Automata on Infinite Words

Definition of Büchi Automata

Let $\Sigma = \{a, b, \ldots\}$ be a finite alphabet.

A *non-deterministic Büchi automaton* (NBA) over Σ is a tuple $A = \langle S, I, T, F \rangle$, where:

- S is a finite set of *states*,
- $I \subseteq S$ is a set of *initial states*,
- $T \subseteq S \times \Sigma \times S$ is a *transition relation*,
- $F \subseteq S$ is a set of *final states*.

Acceptance Condition

A *run* of a Büchi automaton is defined over an infinite word $w : \alpha_1\alpha_2\ldots$ as an infinite sequence of states $\pi : s_0 s_1 s_2 \ldots$ such that:

- $s_0 \in I$ and
- $(s_i, \alpha_{i+1}, s_{i+1}) \in T$, for all $i \in \mathbb{N}$.

$$\boxed{\inf(\pi) = \{s \mid s \text{ appears infinitely often on } \pi\}}$$

Run π of A is said to be *accepting* iff $\inf(\pi) \cap F \neq \emptyset$.

The language of A, denoted $\mathcal{L}(A)$, is the set of all words accepted by A.

A language $L \subseteq \Sigma^\omega$ is *recognizable* if there exists a Büchi automaton A such that $L = \mathcal{L}(A)$.

Examples

Let $\Sigma = \{0, 1\}$. Define Büchi automata for the following languages:

1. $L = \{\alpha \in \Sigma^\omega \mid 0 \text{ occurs in } \alpha \text{ exactly once}\}$
2. $L = \{\alpha \in \Sigma^\omega \mid \text{after each } 0 \text{ in } \alpha \text{ there is } 1\}$
3. $L = \{\alpha \in \Sigma^\omega \mid \alpha \text{ contains finitely many 1's}\}$
4. $L = (01)^*\Sigma^\omega$
5. $L = \{\alpha \in \Sigma^\omega \mid 0 \text{ occurs on all even positions in } \alpha\}$

The Büchi Characterization Theorems

Lemma 1 *If* $L \subseteq \Sigma^*$ *is a rational language, there exists a DFA* $A = \langle S, \{s_0\}, T, F \rangle$ *such that* s_0 *has no incoming transitions and* $L = \mathcal{L}(A)$.

Theorem 1 *Let* $W, V \subseteq \Sigma^*$ *be rational languages. Then the language* $V \cdot W^\omega$ *is Büchi recognizable.*

Is it possible that, for some non-rational language $W \subseteq \Sigma^*$, W^ω is Büchi recognizable?

First Büchi Characterization Theorem

Let $A = \langle S, I, T, F \rangle$ be a Büchi automaton and $s, s' \in S$ be two states.

Let $W_{s,s'} = \{w \in \Sigma^* \mid s \xrightarrow{w} s'\}$.

The language $W_{s,s'} \subseteq \Sigma^*$ is rational, for any $s, s' \in S$.

Theorem 2 *An ω-langage $L \subseteq \Sigma^\omega$ is Büchi recognizable iff L is a finite union of ω-languages $V \cdot W^\omega$, where $V, W \subseteq \Sigma^*$ are rational languages.*

"$\Rightarrow$" $L = \bigcup_{s \in I, s' \in F} W_{s,s'} \cdot W^\omega_{s',s'}$

Second Büchi Characterization Theorem

Theorem 3 *An ω-langage $L \subseteq \Sigma^\omega$ is Büchi recognizable iff L is a finite union of ω-languages $V \cdot W^\omega$, where $V, W \subseteq \Sigma^*$ are rational languages such that $W \cdot W \subseteq W$.*

"⇒" $\mathcal{L}(A) = \bigcup_{s \in I, s' \in F} W_{s,s'} W_{s',s'}^{\omega}$ and $W_{s',s'} \cdot W_{s',s'} \subseteq W_{s',s'}$ for all $s' \in F$

Corollary 1 *Any non-empty Büchi-recognizable language contains an ultimately periodic word of the form $uvvv\ldots$.*

Closure Properties

Closure under union and projection are like in the finite automata case.

Intersection is a bit special.

Complementation of non-deterministic Büchi automata is a complex result.

Deterministic Büchi automata are not closed under complement.

Closure under Intersection

Let $A_1 = \langle S_1, I_1, T_1, F_1 \rangle$ and $A_2 = \langle S_2, I_2, T_2, F_2 \rangle$

Build $A_\cap = \langle S, I, T, F \rangle$:

- $S = S_1 \times S_2 \times \{1, 2, 3\}$,
- $I = I_1 \times I_2 \times \{1\}$,
- the definition of T is the following:
  - $((s_1, s_1', 1), a, (s_2, s_2', 1)) \in T$ iff $(s_i, a, s_i') \in T_i$, $i = 1, 2$ and $s_1 \notin F_1$
  - $((s_1, s_1', 1), a, (s_2, s_2', 2)) \in T$ iff $(s_i, a, s_i') \in T_i$, $i = 1, 2$ and $s_1 \in F_1$
  - $((s_1, s_1', 2), a, (s_2, s_2', 2)) \in T$ iff $(s_i, a, s_i') \in T_i$, $i = 1, 2$ and $s_1' \notin F_2$
  - $((s_1, s_1', 2), a, (s_2, s_2', 3)) \in T$ iff $(s_i, a, s_i') \in T_i$, $i = 1, 2$ and $s_1' \in F_2$
  - $((s_1, s_1', 3), a, (s_2, s_2', 1)) \in T$ iff $(s_i, a, s_i') \in T_i$, $i = 1, 2$
- $F = S_1 \times S_2 \times \{3\}$

The Emptiness Problem

Theorem 4 *Given a Büchi automaton A, $\mathcal{L}(A) \neq \emptyset$ iff there exist $u, v \in \Sigma^*$, $|u|, |v| \leq \|A\|$, such that $uv^\omega \in \mathcal{L}(A)$.*

In practical terms, A is non-empty iff there exists a state s which is reachable both from an initial state and from itself.

Q: Is the membership problem decidable for Büchi automata?

Deterministic Büchi Automata

ω-languages recognized by NBA $\supset$ ω-languages recognized by DBA

Q: Why classical subset construction does not work for Büchi automata?

Let $A = \langle S, I, T, F \rangle$ and $A_d = \langle 2^S, \{I\}, T_d, \{Q \mid Q \cap F \neq \emptyset\} \rangle$.

Let $u_0 u_1 u_2 \ldots \in \mathcal{L}(A)$ be an infinite word. In A_d this gives:

$$I \xrightarrow{u_0} Q_1 \xrightarrow{u_1} Q_2 \xrightarrow{u_2} \ldots$$

where each $Q_i \cap F$. However this does not necessarily correspond to an accepting path in A!

Deterministic Büchi Automata

Let $W \subseteq \Sigma^*$. Define $\overrightarrow{W} = \{\alpha \in \Sigma^\omega \mid \alpha(0,n) \in W \text{ for infinitely many } n\}$

Theorem 5 *A language $L \subseteq \Sigma^\omega$ is recognizable by a deterministic Büchi automaton iff there exists a rational language $W \subseteq \Sigma^*$ such that $L = \overrightarrow{W}$.*

If $L = \mathcal{L}(A)$ then $W = \mathcal{L}(A')$ where A' is the DFA with the same definition as A, and with the finite acceptance condition.

Deterministic Büchi Automata

Theorem 6 *There exists a Büchi recognizable language that can be recognized by no deterministic Büchi automaton.*

$\Sigma = \{a, b\}$ and $L = \{\alpha \in \Sigma^\omega \mid \#_a(\alpha) < \infty\} = \Sigma^* b^\omega$.

Suppose $L = \overrightarrow{W}$ for some $W \subseteq \Sigma^*$.

$b^\omega \in L \Rightarrow b^{n_1} \in W$

$b^{n_1} a b^\omega \in L \Rightarrow b^{n_1} a b^{n_2} \in W$

$$\dots$$

$b^{n_1} a b^{n_2} a \dots \in \overrightarrow{W} = L$, contradiction.

Deterministic Büchi Automata are not closed under complement

Theorem 7 *There exists a DBA A such that no DBA recognizes the language $\Sigma^\omega \setminus \mathcal{L}(A)$.*

$\Sigma = \{a, b\}$ and $L = \{\alpha \in \Sigma^\omega \mid \#_a(\alpha) < \infty\} = \Sigma^* b^\omega$.

Let $V = \Sigma^* a$. There exists a DFA A such that $\mathcal{L}(A) = V$.

There exists a deterministic Büchi automaton B such that $\mathcal{L}(A) = \overrightarrow{V}$

But $\Sigma^\omega \setminus \overrightarrow{V} = L$ which cannot be recognized by any DBA.

Büchi Automata and S1S

Let $\Sigma = \{a, b, \ldots\}$ be a finite alphabet.

Any finite word $w \in \Sigma^*$ induces the *infinite* sets $p_a = \{p \mid w(p) = a\}$.

- $x \leq y$: x is less than y,
- $S(x) = y$: y is the successor of x,
- $p_a(x)$: a occurs at position x in w

Remember that $\leq$ and S can be defined one from another.

Problem Statement

Let $\mathcal{L}(\varphi) = \{w \mid \mathfrak{m}_w \models \varphi\}$

A language $L \subseteq \Sigma^*$ is said to be S1S-*definable* iff there exists a S1S formula φ such that $L = \mathcal{L}(\varphi)$.

1. Given a Büchi automaton A build an S1S formula φ_A such that $\mathcal{L}(A) = \mathcal{L}(\varphi)$
2. Given an S1S formula φ build a Büchi automaton A_φ such that $\mathcal{L}(A) = \mathcal{L}(\varphi)$

The Büchi recognizable and S1S-definable languages coincide

From Automata to Formulae

Let $A = \langle S, I, T, F \rangle$ with $S = \{s_1, \ldots, s_p\}$, and $\Sigma = \{0,1\}^m$.

Build $\Phi_A(X_1, \ldots, X_m)$ such that $\forall w \in \Sigma^* \, . \, w \in \mathcal{L}(A) \iff w \models \Phi_A$

$\Phi_A(X_1, \ldots, X_m) = \exists Y_1 \ldots \exists Y_p \, . \, \Phi_S(\vec{Y}) \wedge \Phi_I(\vec{Y}) \wedge \Phi_T(\vec{Y}, \vec{X}) \wedge \Phi_F(\vec{Y})$

$$\Phi_F(\vec{Y}) = \forall x \exists y \, . \, x \leq y \wedge x \neq y \wedge \bigvee_{s_i \in F} Y_i(y)$$

Consequences

Theorem 8 *A language* $L \subseteq \Sigma^\omega$ *is definable in S1S iff it is Büchi recognizable.*

Corollary 2 *The SAT problem for S1S is decidable.*

Lemma 2 *Any S1S formula* $\phi(X_1, \ldots, X_m)$ *is equivalent to an S1S formula of the form* $\exists Y_1 \ldots \exists Y_p \,.\, \varphi$*, where* φ *does not contain other set variables than* $X_1, \ldots, X_m, Y_1, \ldots, Y_p$.

Müller and Rabin Word Automata

Müller Automata

Let $\Sigma = \{a, b, \ldots\}$ be a finite alphabet.

Definition 1 *A* Müller automaton *over* Σ *is* $A = \langle S, s_0, T, \mathcal{F} \rangle$*, where:*

- S *is the finite* set of states
- $s_0 \in S$ *is the* initial state
- $T : S \times \Sigma \mapsto S$ *is the* transition table
- $\mathcal{F} \subseteq 2^S$ *is the set of* accepting sets

Notice that Müller automata are deterministic and complete by definition.

Acceptance Condition

A *run* of a Müller automaton is defined over an infinite word $w : \alpha_1\alpha_2\ldots$ as an infinite sequence of states $\pi \ : \ s_0s_1s_2\ldots$ such that:

- $T(s_i, \alpha_{i+1}) = s_{i+1}$, for all $i \in \mathbb{N}$.

Let $\inf(\pi) = \{s \mid s$ appears infinitely often on $\pi\}$.

Run π of A is said to be *accepting* iff $\inf(\pi) \in \mathcal{F}$.

$L \subseteq \Sigma^\omega$ is *Müller-recognizable* iff there exists a MA A such that $L = \mathcal{L}(A)$.

Exercise

Let $\Sigma = \{a, b\}$ and $A = \langle S, s_0, T, \mathcal{F} \rangle$, where:

- $S = \{s_0, s_1\}$,
- $T(s_0, a) = s_0$, $T(s_0, b) = s_1$, $T(s_1, a) = s_0$ and $T(s_1, b) = s_1$,
- $\mathcal{F} = \{\{s_0, s_1\}\}$

What is $\mathcal{L}(A)$? What if A was Büchi with $F = \{s_0, s_1\}$?

Deterministic Büchi $\subseteq$ Müller

Theorem 9 *For each deterministic Büchi automaton A there exists a Müller automaton B such that $\mathcal{L}(A) = \mathcal{L}(B)$*

Let $A = \langle S, \{s_0\}, T, F \rangle$ be a deterministic Büchi automaton.

Define $B = \langle S, s_0, T, \{G \in 2^S \mid G \cap F \neq \emptyset\} \rangle$

Müller $\subseteq$ Non-deterministic Büchi

Theorem 10 *For each Müller automaton A there exists a non-deterministic Büchi automaton B such that $\mathcal{L}(A) = \mathcal{L}(B)$.*

Let $A = (S, s_0, T, \mathcal{F})$ be a Müller automaton, with $\mathcal{F} = \{F_1, \ldots, F_n\}$. Then B simulates A and guesses the accepting set F_i.

We introduce finite memory to accumulate F_i states. The Büchi automaton guesses when all the states outside F_i are finished.

When the memory is full we reset it to $\emptyset$, to ensure that we see F_i states again and again.

Müller $\subseteq$ Non-deterministic Büchi

Define the Büchi automaton $B = (S_B, s_0, T_B, F_B)$ where:

- $S_B = S \cup (S \times 2^S \times \{1, \ldots, n\})$
- $F_B = \{(s, \emptyset, i) \mid s \in S,\ i \in \{1, \ldots, n\}\}$
- T_B is defined as follows:
  - $(s, \alpha, t) \in T_B$ and $(s, \alpha, (t, \emptyset, i)) \in T_B$ if $T(s, \alpha) = t$
  - $((s, Q, i), \alpha, (t, Q \cup \{t\}, i)) \in T_B$ if $T(s, \alpha) = t$ and $Q \cup \{t\} \subset F_i$
  - $((s, Q, i), \alpha, (t, \emptyset, i)) \in T_B$ if $T(s, \alpha) = t$ and $Q \cup \{t\} = F_i$

Now we prove that $\mathcal{L}(A) = \mathcal{L}(B)$.

Closure Properties

Theorem 11 *The class of Müller-recognizable languages is closed under union, intersection and complement.*

Let $A = \langle S, s_0, T, \mathcal{F} \rangle$ be a Müller automaton.

Define $B = \langle S, s_0, T, 2^S \setminus \mathcal{F} \rangle$.

We have $\mathcal{L}(B) = \Sigma^\omega \setminus \mathcal{L}(A)$.

Closure Properties

Let $A_i = \langle S_i, s_{0,i}, T_i, \mathcal{F}_i \rangle$, $i = 1, 2$ be Müller automata.

Define $B = \langle S, s_0, T, \mathcal{F} \rangle$ where:

- $S = S_1 \times S_2$,
- $s_0 = \langle s_{0,1}, s_{0,2} \rangle$,
- $T(\langle s_1, s_2 \rangle, a) = \langle T(s_1, a), T(s_2, a) \rangle$
- $\mathcal{F} = \{\{\langle s_1, s'_1 \rangle, \ldots, \langle s_k, s'_k \rangle\} \mid \{s_1, \ldots, s_k\} \in \mathcal{F}_1$ or $\{s'_1, \ldots, s'_k\} \in \mathcal{F}_2\}$

We have $\mathcal{L}(B) = \mathcal{L}(A_1) \cup \mathcal{L}(A_2)$.

For intersection it is enough to set

$\mathcal{F} = \{\{\langle s_1, s'_1 \rangle, \ldots, \langle s_k, s'_k \rangle\} \mid \{s_1, \ldots, s_k\} \in \mathcal{F}_1$ and $\{s'_1, \ldots, s'_k\} \in \mathcal{F}_2\}$

Characterization of Müller-recognizable languages

A language $L \subseteq \Sigma^\omega$ is Müller-recognizable iff L is a Boolean combination of sets $\overrightarrow{W}$, $W \subseteq \Sigma^*$, i.e. $L = \bigcup_i \left(\bigcap_j \overrightarrow{W_{ij}} \cap \bigcap_k (\Sigma^\omega \setminus \overrightarrow{W_{ik}}) \right)$.

"$\Leftarrow$" Any set $\overrightarrow{W_{ij}}$ is recognized by a deterministic Büchi automaton, hence also by a Müller automaton.

"$\Rightarrow$" Let $A = \langle S, s_0, T, \mathcal{F} \rangle$ be a Müller automaton recognizing L.

Let $A_q = \langle S, s_0, T, \{q\} \rangle$, $q \in S$, and $W_q = \mathcal{L}(A_q)$.

$$L = \bigcup_{Q \in \mathcal{F}} \left(\bigcap_{q \in Q} \overrightarrow{W_q} \cap \bigcap_{q \in S \setminus Q} (\Sigma^\omega \setminus \overrightarrow{W_q}) \right)$$

Rabin Word Automata

Let $\Sigma = \{a, b, \ldots\}$ be a finite alphabet.

Definition 2 *A* Rabin automaton *over* Σ *is* $A = \langle S, s_0, T, \Omega \rangle$, *where:*

- S *is the finite* set of states
- $s_0 \in S$ *is the* initial state
- $T : S \times \Sigma \mapsto S$ *is the* transition table
- $\Omega = \{\langle N_1, P_1 \rangle, \ldots, \langle N_k, P_k \rangle\}$ *is the set of* accepting pairs, $N_i, P_i \subseteq S$.

Run π of A is said to be *accepting* iff

$$\inf(\pi) \cap N_i = \emptyset \text{ and } \inf(\pi) \cap P_i \neq \emptyset$$

for some $1 \leq i \leq k$.

The Streett acceptance condition

The Rabin acceptance condition is of the form:

$$\bigvee_{1 \leq i \leq k} inf(\pi) \cap N_i = \emptyset \;\wedge\; inf(\pi) \cap P_i \neq \emptyset$$

The Streett acceptance condition is the negation:

$$\bigwedge_{1 \leq i \leq k} inf(\pi) \cap N_i \neq \emptyset \;\rightarrow\; inf(\pi) \cap P_i \neq \emptyset$$

From Rabin to Müller

Given a Rabin automaton $A = \langle S, s_0, T, \Omega \rangle$, there exists a Müller automaton B such that $\mathcal{L}(A) = \mathcal{L}(B)$

Let $\Omega = \{\langle N_1, P_1 \rangle, \ldots, \langle N_k, P_k \rangle\}$.

Let $A_i = \langle S, s_0, T, P_i \rangle$, and $B_i = \langle S, s_0, T, N_i \rangle$.

$$\mathcal{L}(A) = \bigcup_{i=1}^{k} \left(\overrightarrow{\mathcal{L}(A_i)} \cap (\Sigma^\omega \setminus \overrightarrow{\mathcal{L}(B_i)}) \right)$$

From Rabin to Müller (2)

Given a Rabin automaton $A = \langle S, s_0, T, \Omega \rangle$, such that

$$\Omega = \{\langle N_1, P_1 \rangle, \ldots, \langle N_k, P_k \rangle\}$$

let $B = \langle S, s_0, T, \mathcal{F} \rangle$ be the Müller automaton, where

$$\mathcal{F} = \{F \subseteq S \mid F \cap N_i = \emptyset \text{ and } F \cap P_i \neq \emptyset \text{ for some } 1 \leq i \leq k\}$$

From Müller to Rabin

Given a Müller automaton $A = \langle S, s_0, T, \mathcal{F} \rangle$, there exists a Rabin automaton B such that $\mathcal{L}(A) = \mathcal{L}(B)$

Let $\mathcal{F} = \{Q_1, \ldots, Q_k\}$

Let $B = \langle S', s'_0, T', \Omega' \rangle$ where:

- $S' = 2^{Q_1} \times \ldots \times 2^{Q_k} \times S$
- $s'_0 = \langle \emptyset, \ldots, \emptyset, s_0 \rangle$

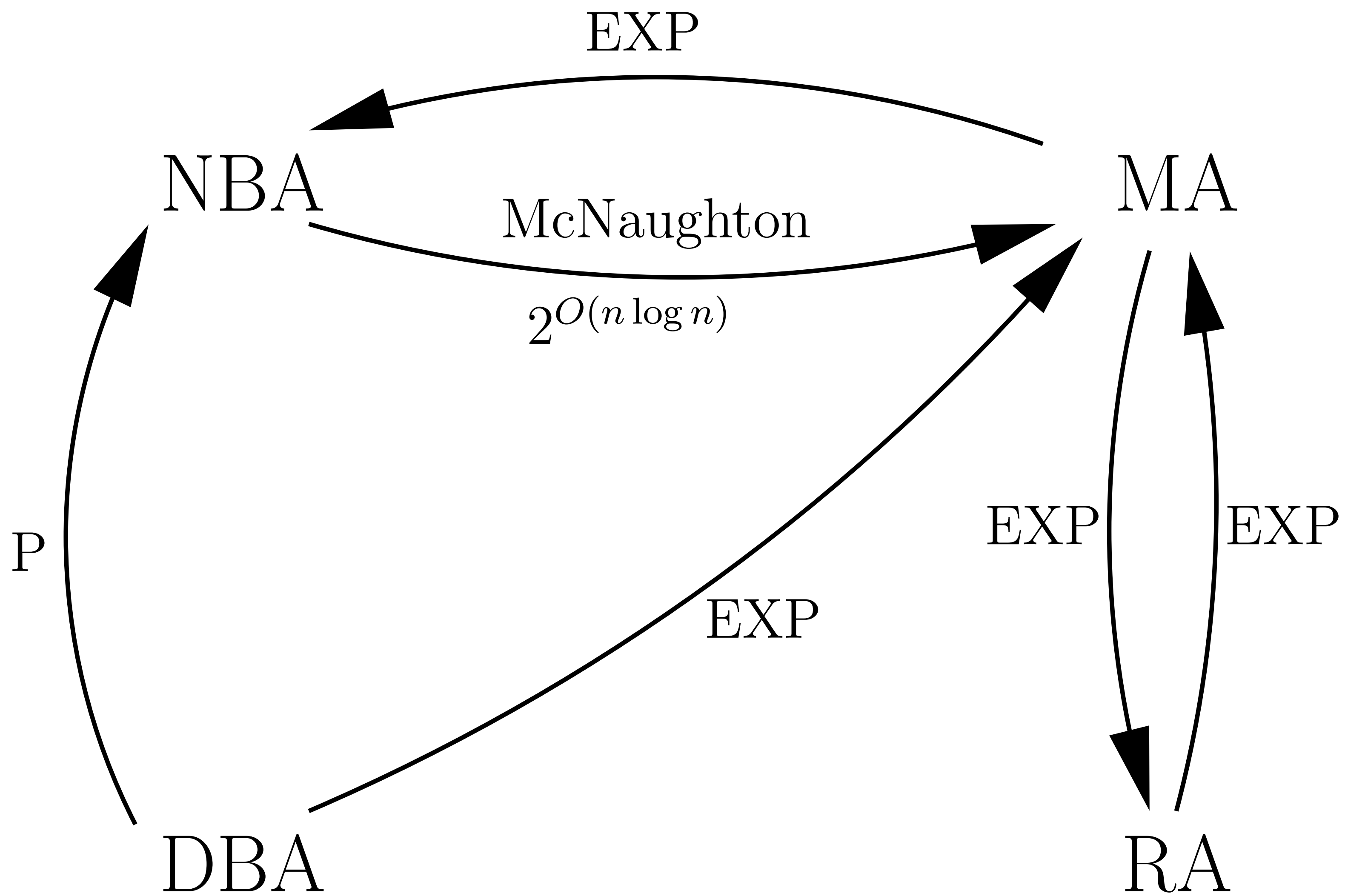
From Müller to Rabin

- $T'(\langle S_1, \ldots, S_k, s\rangle, a) = \langle S'_1, \ldots, S'_k, s'\rangle$ where:
  - $s' = T(s, a)$
  - $S'_i = \emptyset$ if $S_i = Q_i$, $1 \leq i \leq k$
  - $S'_i = (S_i \cup \{s'\}) \cap Q_i$, $1 \leq i \leq k$

- $P_i = \{\langle S_1, \ldots, S_i, \ldots, S_k, s\rangle \mid S_i = Q_i\}$, $1 \leq i \leq k$

- $N_i = \{\langle S_1, \ldots, S_i, \ldots, S_k, s\rangle \mid s \notin Q_i\}$, $1 \leq i \leq k$

The Big Picture



Exercises

Write down Müller automata for the following languages:

1. $\Sigma = \{a, b\}$, $L = (a + b)^* a^\omega$
2. $\Sigma = \{a, b, c\}$, $L =$
   $\{w \mid \text{if } a \text{ occurs infinitely often then } b \text{ occurs infinitely often in } w\}$

Let $A = \langle S, s_0, T, \{Q_1, \ldots, Q_t\}\rangle$ be a Müller automaton. Consider the Rabin automaton $A' = \langle S, s_0, T, \Omega\rangle$ where

$$\Omega = \{(S \setminus Q_1, Q_1), \ldots, (S \setminus Q_t, Q_t)\}$$

Give an example of A such that $\mathcal{L}(A) \neq \mathcal{L}(A')$.