# Obligation and Weak-Parity Games

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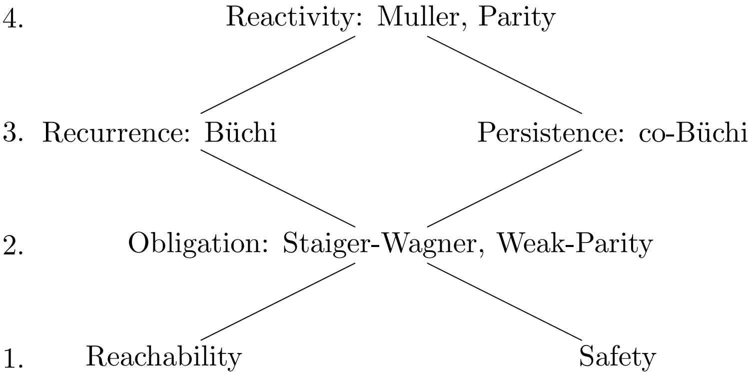
# Hierarchy


4.
Reactivity: Muller, Parity
3.
Recurrence: Büchi
Persistence: co-Büchi
2.
Obligation: Staiger-Wagner, Weak-Parity
1.
Reachability
Safety

# Obligation Games

We consider games where the winning condition for Player 0 (on the play) is

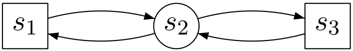
- a Boolean combination of reachability conditions
- equivalently: a condition on the set Occ

Standard form: Staiger-Wagner winning condition, using $F = \{F_1, \ldots, F_k\}$

Player 0 wins play $\rho$ iff $\mathrm{Occ}(\rho) \in F$. We call these games obligation games (or Staiger-Wagner games).

# Example

$S = \{s_1, s_2, s_3\}$ $F = \{\{s_1, s_2, s_3\}\}$




No winning strategy is positional.

There is a finite-state winning strategy.

# Weak Parity Games

Method for solving Staiger-Wagner games:

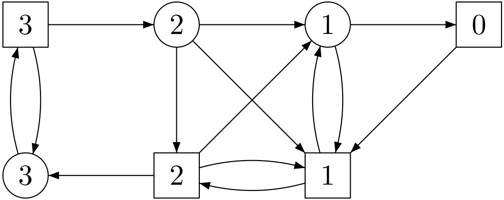
1. Solve weak parity games.
2. Reduce Staiger-Wagner games to weak parity games.

A weak parity game is a pair $(G, p)$, where

- $G = (S, S_0, E)$ is a game graph and
- $p : S \to \{0, \ldots, k\}$ is a priority function mapping every state in $S$ to a number in $\{0, \ldots, k\}$.

A play $\rho$ is winning for Player 0 iff the minimum priority occurring in $\rho$ is even: $\min_{s \in \mathrm{Occ}(\rho)} p(s)$ is even

# Example



# Weak Parity Games

### Theorem

*For a weak parity game one can compute the winning regions $W_0$, $W_1$ and also construct corresponding positional winning strategies.*

### Proof.

Let $G = (S, S_0, E)$ be a game graph, $p : S \to \{0, \ldots, k\}$ a priority function. Let $P_i = \{s \in S \mid p(s) = i\}$.

First steps if $P_0 \neq \emptyset$: We first compute $A_0 = \mathrm{Attr}_0(P_0)$, clearly from here Player 0 can win.

In the rest game, we compute $A_1 = \mathrm{Attr}_1(P_1 \setminus A_0)$ from here Player 1 can win.

# General Construction

Aim: Compute $A_0, A_1, \dots A_k$

Let $G_i$ be the game graph restricted to $S \setminus (A_0 \cup \dots A_{i-1})$.

$\mathrm{Attr}_0^{G_i}(M)$ is the 0-attractor of $M$ in the subgraph induced by $G_i$

$$
\begin{aligned}
A_0 \quad & := \mathrm{Attr}_0(P_0) \\
A_1 \quad & := \mathrm{Attr}_1^{G_1}(A_0 \setminus P_1) \\
\text{for } i > 1 : & \\
A_i \quad & := \begin{cases} \mathrm{Attr}_0^{G_i}(P_i \setminus (A_0 \cup .. \cup A_{i-1})) & \text{if } i \text{ is even} \\ \mathrm{Attr}_1^{G_i}(P_i \setminus (A_0 \cup .. \cup A_{i-1)}) & \text{if } i \text{ is odd} \end{cases}
\end{aligned}
$$

# Correctness

Correctness Claim:

$$W_0 = \bigcup_{i \text{ even}} A_i \text{ and } W_1 = \bigcup_{i \text{ odd}} A_i$$

and the union of the corresponding attractor strategies are positional winning strategies for the two players on their respective winning regions.

Prove by induction on $j = 0, \ldots, k$ the following:

$$\bigcup_{i=0..j, i \text{ even}} A_i \subseteq W_0 \text{ and } \bigcup_{i=1..j, i \text{ odd}} A_i \subseteq W_1$$

# Correctness (cont.)

Base:

- i=0: $A_0 = \text{Attr}_0(P_0) \subseteq W_0$
- i=1: $A_1 = \text{Attr}_1(P_1 \setminus A_0) \subseteq W_1$

Induction step:

- i even: Consider play $\rho$ starting $A_i$ that complies to attractor strategy.
  - Case 1: $\rho$ eventually leaves $A_i$ to some $A_j$ (from a Player-1 state), which $j < i$ and even, then Player 0 wins by induction hypothesis.
  - Case 2: $\rho$ visits $P_i$, then we need to show that $\rho$ visits only states with $p(s) \geq i$. Consider a state $s$ that visits $P_i$, then
    - if $s \in S_0$, then not all edges lead to states with lower priority, otherwise $s \in A_j$ for some $j < i$. Contradiction.

# Correctness (cont.)

- Case 2 (cont.):
  - if $s \in S_1$, then all edges lead to states with priority $\geq i$. Any edge to a lower priority must lead to $A_j$ with even $j$ (Case 1). If there were edges to states $s'$ with priority $j < i$ and $j$ odd, then $s'$ would already be in $A_j$. Contradiction.
- i odd: switch players

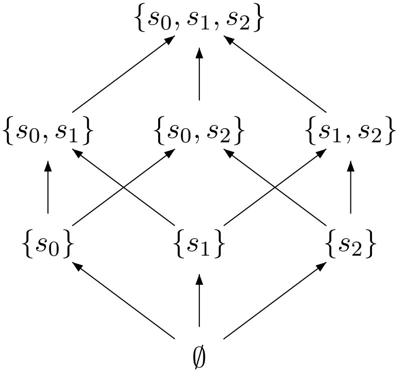
# Obligation/Staiger-Wagner to Weak-Parity Games

- How to translate a Staiger-Wagner automaton to Weak-Parity automaton?
- Idea: record visited states during a run
- Record set: $R \subseteq S$
- Question: How to give priorities?

# Record Sets and Priorities

Assume automaton with states $\{s_0, s_1, s_2\}$.

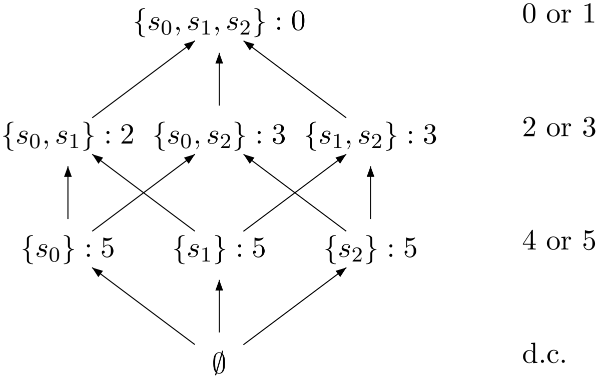
Consider possible record sets:




Assume the following run $s_1, s_0, s_1, s_0, s_2, \ldots$ and the acceptance condition $F = \{\{s_0, s_1\}, \{s_0, s_1, s_2\}\}$. How to assign priorities?

# Record Sets and Priorities

$F = \{\{s_0, s_1\}, \{s_0, s_1, s_2\}\}$. How would you assign priorities?



# From Staiger-Wagner to Weak Parity Automata

Given a deterministic Staiger-Wagner automaton $A = (S, I, T, F)$, we can construct an equivalent weak parity automaton $A' = (S', I', T', p)$ as follows:

$$
\begin{aligned}
S' &:= S \times 2^S \\
I' &:= (I, \{I\}) \\
T'((s, R), a) &:= (T(s, a), R \cup \{T(s, a)\} \\
p((s, R)) &:= 2 \cdot |S| - \begin{cases} 2 \cdot |R| & \text{if } R \in F \\ 2 \cdot |R| - 1 & \text{if } R \notin F \end{cases}
\end{aligned}
$$

# Idea of Game Reduction

We want to solve Staiger-Wagner games. We use a reduction to weak parity games (and the positional winning strategies of weak parity games).

Reduction will transform a game $(G, \phi)$ into a game $(G', \phi')$ such that usually

- $G'$ is (usually) larger than $G$
- $\phi'$ is simpler than $\phi$ (so the solution of $(G', \phi')$ is simpler than that of $(G, \phi)$)
- from a solution of $(G', \phi')$ we can construct a solution of $(G, \phi)$.

Concrete application: Transform Staiger-Wagner game into a weak parity game over a larger graph (from $S$ proceed to $S \times 2^S$)

# Game Reduction

Let $G = (S, S_0, E)$ and $G' = (S', S'_0, E')$ be game graphs with winning conditions $\phi$ and $\phi'$, respectively.

$(G, \phi)$ is reducible to $(G', \phi')$ if:

1. $S' = S \times M$ for a finite set $M$ and $S'_0 = S_0 \times M$
2. Each play $\rho = s_0 s_1 \ldots$ over $G$ is translated into a play $\rho' = s'_0 s'_1 \ldots$ over $G'$ by
   - a function $f : S \to S \times M$ (the beginning of $\rho'$).
   - forall states $(m, s) \in S \times M$ in $G'$ and all states $s' \in S$ in $G$, if there exists an edge $(s, s') \in E$, then there is a unique $m'$ with $((m, s), (m', s')) \in E'$
   - forall edges $((m, s), (m', s')) \in E'$ in $G'$, there is an edges $(s, s') \in E$ in $G$
3. For all plays $\rho$ and $\rho'$ according to 2.: $\rho \in \phi$ iff $\rho' \in \phi'$

# Application of Game Reduction

Theorem

*Suppose* $(G, \phi)$ *is reducible to* $(G', \phi')$ *with extension set* $M$*, initial function* $g$*, and* $G$ *and* $G'$ *defined as before. Then, if Player 0 wins in* $(G', \phi')$ *from* $g(s)$ *with a memoryless winning strategy, then Player 0 wins in* $(G, \phi)$ *from* $s$ *with a finite-state strategy.*

Idea: Given a memoryless winning strategy $f : S'_0 \to S'$ from $g(s)$ for Player 0 in $(G', \phi')$, we can construct a strategy automaton $A = (M, m_0, \delta, \lambda)$ for Player 0 in $(G, \phi)$.

# Obligation/Staiger-Wagner Games

## Theorem

*Given a Staiger-Wagner game $(G, \phi)$, one can compute the winning regions of Player 0 and 1 and corresponding finite state strategies.*

## Proof.

We can apply game reduction with $(G', \phi')$ as follows:

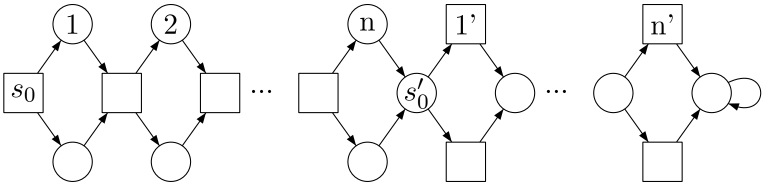
$$
\begin{aligned}
G' &:= (S', S'_0, E') \\
S' &:= 2^S \times S \\
((R, s), (R', s')) \in E') &\quad \text{iff } (s, s') \in E, R' = R \cup \{s'\} \\
g(s) &= (\{s\}, s) \\
p((R, s)) &:= 2 \cdot |S| - \begin{cases} 2 \cdot |R| & \text{if } P \in \phi \\ 2 \cdot |R| - 1 & \text{if } P \notin \phi \end{cases}
\end{aligned}
$$

# Exponential-Size Memory

## Theorem

*There is a family of Staiger-Wagner games over game graphs $G_1, G_2, G_3, \ldots$ which grow linearly in $n$ such that*

- *Player 0 wins from a certain initial vertex of $G_n$*
- *any finite-state strategy for Player 0 needs at least $2^n$ states*




Winning condition:

$$\phi = \{\rho \mid \forall i = 1 \ldots n : i \in \mathrm{Occ}(\rho) \leftrightarrow i' \in \mathrm{Occ}(\rho)\}$$

# Exponential Memory (cont.)

Claim:

Over $G_n$ there is an automaton winning strategy for Player 0 from vertex $s_0$ with a memory of size $2^n$. (Remember the visited vertices $i$, for the appropriate choice from vertex $s_0'$ onwards.)

Each automaton winning strategy for Player 0 from $s_0$ in $G_n$ has a memory of $2^n$ many states.

Proof.

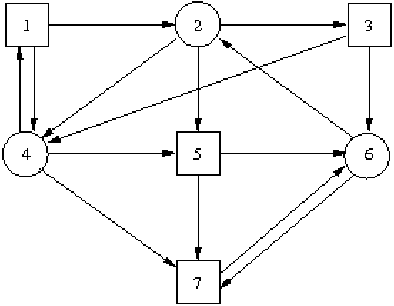
Assume $|\text{states}| < 2^n$ is sufficient.

Then two play prefixes $u \neq v$ exist leading to the same memory states at $s_0'$. The rest $r$ of the play is then the same after $u$ and $v$.

One of the two player $ur$, $vr$ is lost by Player 0. Contradiction.

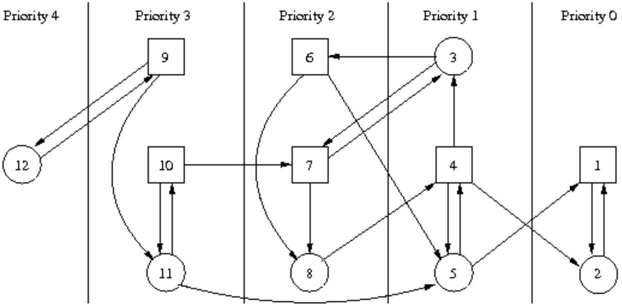
# Exercise

1. Consider the game graph shown below. Let the winning condition for Player 0 be $\mathrm{Occ}(\rho) = \{1, 2, 3, 4, 5, 6, 7\}$.
    1. Find the winning region for Player 0 and describe a winning strategy
    2. Show that there is no positional winning strategy for Player 0.



# Exercise

2. Compute the winning regions and the corresponding positional winning strategies for Player 0 and 1 in this weak-parity game.



# Exercise

3. A winning strategy is called *uniform* if it is a winning strategy from every winning state in the game. Let $(G, p)$ be a weak parity game and let $W_0$ be the winning region of Player 0. For all $s \in W_0$ let $f_s$ be a positional winning strategy from $s$ for Player 0. Construct a uniform winning strategy $f$ from the strategies $f_s$ meaning that for every $s \in W_0$ there is a $t \in W_0$, s.t. $f(s) = f_t(s)$.