Efficient Computation of Reachable Sets of Linear Time-Invariant Systems with Inputs

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joint work with

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Motivations

- Discrete Linear Time Invariant System:

$$x_{k+1} = \Phi x_k + u_k \quad x_0 \in \Omega_0, \forall i \, u_i \in U$$

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We will not detail here how Ω_0, Φ and U can be obtained from a continuous time system.

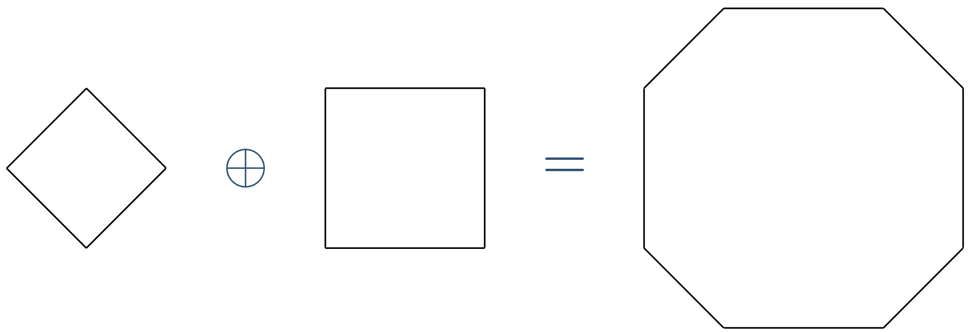
DLTI

We want to compute the N first sets of the sequence defined by:

$$\Omega_{n+1} = \Phi\Omega_n \oplus U$$

- Ω_0 is the set of initial points
- U is the set of inputs
- Φ is a $d \times d$ matrix
- $\oplus$ is the Minkowski sum

$$A \oplus B = \{a + b | a \in A \text{ and } b \in B\}$$



A naive algorithm

Direct use of the recurence relation:

$$\Omega_{n+1} = \Phi\Omega_n \oplus U$$

For that, we need a class of sets closed under linear transformation and Minkowski sum, for example: convex polytopes represented by their vertices.

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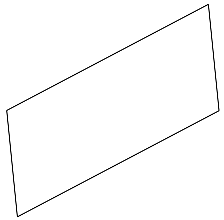

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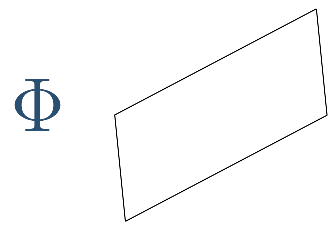

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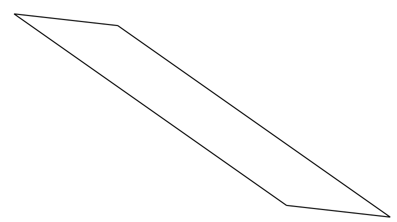

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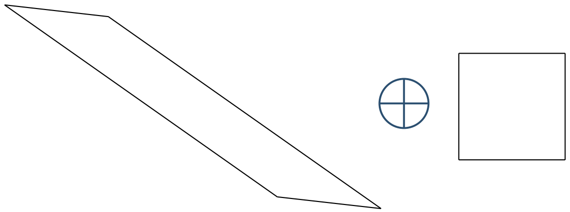

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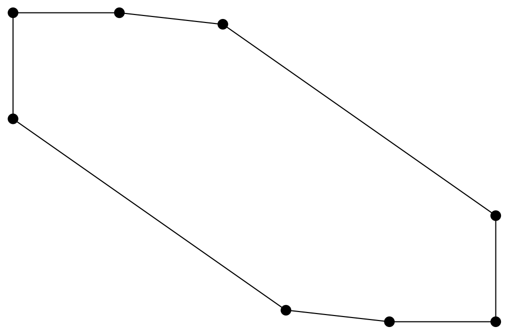

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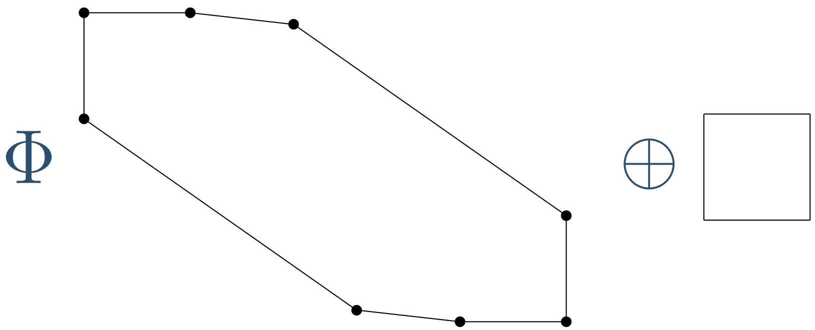

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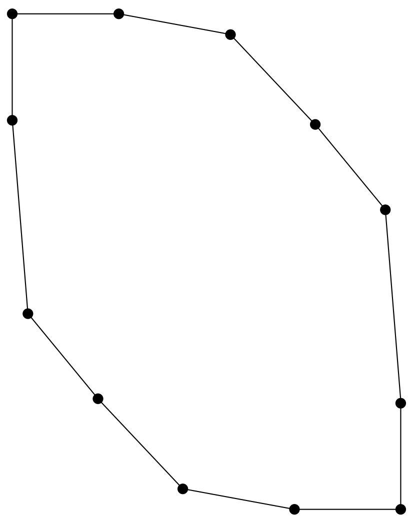

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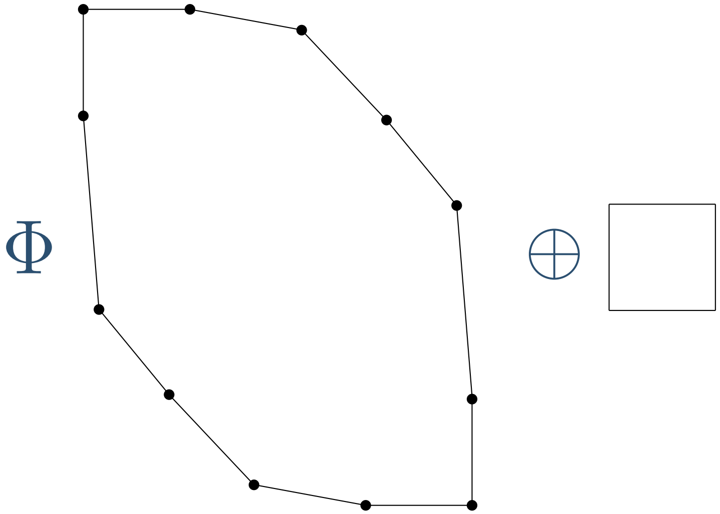

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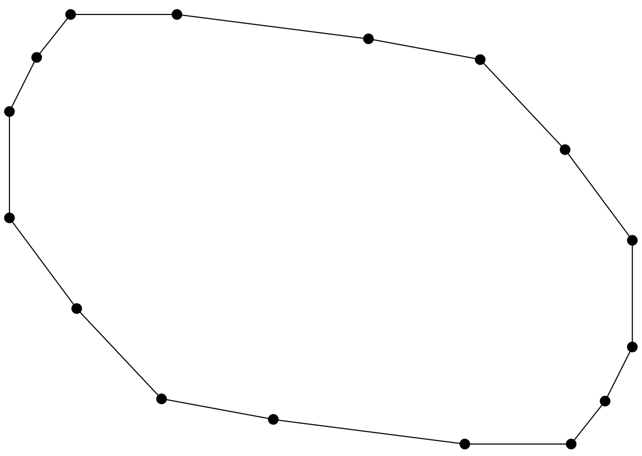

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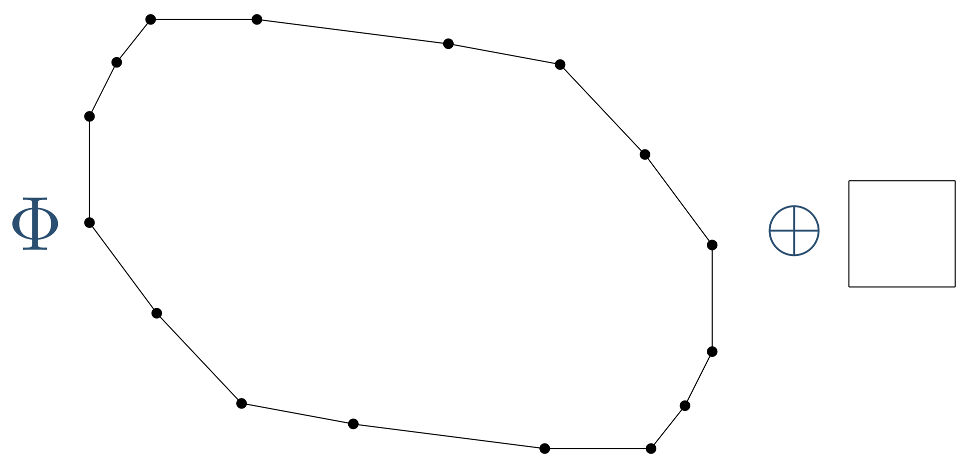

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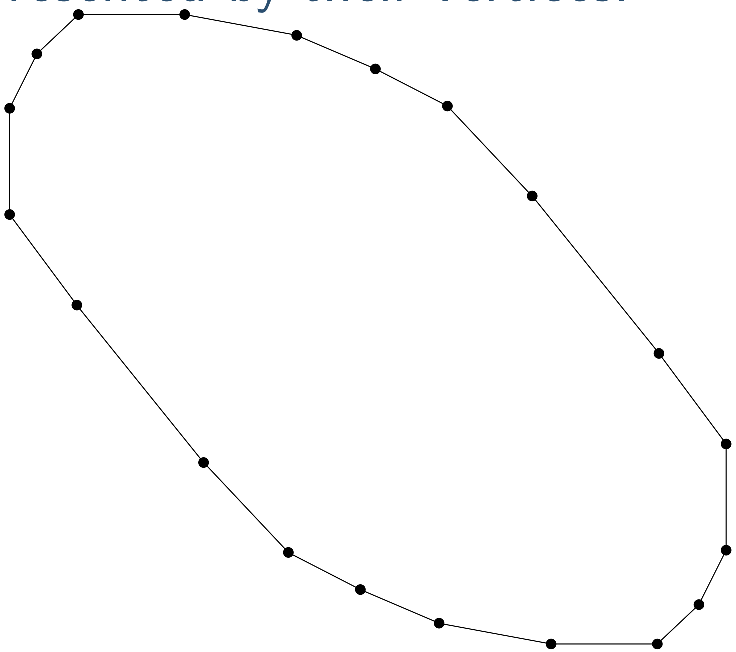

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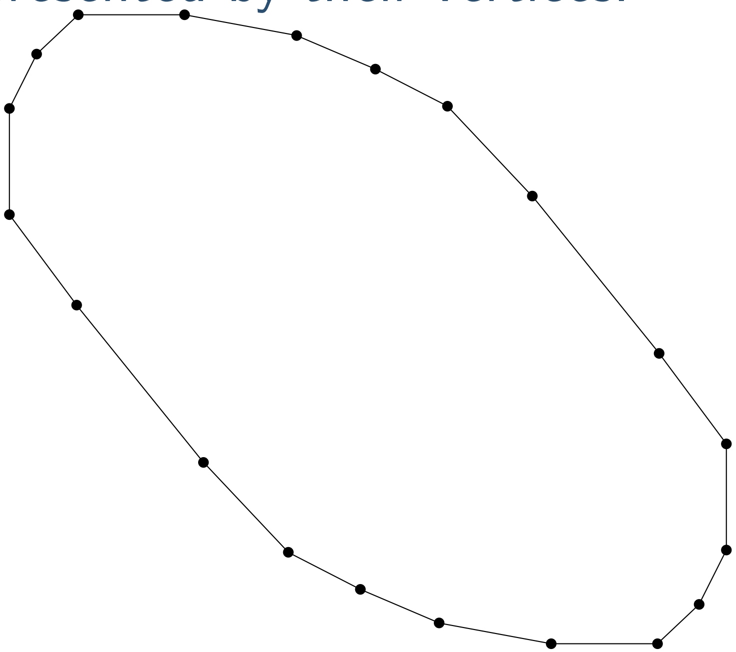

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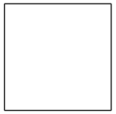

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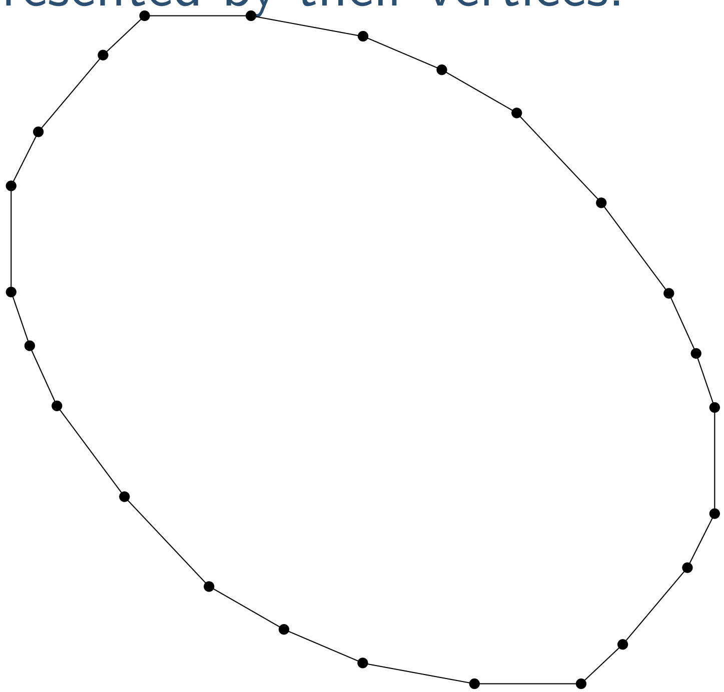

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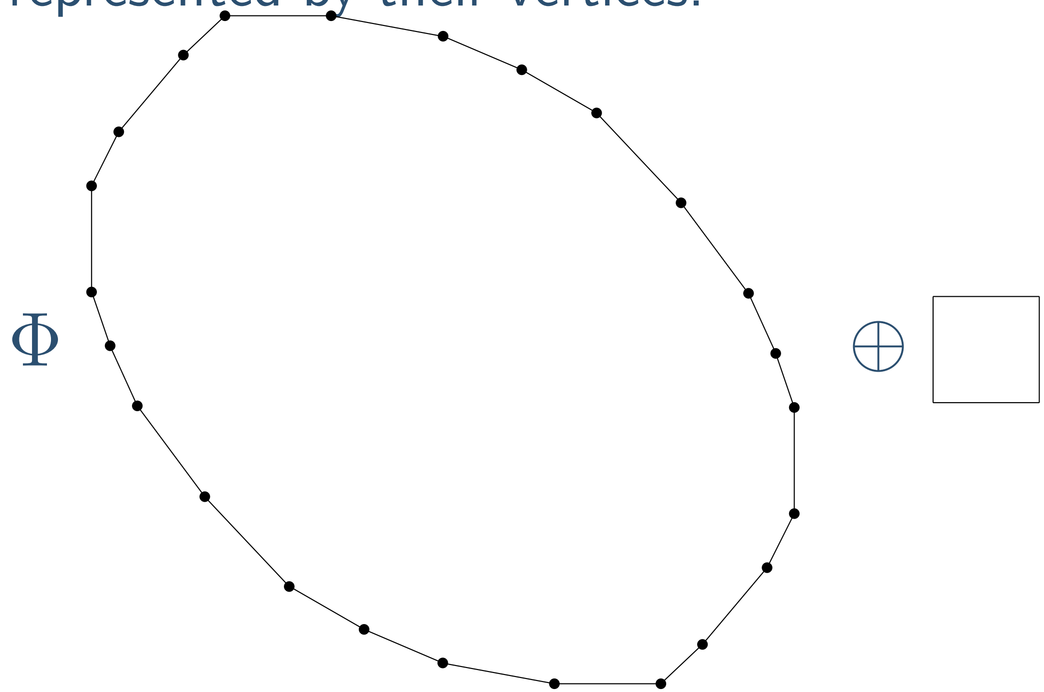

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But:

Ω_{n-1} may have more than $\frac{(2n)^{d-1}}{\sqrt{d}}$ vertices.

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$\Phi\Omega_{n-1}$ needs more than $(2n)^{d-1}d\sqrt{d}$ multiplications.

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This naive algorithm has complexity about N^{d-1}.

where:

- N is the number of steps considered. ($N \in [100; 1000]$)
- d is the dimension of the system. ($d \in [2; 100]$)

Usual Solution: Approximation

$$\overline{\Omega}_{n+1} = \mathsf{APPROX}(\Phi\overline{\Omega}_n \oplus U)$$

APPROX takes a set and computes an over-approximation with bounded representation size.
For example: APPROX can be the Interval Hull.
Then, the algorithm is linear in the number of steps considered.

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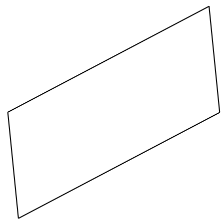

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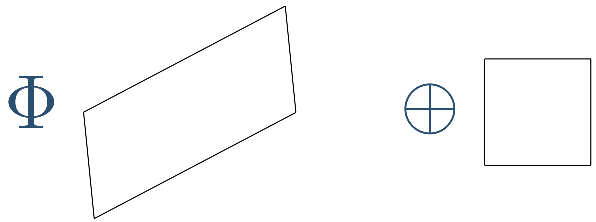

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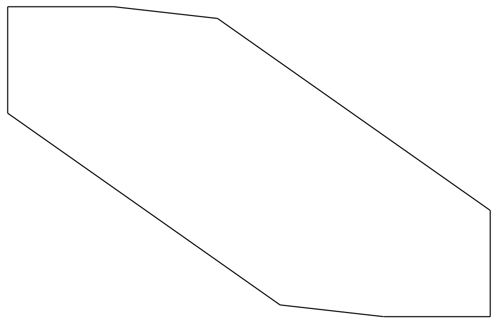

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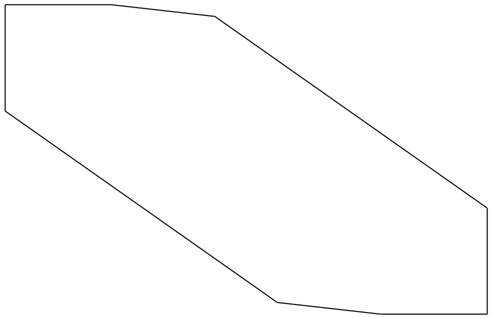
APPROX()

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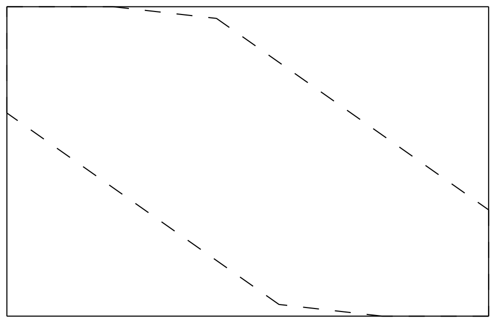

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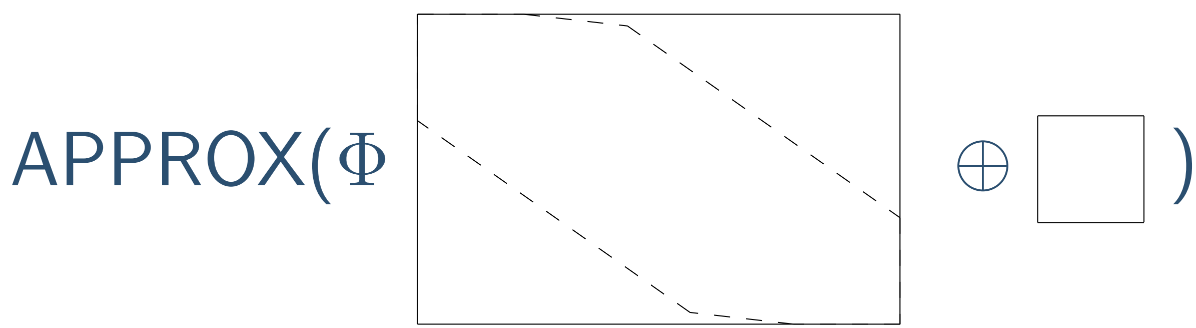

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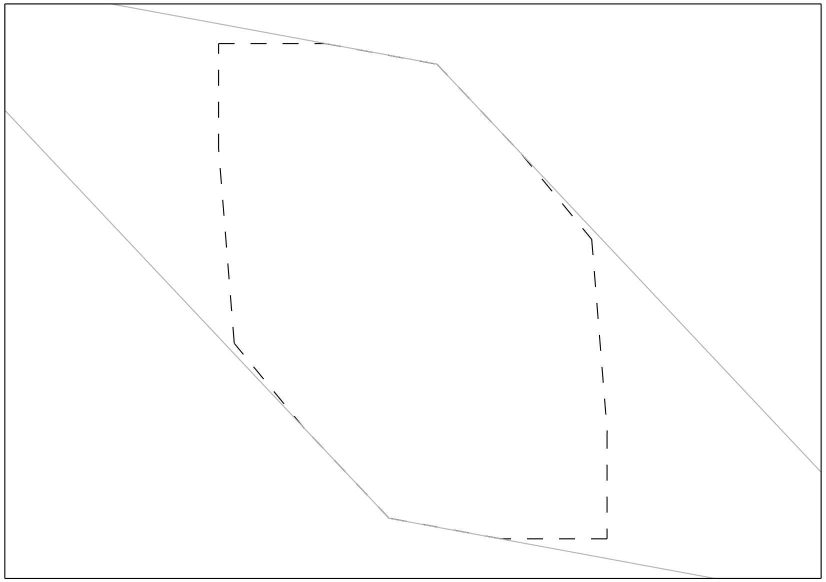

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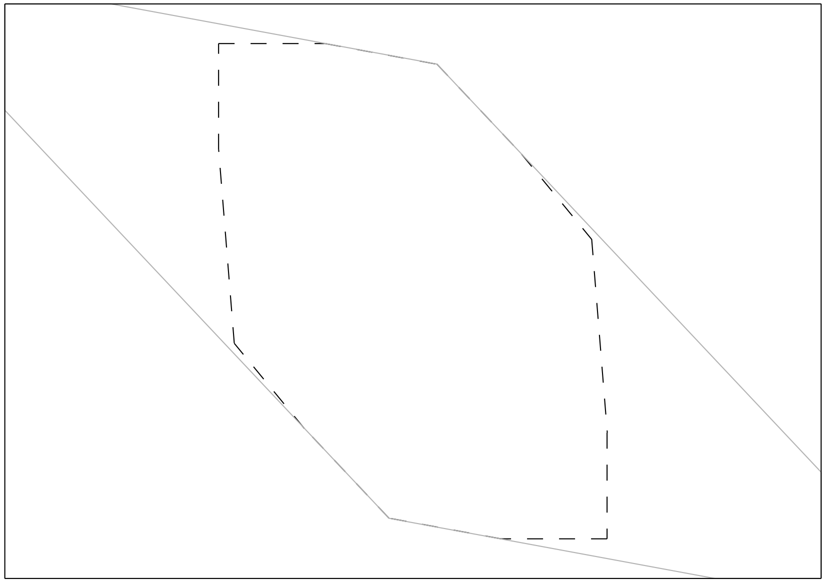
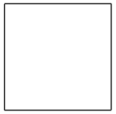
APPROX(Φ  $\oplus$ )

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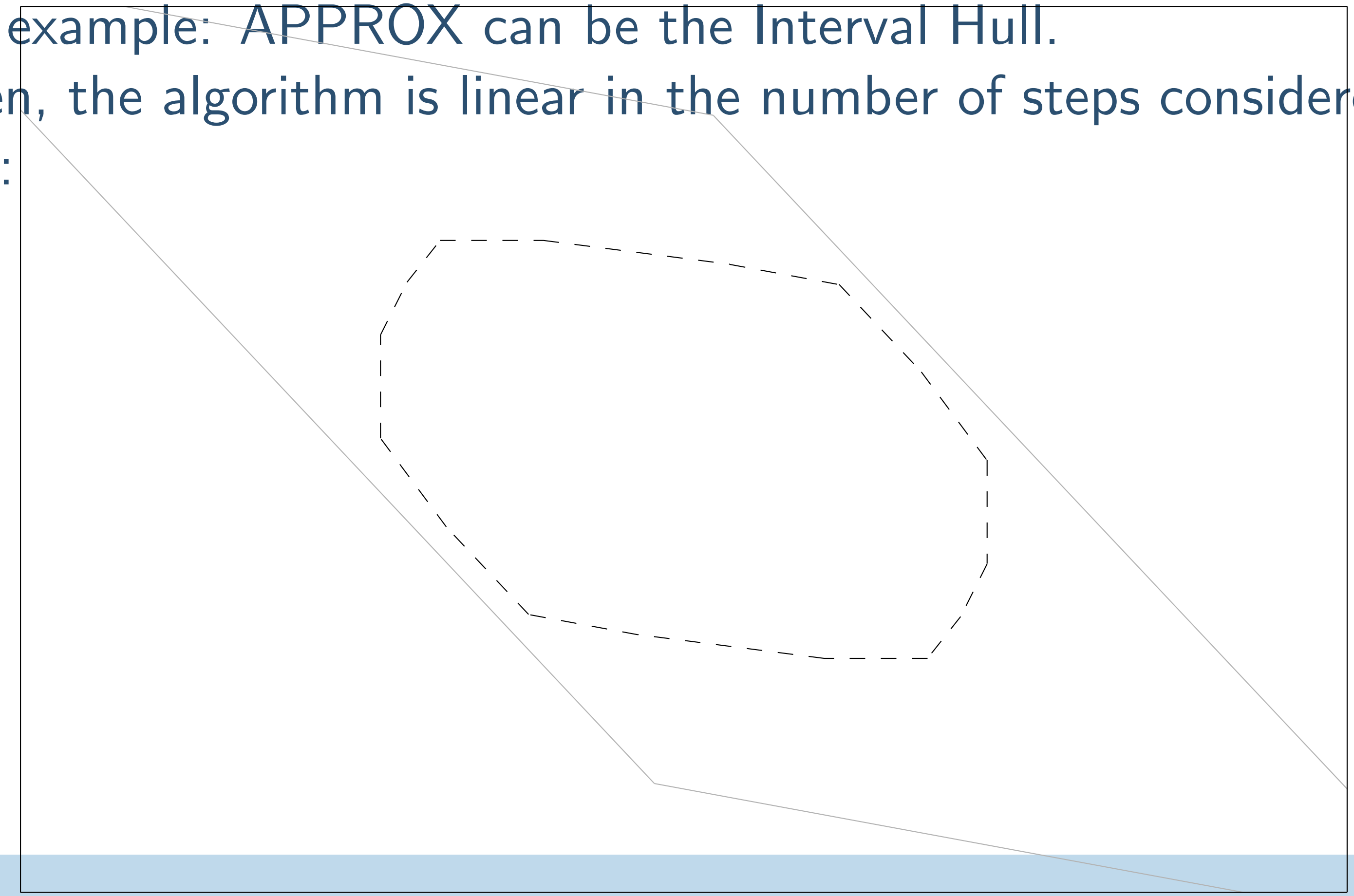

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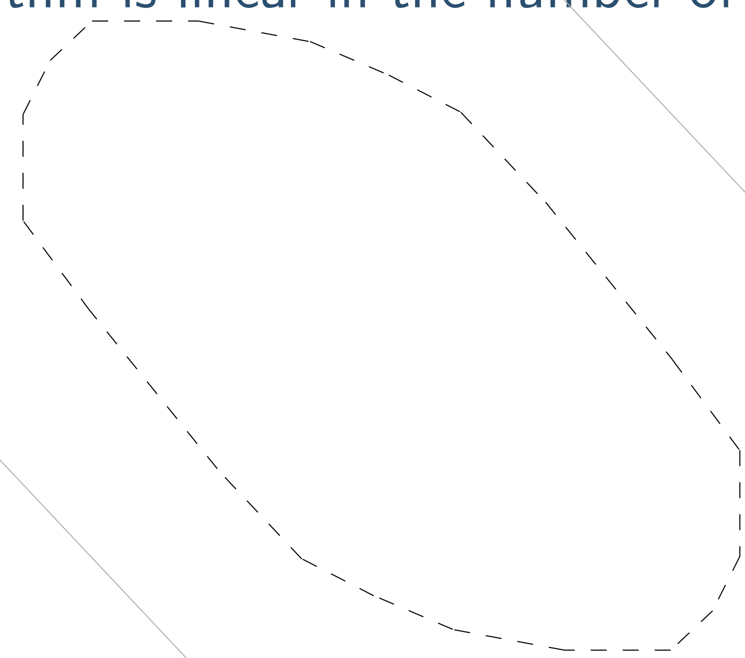

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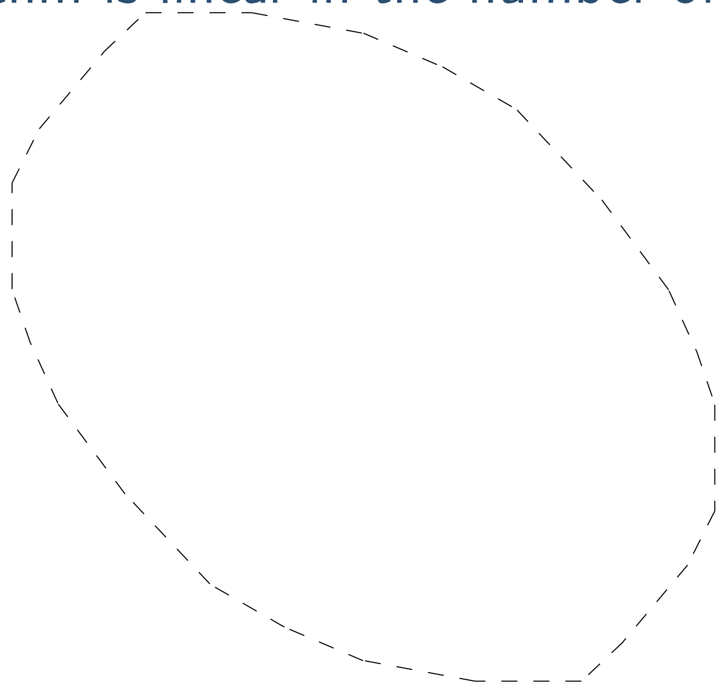

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Most of the effort has been made on looking for a suitable APPROX function.

Tight approximation

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One that minimizes the volume? the Hausdorff distance?...

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These criteria are often hard to evaluate, because they are not conserved by linear transformation.

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An easy to check criterion: Tightness [Kurzhanskiy,Varaiya].
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Thus $\ell_{n+1} = (\Phi^{-1})^T\ell_n$, and APPROX only needs to be tight in the direction given by $(\Phi^{-n})^T\ell_0$.

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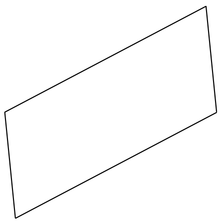

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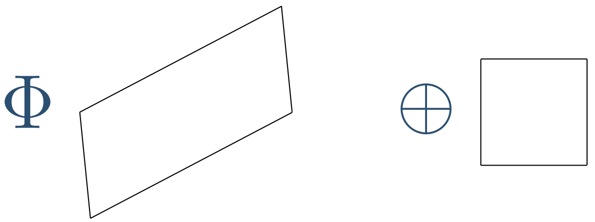

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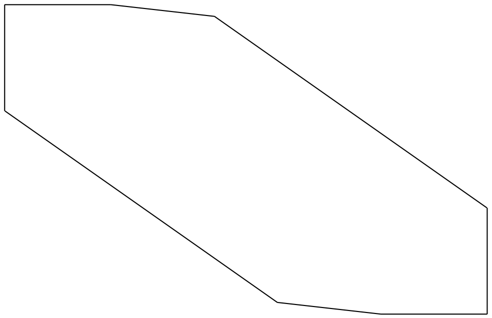

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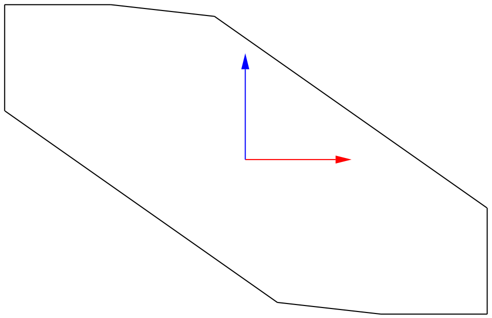
APPROX()

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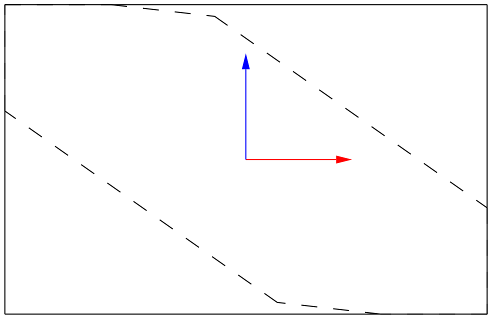

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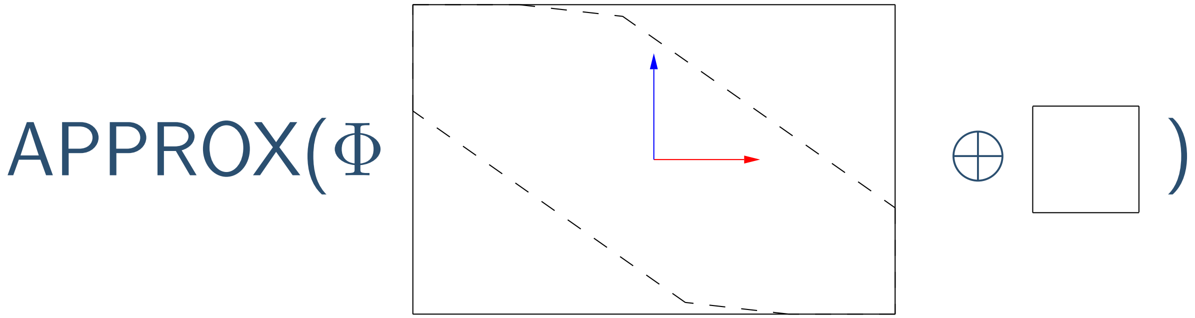

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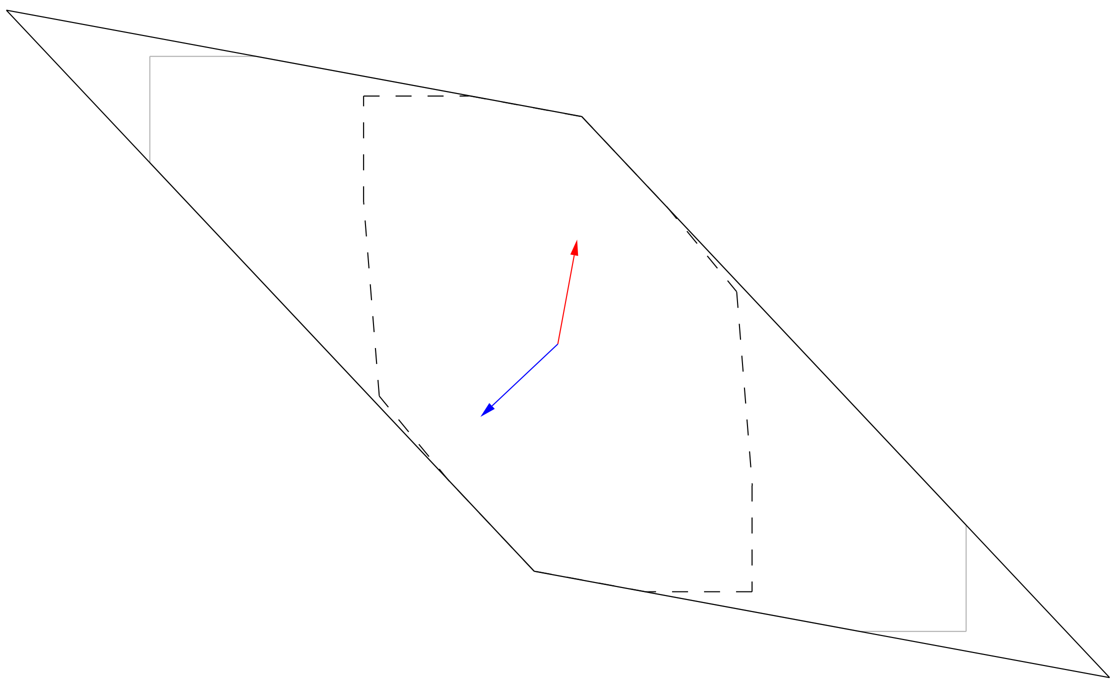

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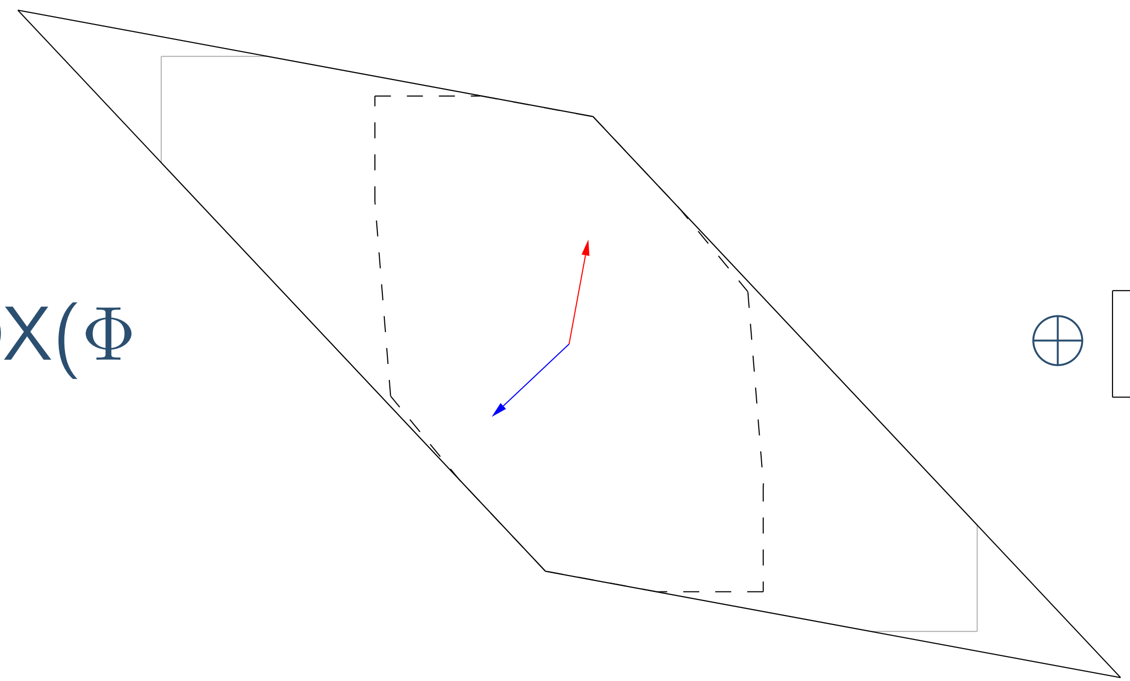

$\oplus \; \square \;)$

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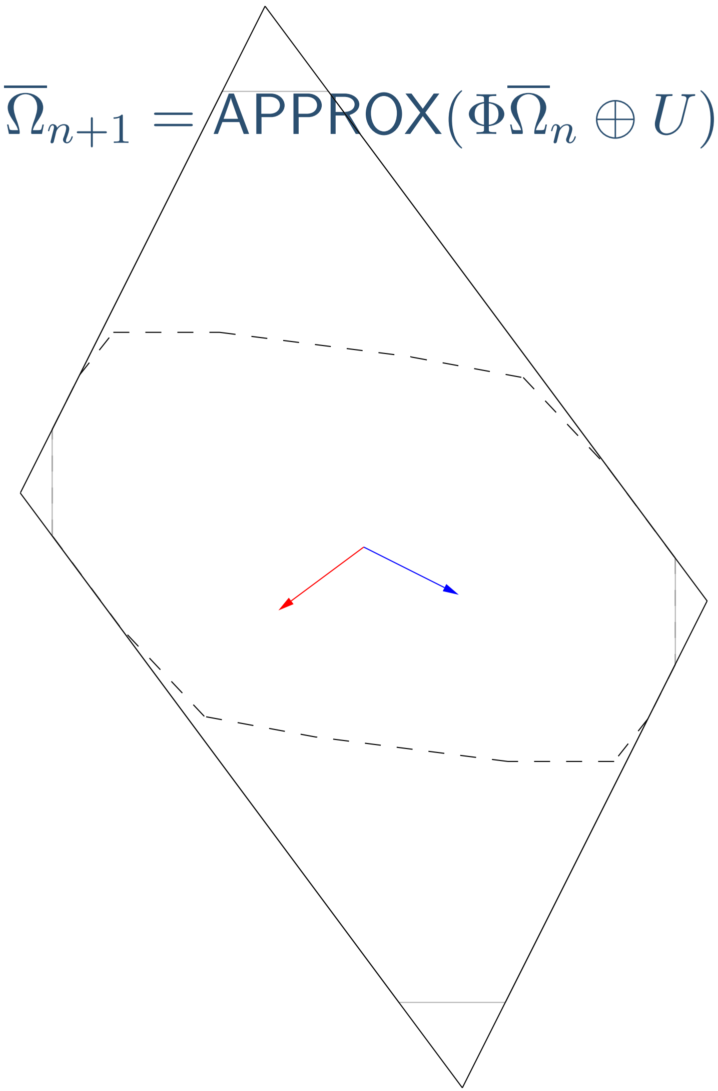

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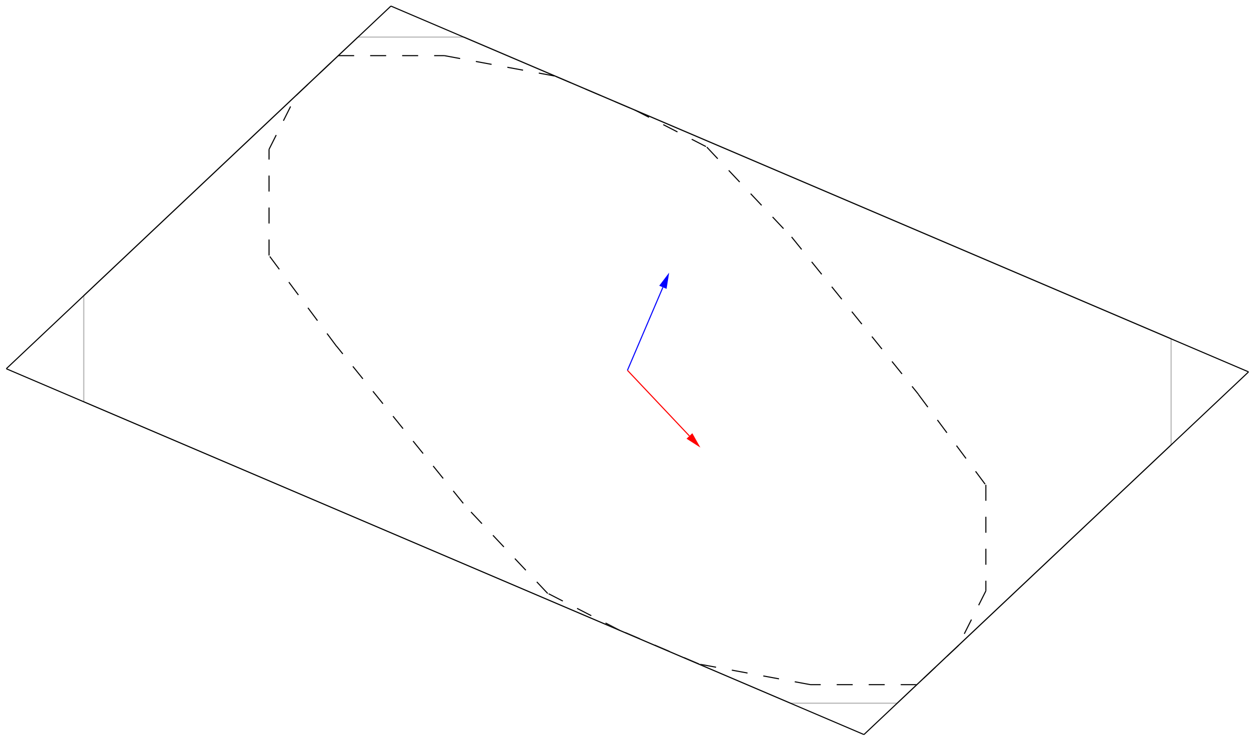

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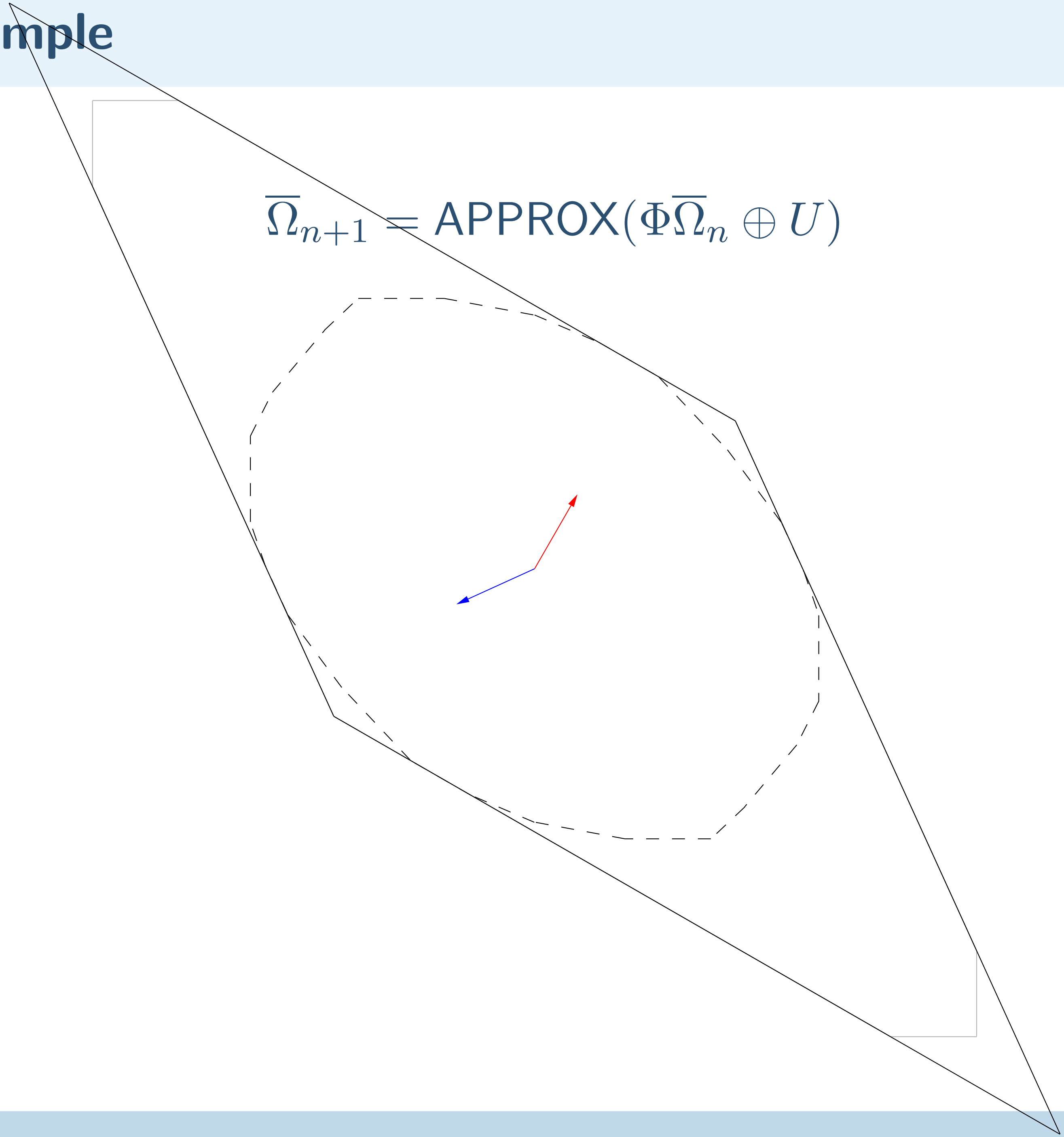

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...

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This is much better.

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This is much better.

But:

- no reported algorithm has bound on the error in terms of diameter, volume, distance,...
- in some case, all approximation directions may converge toward the same vector.

A new algorithm

A simple idea

$$\Omega_{n+1} = \Phi\Omega_n \oplus U$$

The problem comes from the mixing of the Minkowski sum (increases the complexity of the considered sets) and linear transformation (propagates the errors).
We should separate these two operations.

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$$\Omega_1 = \Phi\Omega_0 \oplus U$$

A simple idea

$$\Omega_{n+1} = \Phi\Omega_n \oplus U$$

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$$\Omega_2 = \Phi(\Phi\Omega_0 \oplus U) \oplus U$$

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$$\Omega_2 = \Phi^2\Omega_0 \oplus \Phi U \oplus U$$

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$$\Omega_3 = \Phi(\Phi^2\Omega_0 \oplus \Phi U \oplus U) \oplus U$$

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$$\dots$$

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To compute Ω_n you need two linear transformations (on $\Phi^{n-1}\Omega_0$ and $\Phi^{n-2}U$) and two Minkowski sums.

Exact Algorithm

It is enough to compute the three following sequences:

- $X_0 = \Omega_0$, $X_n = \Phi X_{n-1}$ $\qquad (X_n = \Phi^n \Omega_0)$
- $V_0 = U$, $V_n = \Phi V_{n-1}$ $\qquad (V_n = \Phi^n U)$
- $S_0 = \{0\}$, $S_n = S_{n-1} \oplus V_{n-1}$ $\qquad (S_n = \bigoplus_{i=0}^{n-1} \Phi^i U)$

then $\Omega_n = X_n \oplus S_n$.

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then $\Omega_n = X_n \oplus S_n$.
We can now forget about linear transformations (they are performed on constant complexity sets)
We should focus on Minkowski sum:

- we can use Zonotopes [Girard] time complexity is $\mathcal{O}(Nd^3)$, space complexity is $\mathcal{O}(Nd^2)$
  - recall that the naive algorithm with vertices representation has time complexity $\mathcal{O}(N^{d-1})$
- or approximate

Interval Hull Approximation

- $X_0 = \Omega_0,\ X_n = \Phi X_{n-1}$ $\qquad (X_n = \Phi^n \Omega_0)$
- $V_0 = U,\ V_n = \Phi V_{n-1}$ $\qquad (V_n = \Phi^n U)$
- $S_0 = \{0\},\ S_n = S_{n-1} \oplus \mathsf{BOX}(V_{n-1})$ $\qquad (\bigoplus_{i=0}^{n-1} \mathsf{BOX}(\Phi^i U)\ )$

and $\overline{\Omega}_n = \mathsf{BOX}(X_n) \oplus S_n$.

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No wrapping effect!

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but for any sets A and B: $\text{BOX}(A) \oplus \text{BOX}(B) = \text{BOX}(A \oplus B)$
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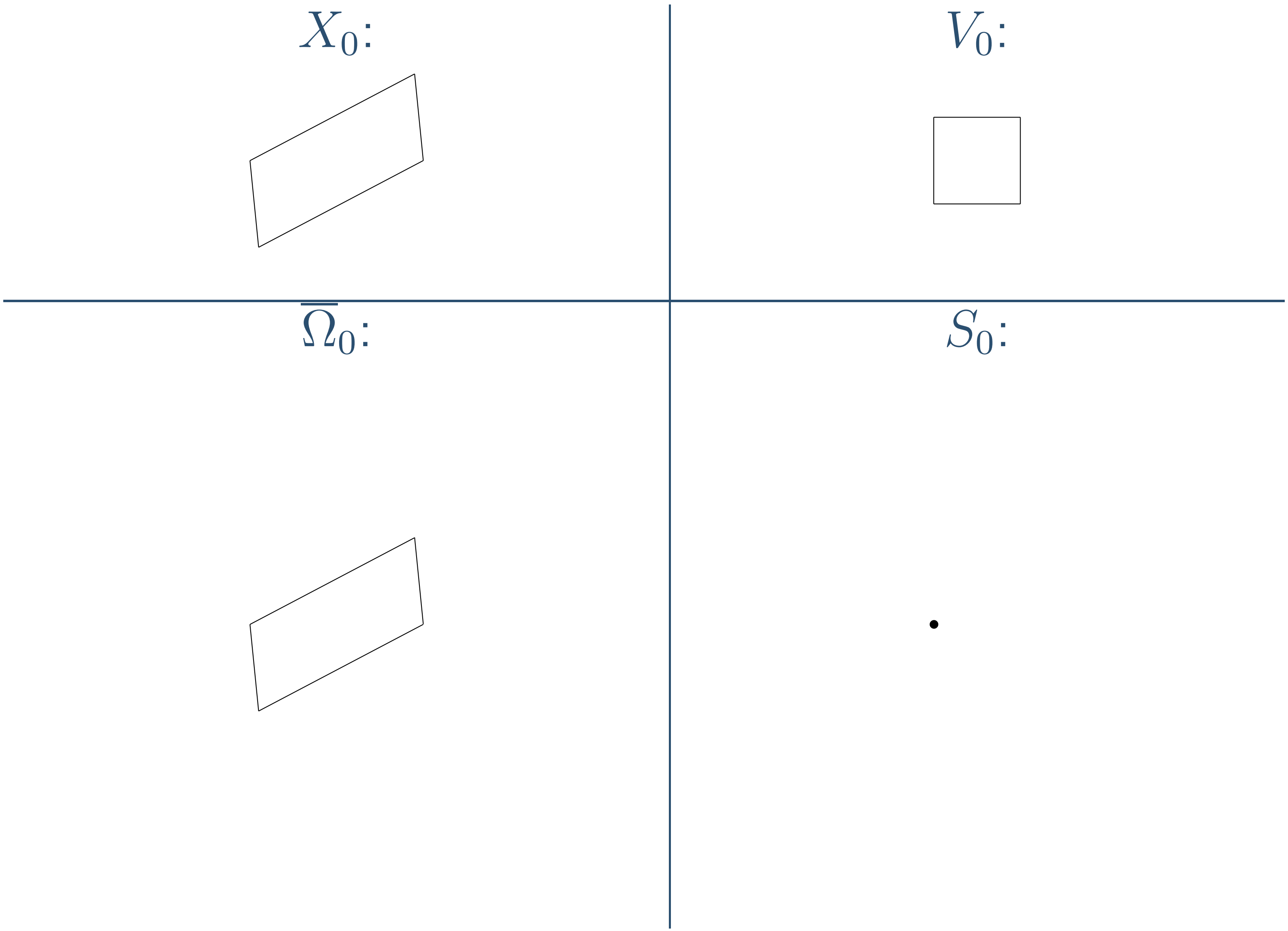
No wrapping effect!

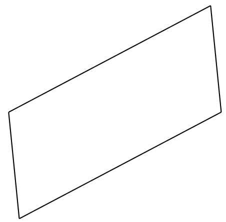
- time complexity: $\mathcal{O}(Nd^3)$ (as the exact algorithm)
- space complexity: $\mathcal{O}(d^2 + Nd)$ (d times smaller)

Example

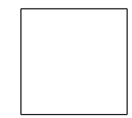


Example

X_1:

Φ 

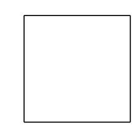
V_1:

Φ 

$\overline{\Omega}_1$:

$\mathrm{BOX}(X_1) \oplus S_1$

S_1:

$\bullet \quad \oplus$ 

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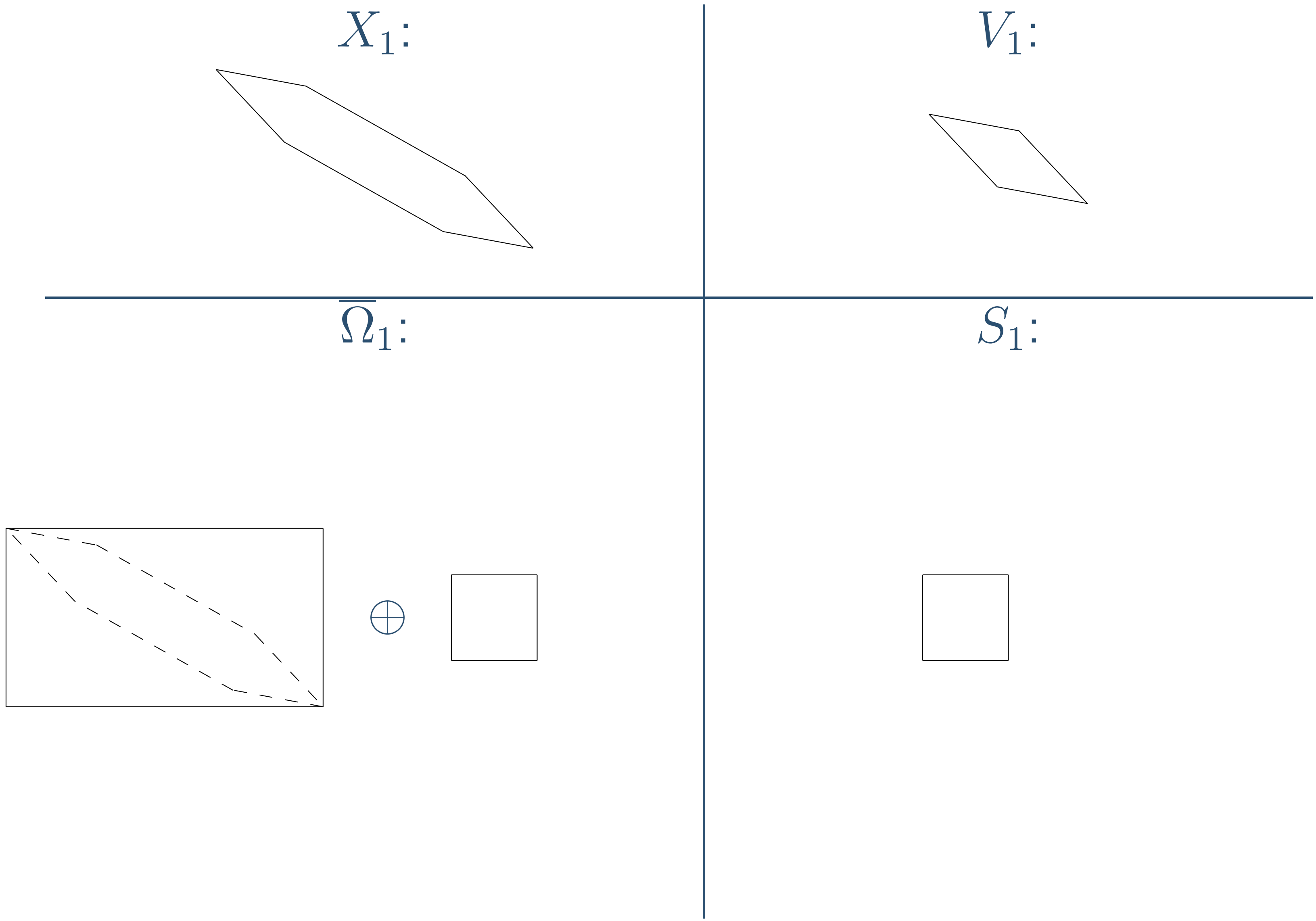
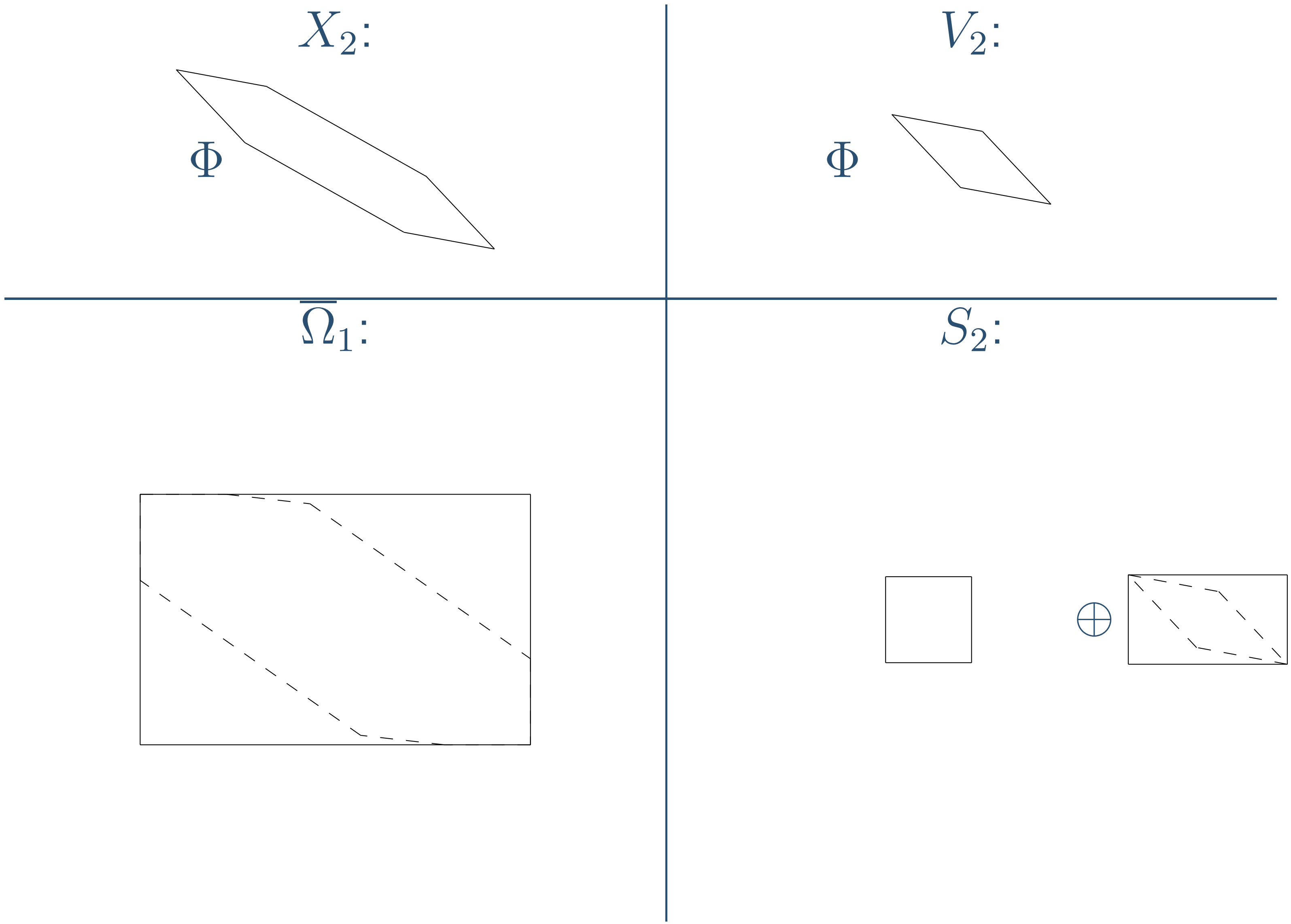


Example



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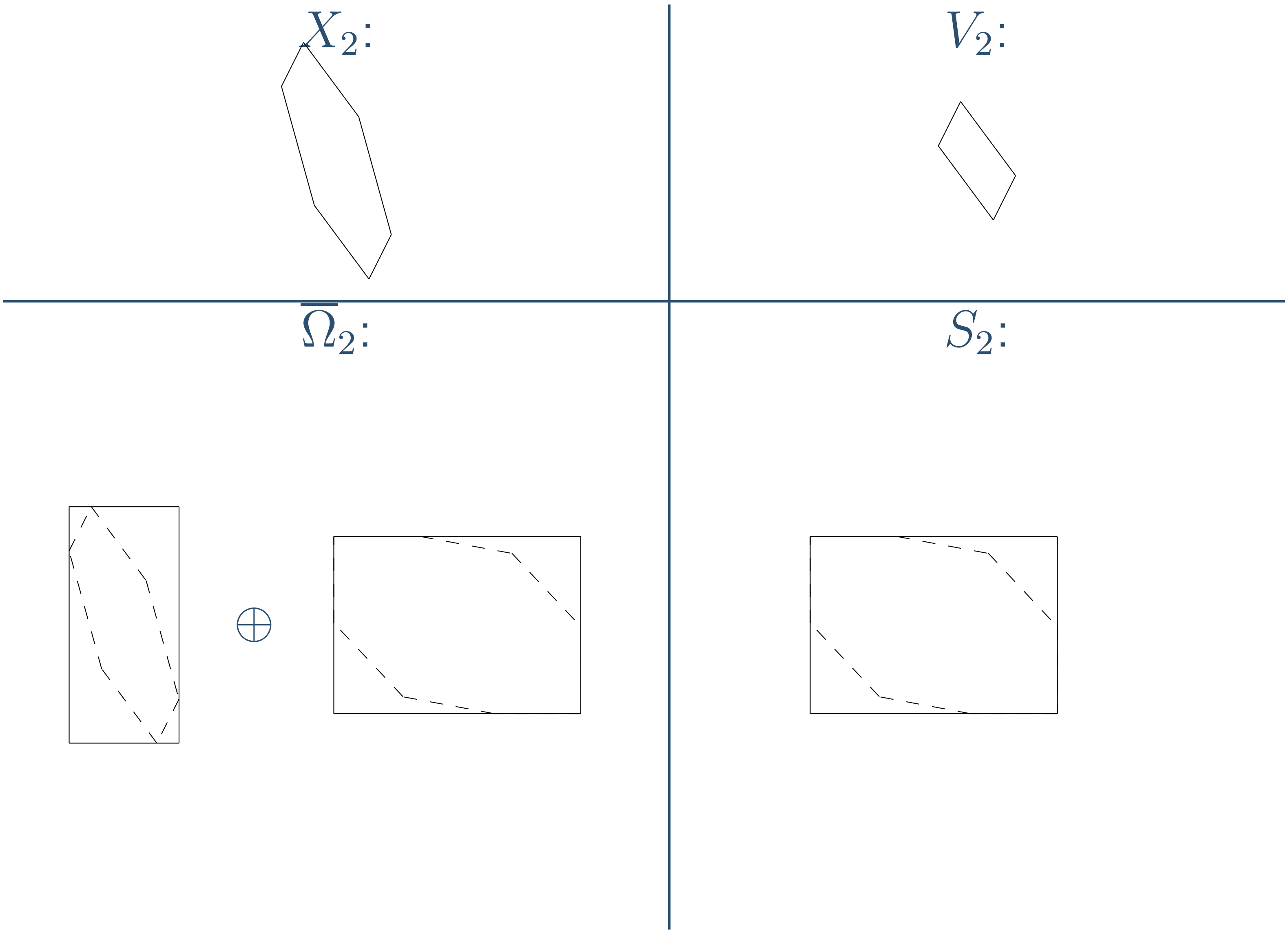

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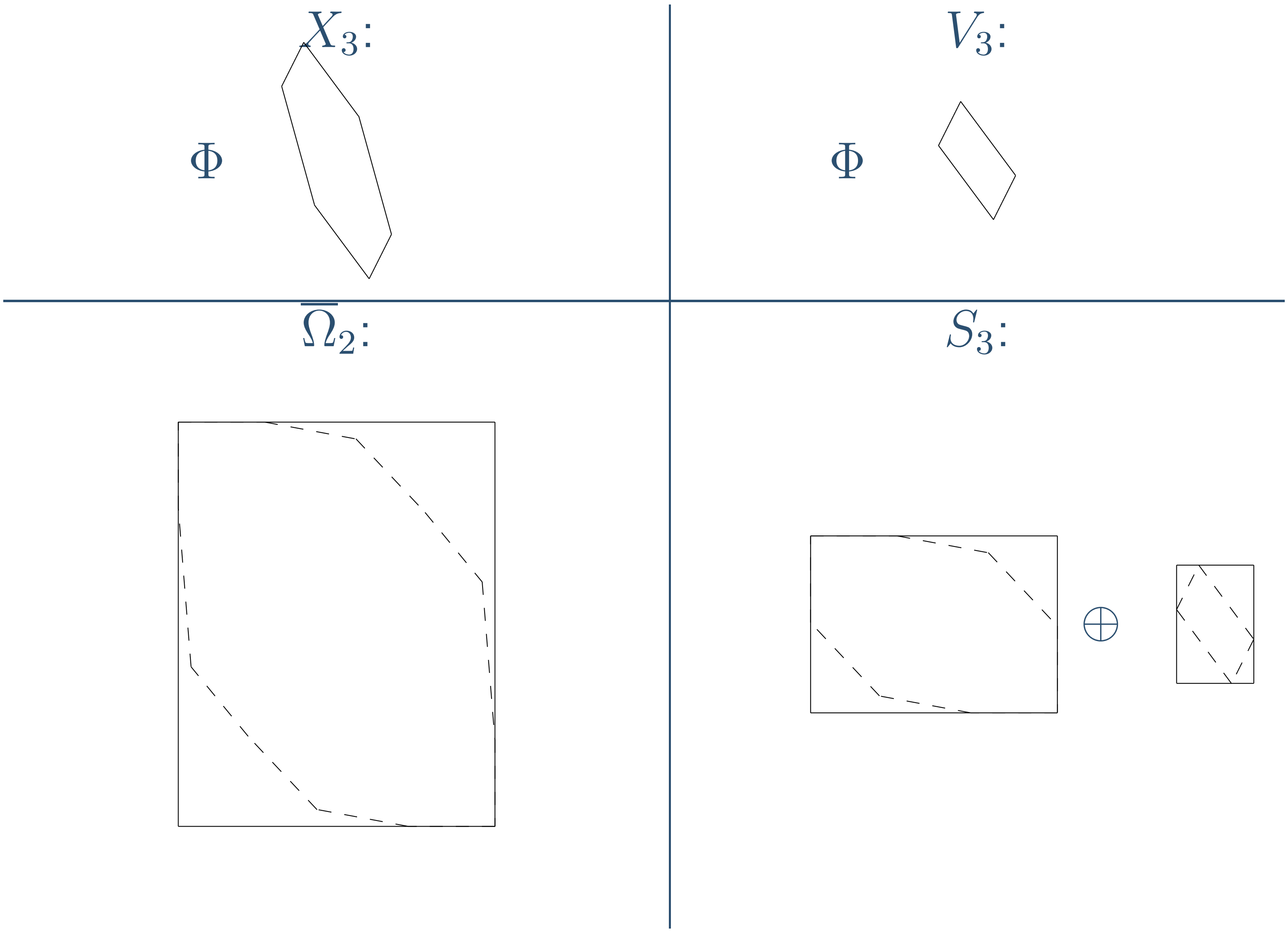

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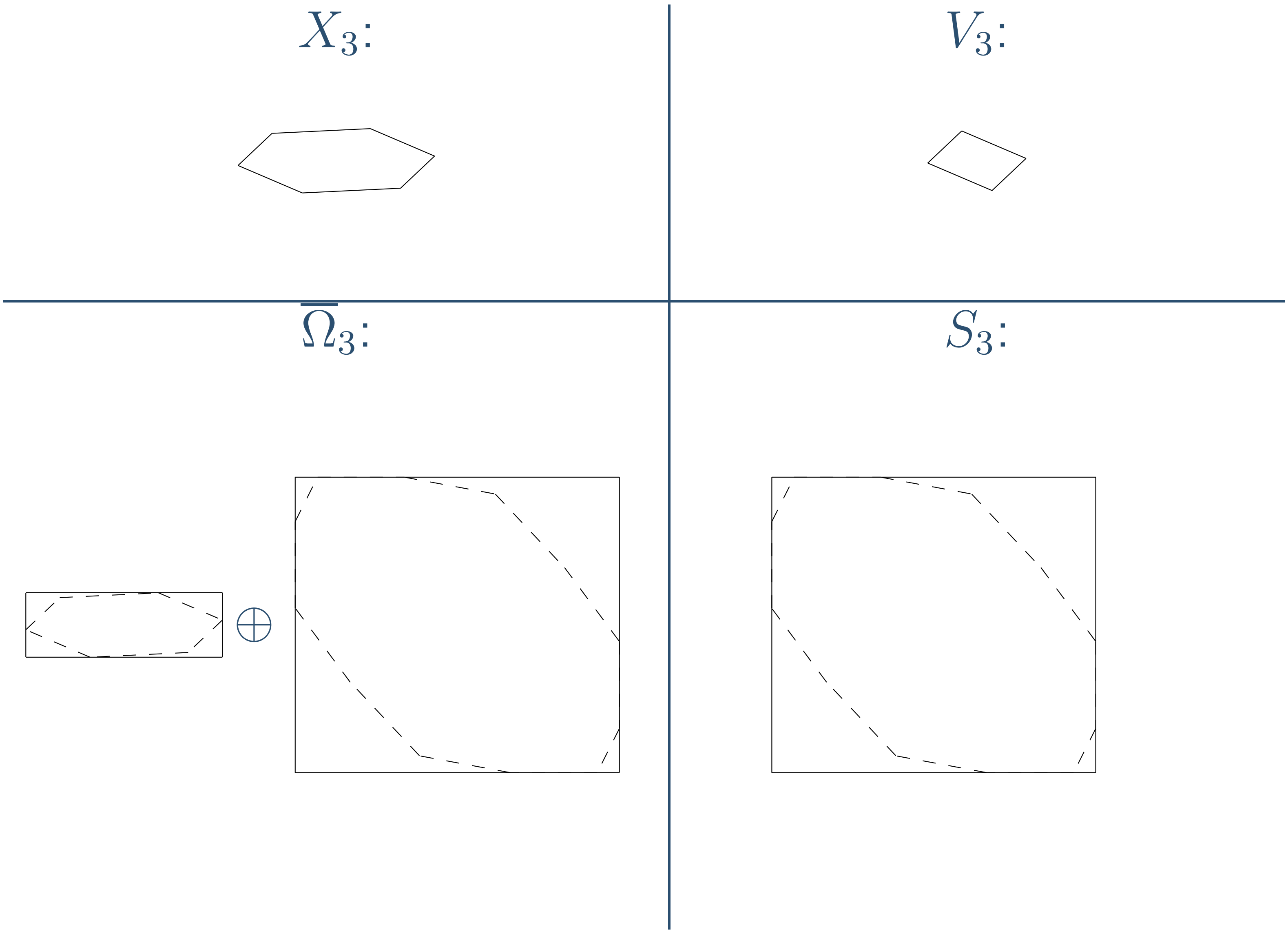
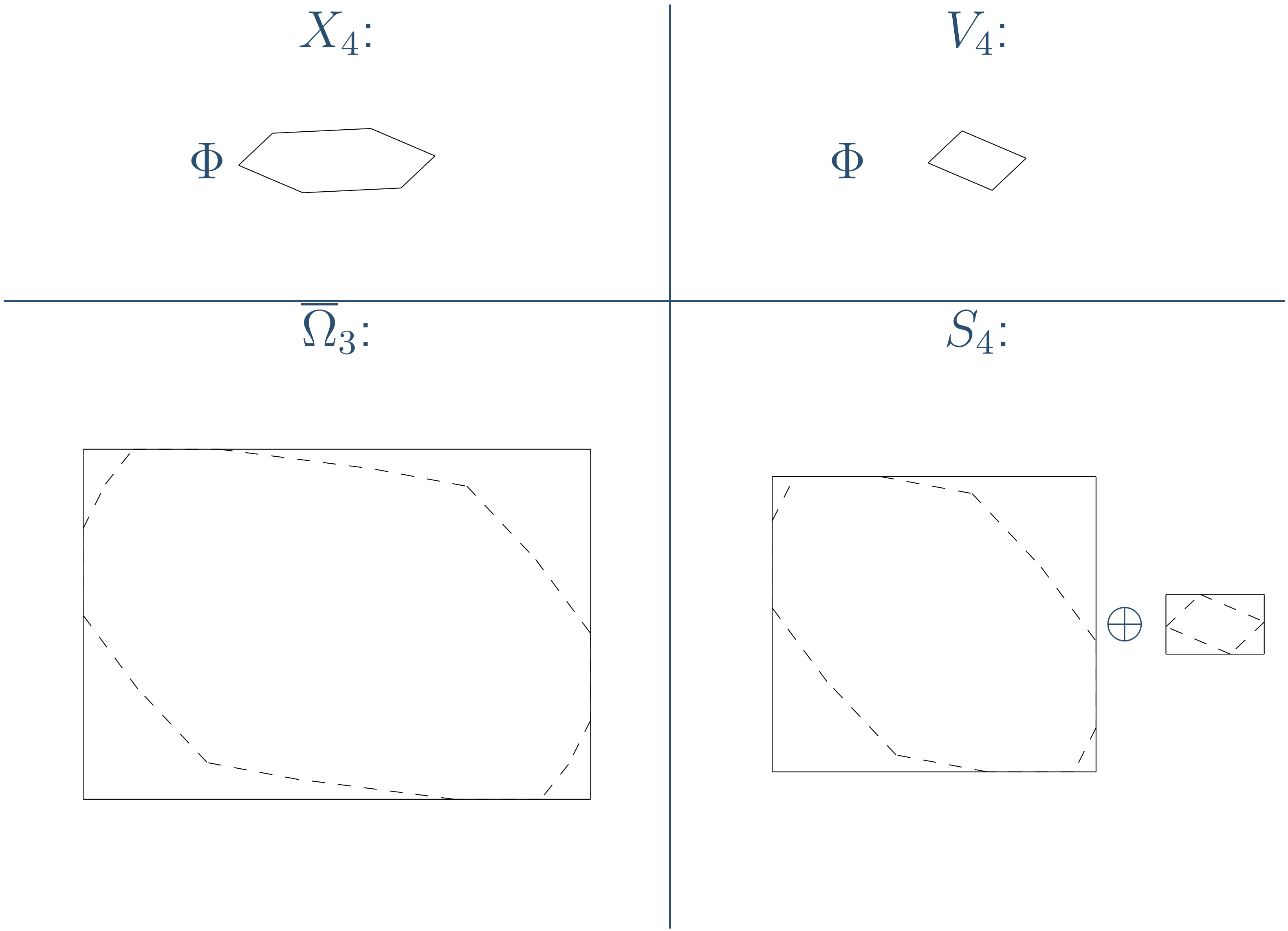
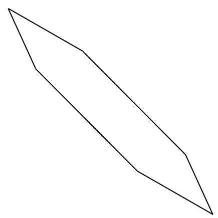


Example

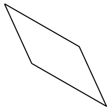


Example

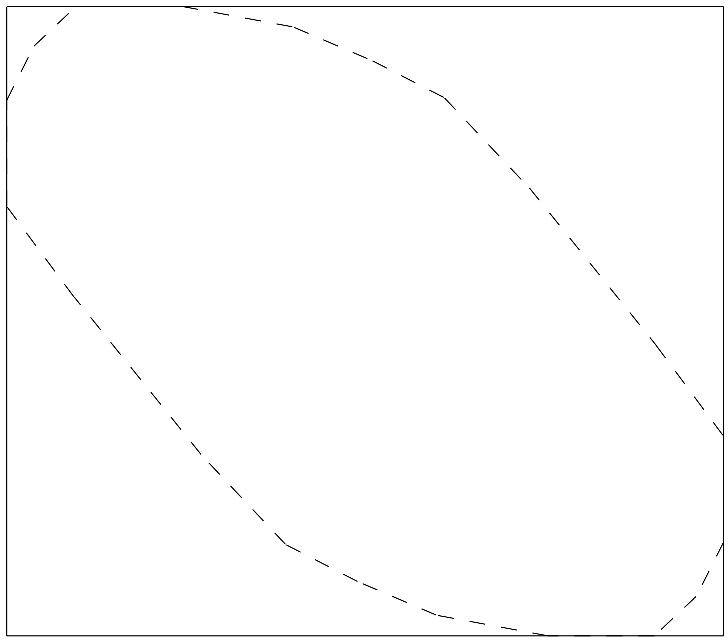
X_4:



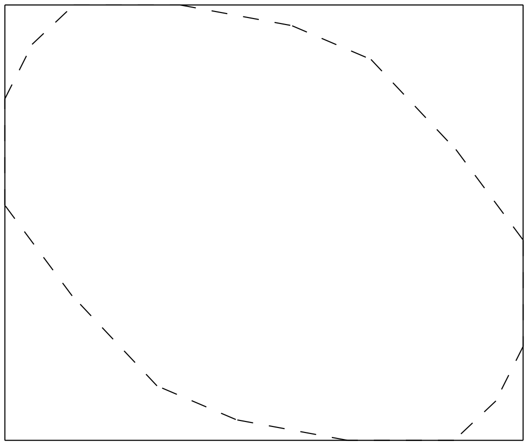
V_4:



$\overline{\Omega}_4$:



S_4:



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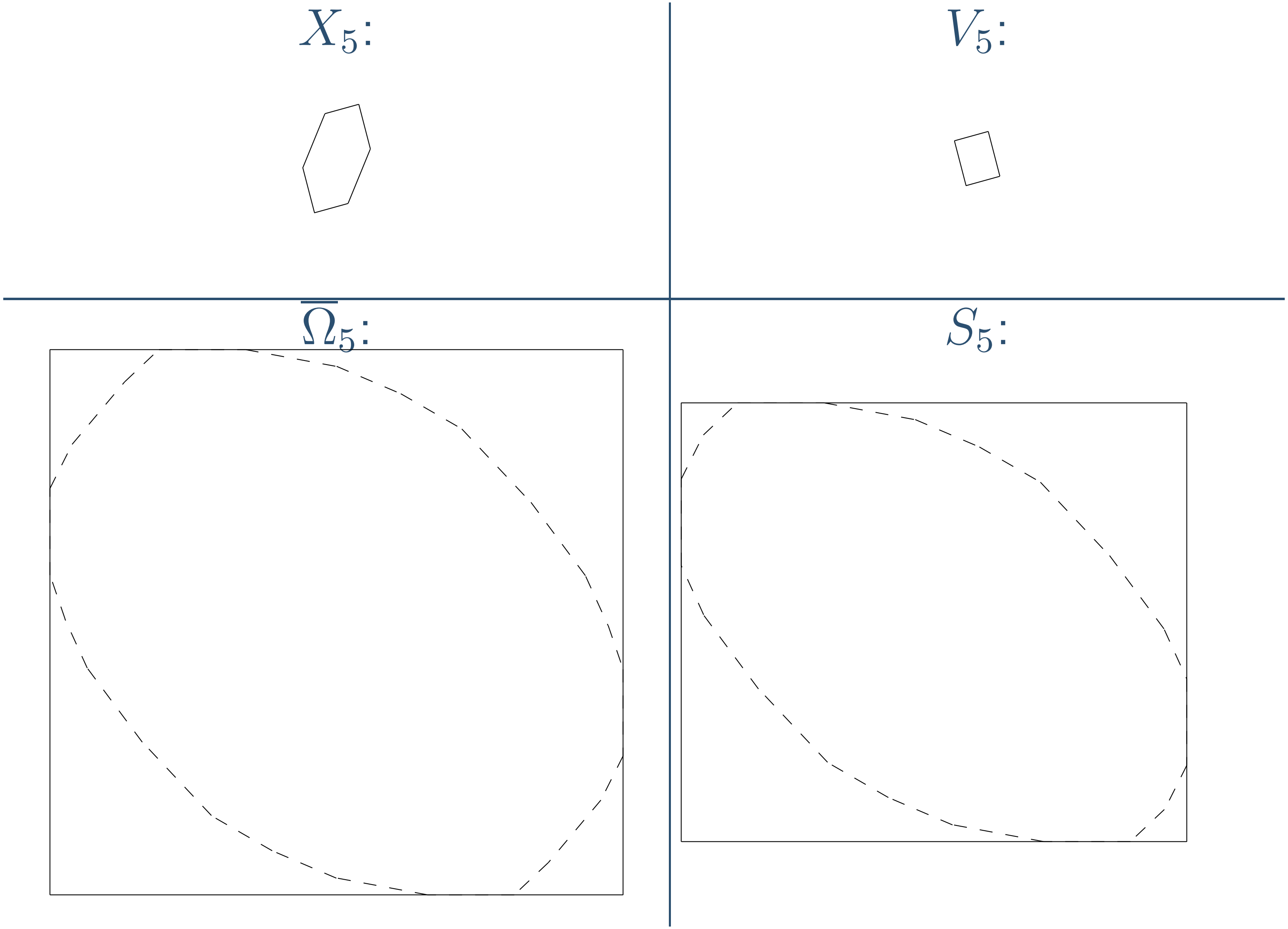


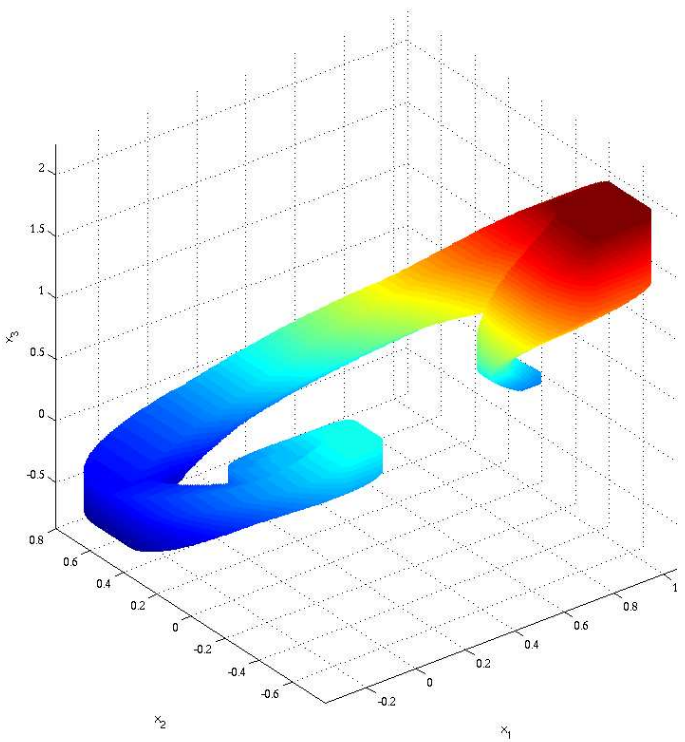
If we are tight in the direction given by the normal to the guards:

$$\overline{\Omega}_i \text{ intersects } G_e \Longleftrightarrow \Omega_i \text{ intersects } G_e.$$

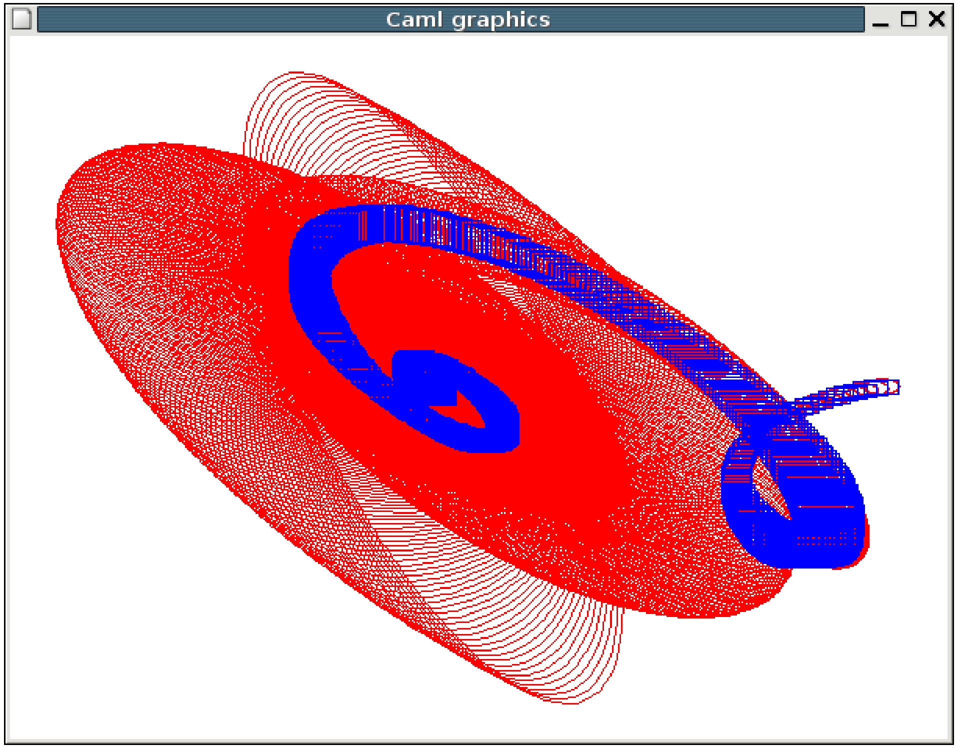
Experimental Results

Dim 5

Result can be exported to the Multi-Parametric Toolbox (MPT).



Interval Hull vs ET (tight in one random direction) [Kurzhanskiy,Varaiya]



dimension 5, 1000 time steps in 0.01s.

Benchmarks

$d =$	5	10	20	50	100	150	200
Exact	0.0s	0.02s	0.11s	1.11s	8.43s	35.9s	136s
BOX	0.0s	0.01s	0.07s	0.91s	8.08s	28.8s	131s

$d =$	5	10	20	50	100	150	200
Exact	246KB	492KB	1.72MB	8.85MB	33.7MB	75.2MB	133MB
BOX	246KB	246KB	246KB	492KB	983KB	2.21MB	3.69MB

Table 1: Time and memory consumption for $N = 100$ for several linear time-invariant systems of different dimensions

Conclusion

Summary

- as fast as Kurzhanskiy and Varaiya's algorithm (tight in two directions)
- needs very little memory
- can deal with any kind of input
- can produce nearly any kind of output (polytopes, ellipsoids,. . .)
- tight over- and under-approximation in user specified directions
  - better approximation
  - guard optimal

Future work

- implementation of $\mathcal{S}$-band intersections
- intersection with the guards
- use of the support function
  - drop complexity
  - parallelization