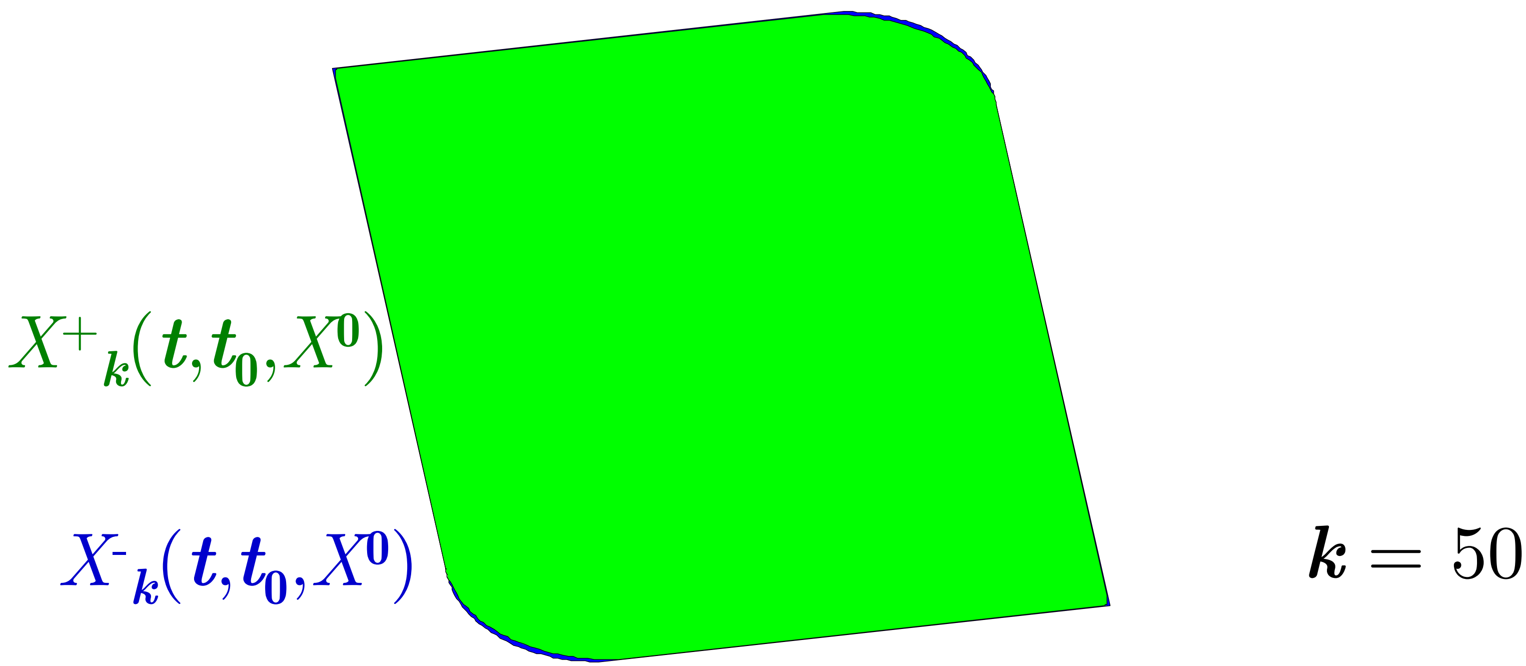


$k = 1$

Piecewise Open-Loop

$$\boxed{X^{+}(t,t_0,X^0) \subseteq X^{+}_{k}(t,t_0,X^0)} \subseteq \boxed{X^{-}_{k}(t,t_0,X^0) \subseteq X^{-}(t,t_0,X^0)}$$

MINMAX MAXMIN



Piecewise Open-Loop

$$X^{+}(t,t_0,X^0) \subseteq X^{+}{}_k(t,t_0,X^0) \subseteq X^{-}{}_k(t,t_0,X^0) \subseteq X^{-}(t,t_0,X^0)$$

MINMAX MAXMIN

$$k \to \infty$$

$$X^{+}{}_\infty(t,t_0,X^0) = X^{-}{}_\infty(t,t_0,X^0) = X(t,t_0,X^0)$$

CLRS

- *On reachability under uncertainty*
  by A.B.Kurzhanskiy, P.Varaiya

Closed-Loop Reach Set (CLRS)

Given initial set X^0, time $t > t_0$,

$X(t, t_0, X^0)$ is the set of all x, for each of which there exist $x^0 \in X^0$ and $u(\tau, x(\tau)) \in \mathcal{P}(\tau)$ that for every $v(\tau) \in \mathcal{Q}(\tau)$ assigns trajectory $x(\tau)$:

$$\dot{x}(\tau) \in A(\tau)x(\tau) + B(\tau)u(\tau, x(\tau)) + G(\tau)v(\tau)$$

where $t_0 \leq \tau < t$, such that $x(t_0) = x_0$ and $x(t) = x$

CLRS Computation

- Tight ellipsoidal approximations for $X(\boldsymbol{t},\boldsymbol{t_0},X^{\mathbf{0}})$:

$$X(\boldsymbol{t},\boldsymbol{t_0},X^{\mathbf{0}}) = \bigcap \mathcal{E}(\boldsymbol{x}_c(\boldsymbol{t}),X_{\boldsymbol{l}}^{+}(\boldsymbol{t})) = \bigcup \mathcal{E}(\boldsymbol{x}_c(\boldsymbol{t}),X_{\boldsymbol{l}}^{-}(\mathrm{t}))$$

where $\boldsymbol{x}_c(\boldsymbol{t})$ satisfies

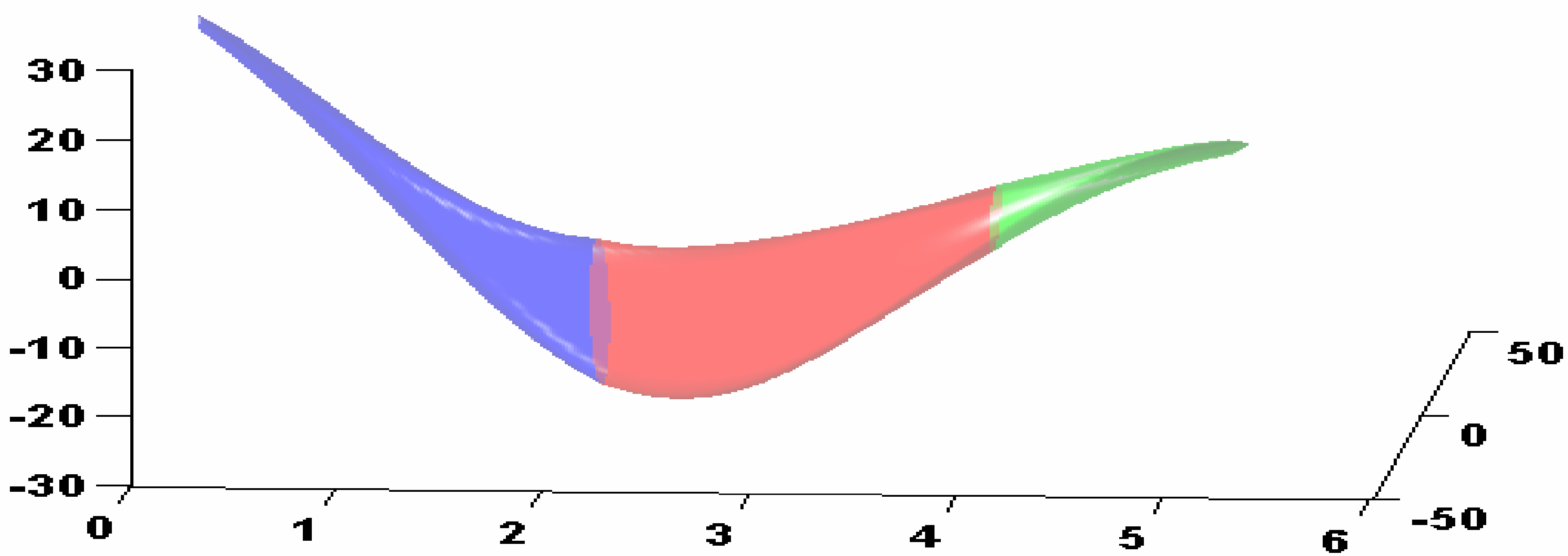
$$\dot{\boldsymbol{x}}_c(\boldsymbol{t}) = A(\boldsymbol{t})\boldsymbol{x}(\boldsymbol{t}) + B(\boldsymbol{t})\boldsymbol{p}(\boldsymbol{t}) + G(\boldsymbol{t})\boldsymbol{q}(\boldsymbol{t})$$

and $X_{\boldsymbol{l}}^{+}(\boldsymbol{t})$, $X_{\boldsymbol{l}}^{-}(\boldsymbol{t})$ are obtained from ODEs

- Implemented in $\boldsymbol{ET}$

Example

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -0.2 & 1 \\ -1 & -0.2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v(t)$$



Steering the System to a Target

SPRING-MASS SYSTEM

- *Reachability approaches and ellipsoidal techniques for closed-loop control of oscillating systems under uncertainty*

  by A.N.Daryin, A.B.Kurzhanski, I.V.Vostrikov

Outline

- Problem setting and basic definitions
- Overview of existing methods and tools
- Ellipsoidal approach
- Systems with disturbances
- Hybrid systems
- Summary and outlook

Hybrid Setting

- Discrete states (modes)
- Continuous dynamics – affine
- Enabling zones (guards) – hyperplanes, ellipsoids, polyhedra
- Resets – affine

- No Zeno

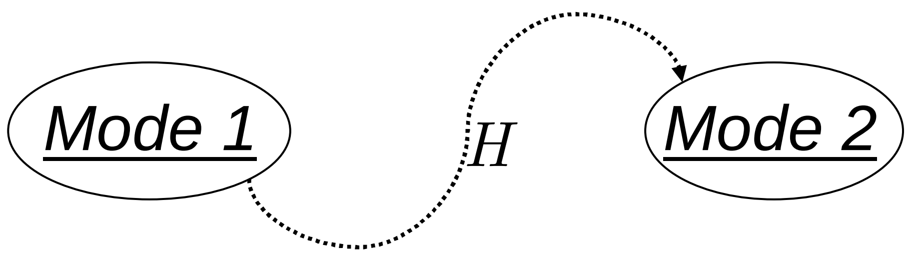
Hybrid System Example

- *Mode 1*:

$$\dot{\boldsymbol{x}}(\boldsymbol{t}) = A_1\boldsymbol{x}(\boldsymbol{t})+B_1\boldsymbol{u}(\boldsymbol{t}),\ \boldsymbol{u}(\boldsymbol{t})\in\mathcal{P}_1(\boldsymbol{t})$$

- *Mode 2*:

$$\dot{\boldsymbol{x}}(\boldsymbol{t}) = A_2\boldsymbol{x}(\boldsymbol{t})+B_2\boldsymbol{u}(\boldsymbol{t})+G_2\boldsymbol{v}(\boldsymbol{t}),\ \boldsymbol{u}(\boldsymbol{t})\in\mathcal{P}_2(\boldsymbol{t}),\ \boldsymbol{v}(\boldsymbol{t})\in \mathcal{Q}_2(\mathrm{t})$$



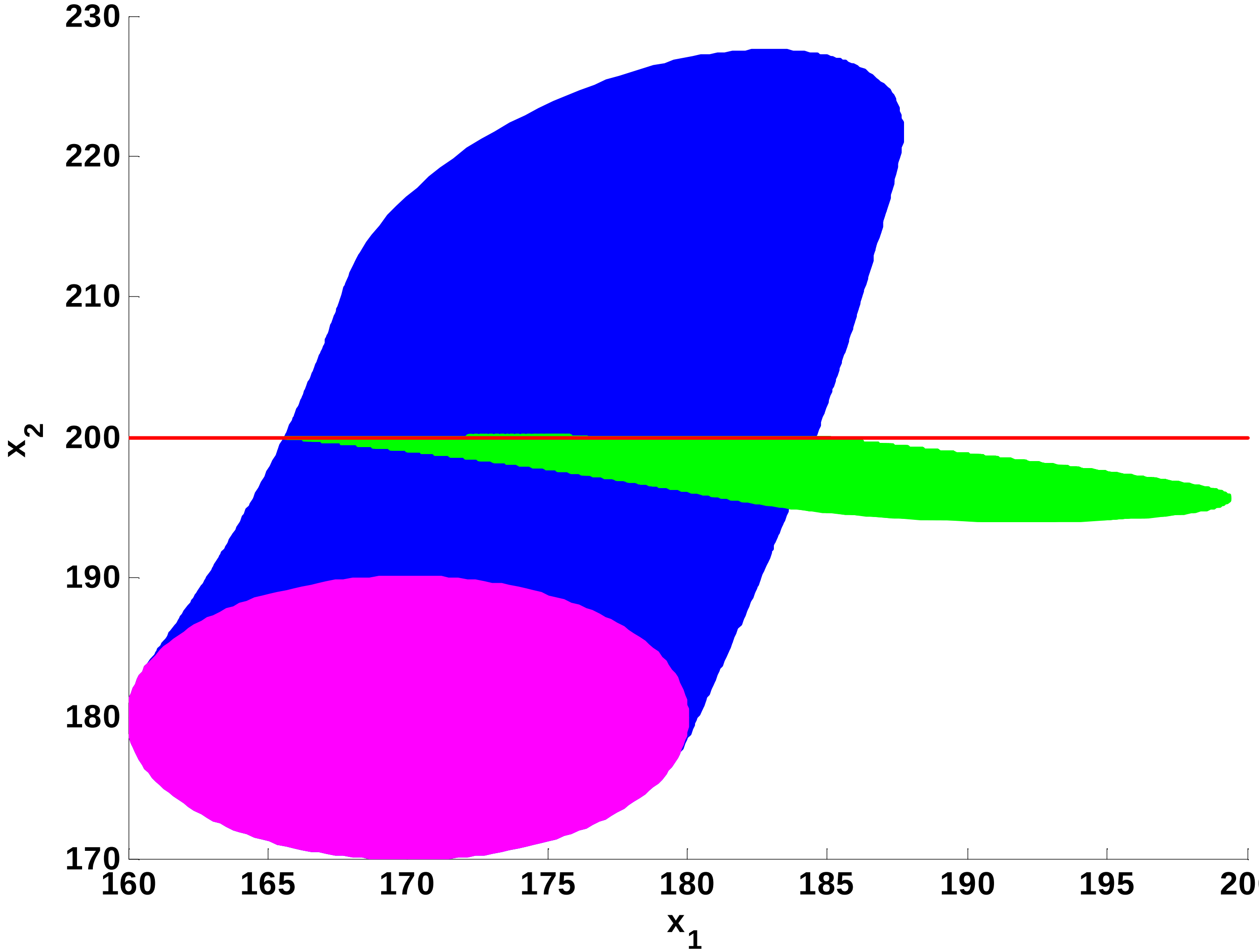
- Guard: hyperplane H
- Reset: identity

Hybrid Reach Set Computation

- Initial conditions: ***Mode 1***, $\boldsymbol{t_0}$, $X^{\mathbf{0}}$
- Compute reach set for ***Mode 1***: $X_1(\boldsymbol{t}, \boldsymbol{t_0}, X^{\mathbf{0}})$
- Detect when $X_1(\boldsymbol{\tau}, \boldsymbol{t_0}, X^{\mathbf{0}}) \cap H \neq \emptyset$, $\boldsymbol{t_0} \leq \boldsymbol{\tau} \leq \boldsymbol{t}$
- For each such $\boldsymbol{\tau}$, compute reach set for ***Mode 2***: $X_2(\boldsymbol{t}, \boldsymbol{\tau}, (X_1(\boldsymbol{\tau}, \boldsymbol{t_0}, X^{\mathbf{0}}) \cap H))$
- Reach set of the whole system:

$$X_1(\boldsymbol{t}, \boldsymbol{t_0}, X^{\mathbf{0}}) \bigcup_{\boldsymbol{\tau}} X_2(\boldsymbol{t}, \boldsymbol{\tau}, (X_1(\boldsymbol{\tau}, \boldsymbol{t_0}, X^{\mathbf{0}}) \cap H))$$

Reach Set Trace Projection



2D

Outline

- Problem setting and basic definitions
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Road Ahead

- State estimation
- Discrete-time systems with disturbance
- Obstacle problems
- Stochastic systems