Bounded-degree factorization of lacunary polynomials

Bonus: PIT algorithms

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Find $f_1, \ldots, f_t$, irreducible, s.t. $f = f_1 \times \cdots \times f_t$.

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Definition

$$f(X_1, \ldots, X_n) = \sum_{j=1}^{k} c_j X_1^{\alpha_{1j}} \cdots X_n^{\alpha_{nj}}$$

- $\text{size}(f) \simeq k\left(\max_j(\text{size}(c_j)) + n\underline{\log(\deg f)}\right)$

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Theorems

There exist deterministic polynomial-time algorithms computing

- **integer roots** of $f \in \mathbb{Z}[X]$; [Cucker-Koiran-Smale'98]
- **low-degree** factors of $f \in \mathbb{Q}(\alpha)[X]$; [Lenstra'99]
- **low-degree** factors of $f \in \mathbb{Q}(\alpha)[X_1, \ldots, X_n]$. [Kaltofen-Koiran'06]

It is NP-**hard** (under BPP reductions) to compute **roots of** $f \in \mathbb{F}_p[X]$. [Bi-Cheng-Rojas'13]

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Let $f = f_1 + f_2 \in \mathbb{Z}[X]$, with coefficients of absolute value $\leqslant 2^s$, s.t. $\mathrm{val}(f_2) - \deg(f_1) > 1 + s$. Then for $|x| \geqslant 2$,
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$$-9 + X^2 + 6X^7 + 2X^8 = -9 + X^2 + X^7(6 + 2X)$$

Our settings

- Generalization to other fields?
- Simpler, more combinatorial proofs?

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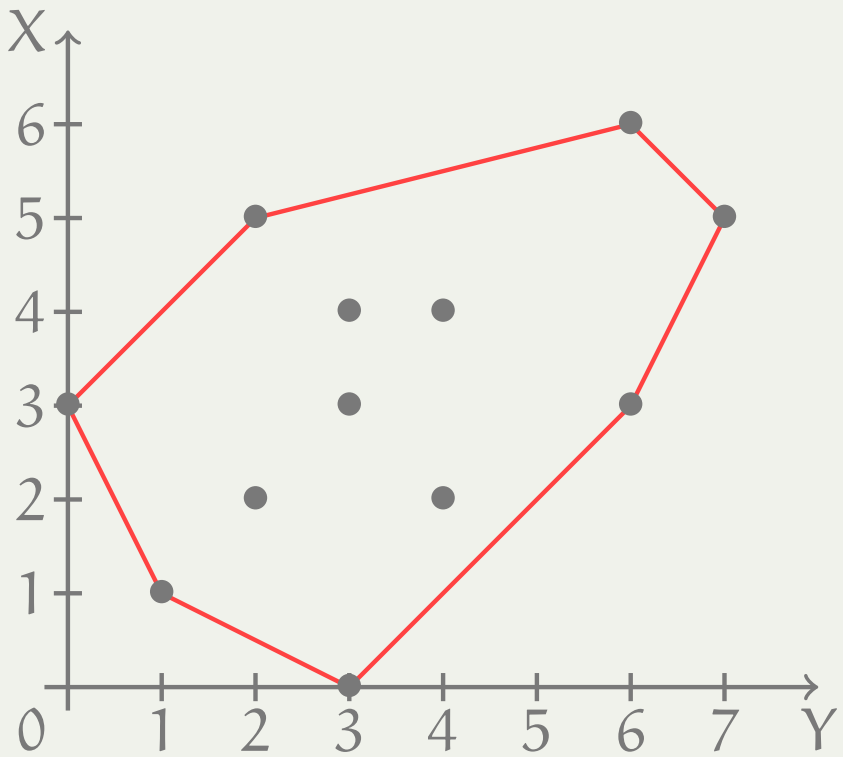
Main (informal) result

Inputs:	$f \in \mathbb{K}[X_1, \ldots, X_n]$; bound d
Assumption:	$\mathbb{K}$ of characteristic 0 or $> \deg(f)$
Output:	Degree-d factors of f
Algorithms:	"Combinatorial" reductions to
	▶ the univariate case
	▶ low-degree multivariate factorization
Complexity:	Deterministic $\mathrm{poly}(\mathrm{size}(f), d)$

Newton polygon/tope



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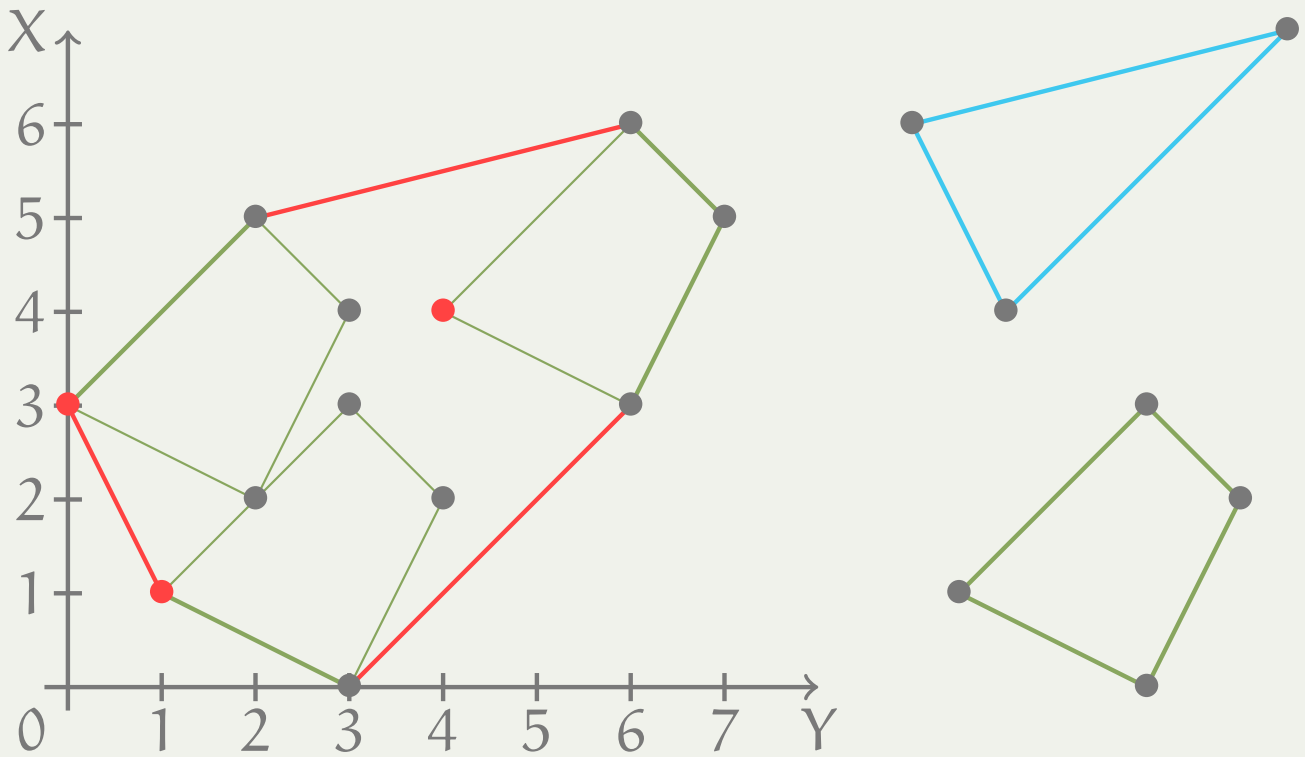

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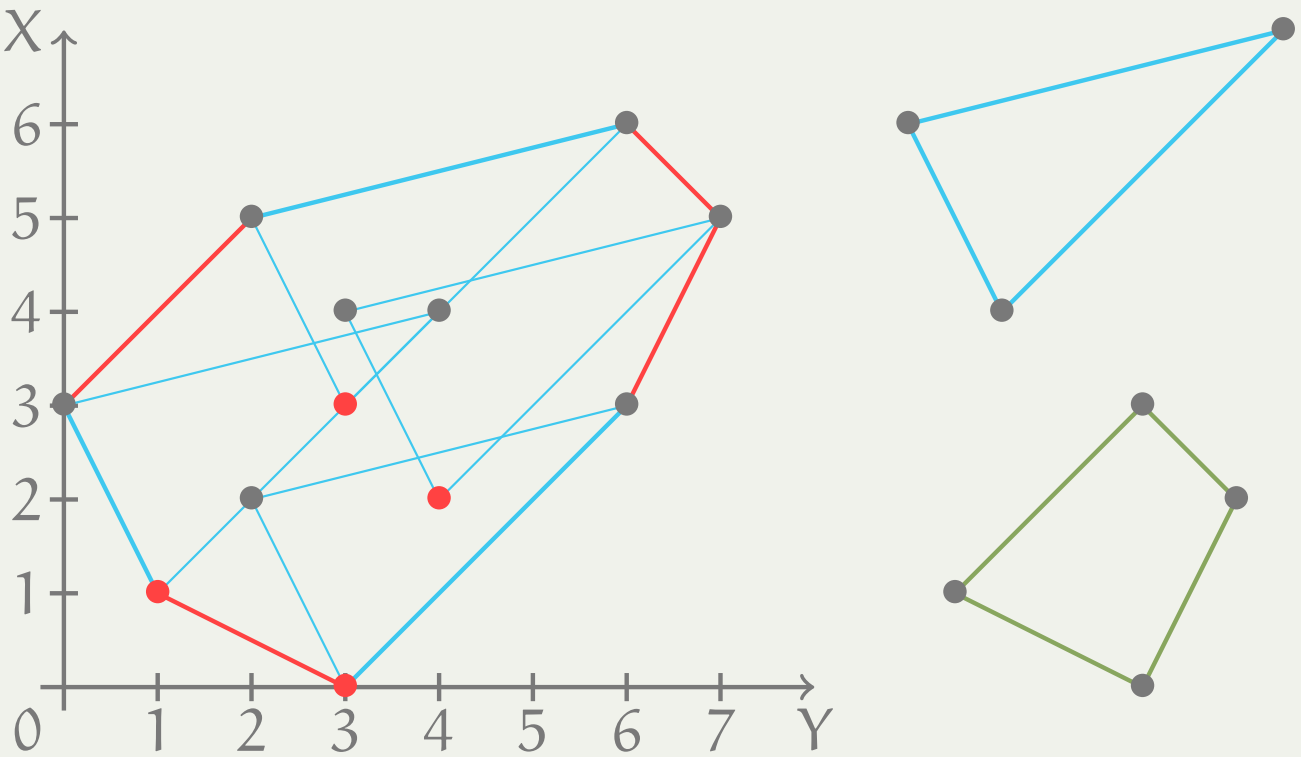

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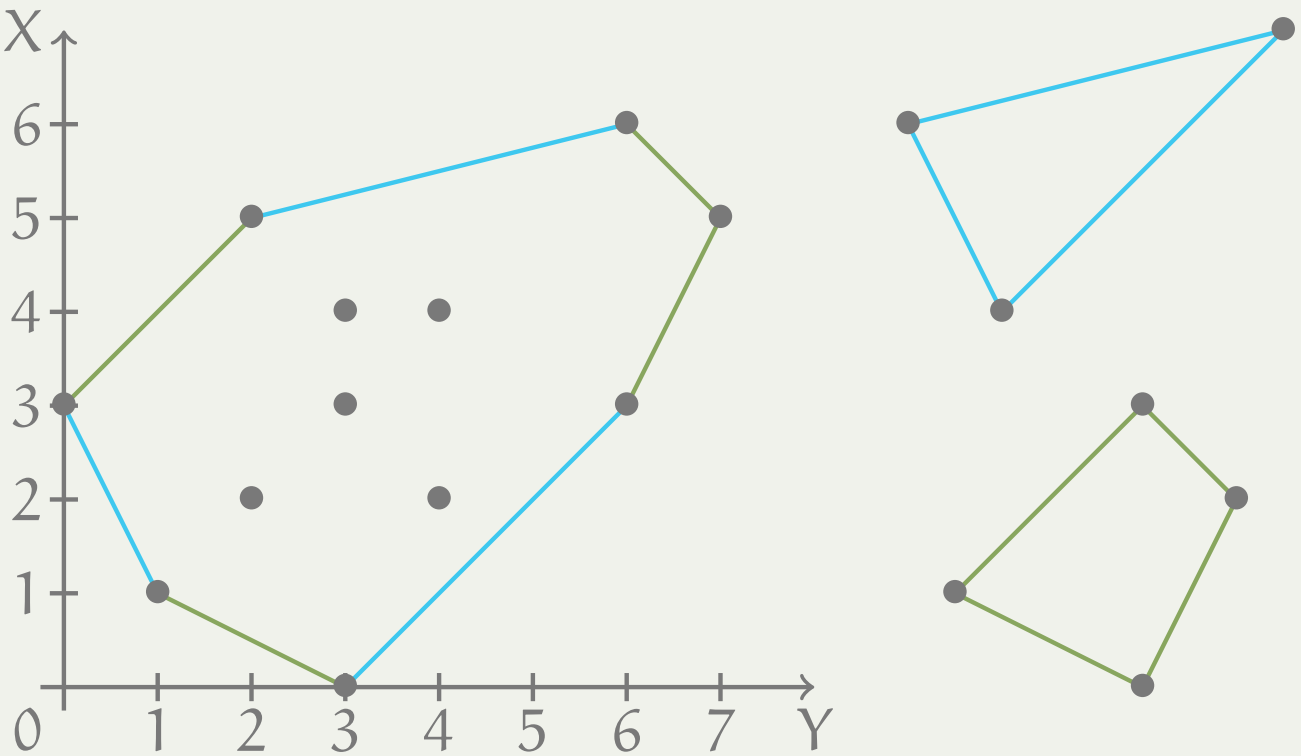

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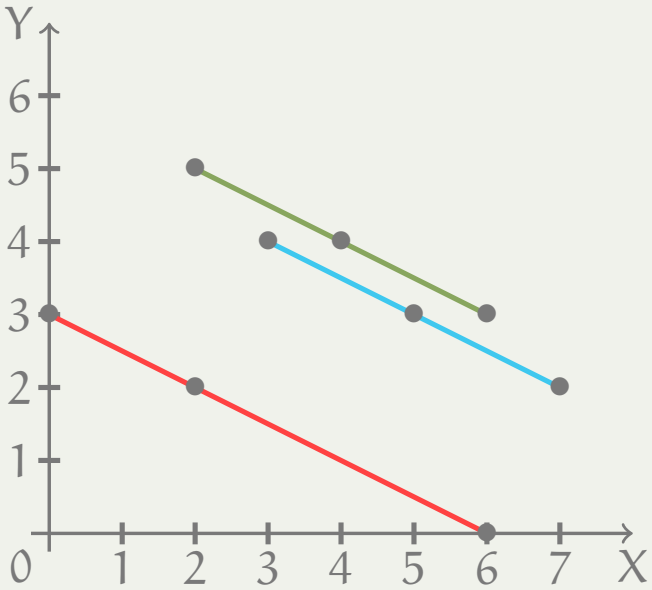
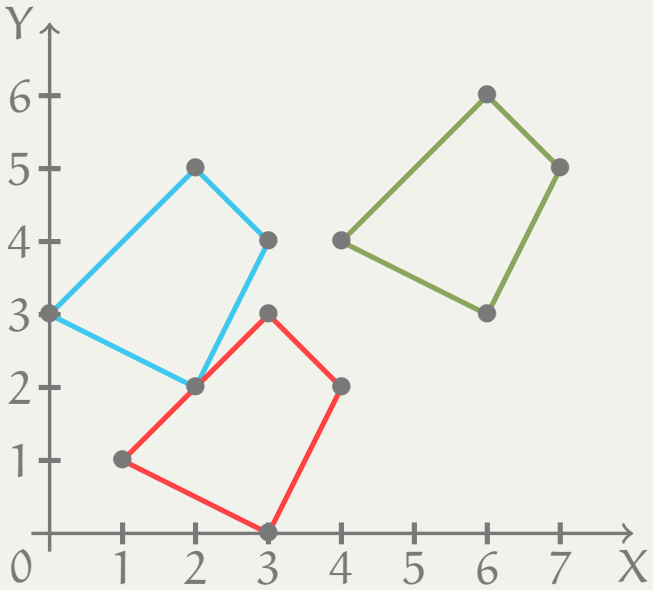
Two kinds of factors

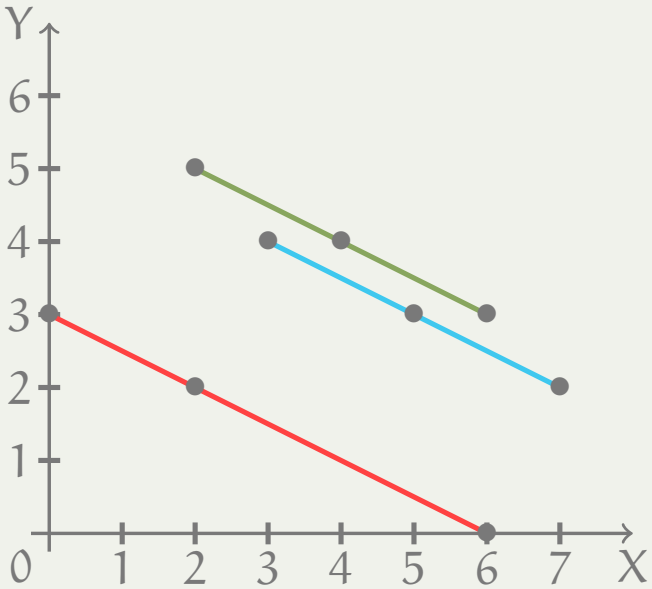


Unidimensional factors



Multidimensional factors

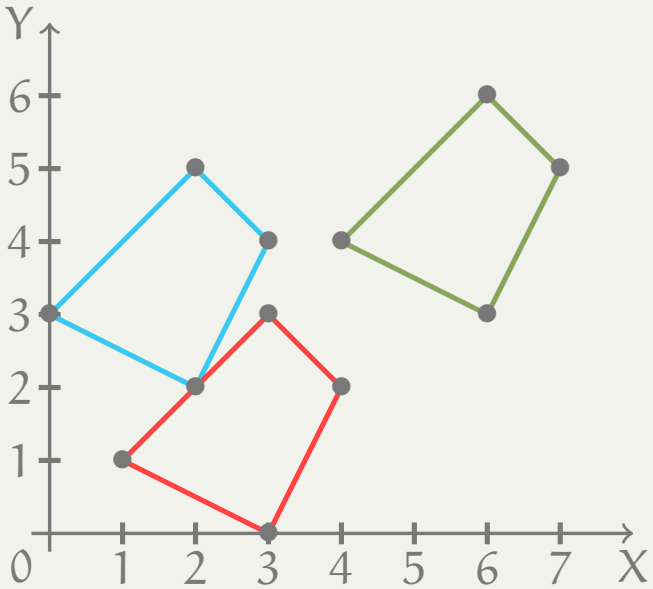
Two kinds of factors



Unidimensional factors

⇓

Univariate lacunary factorization

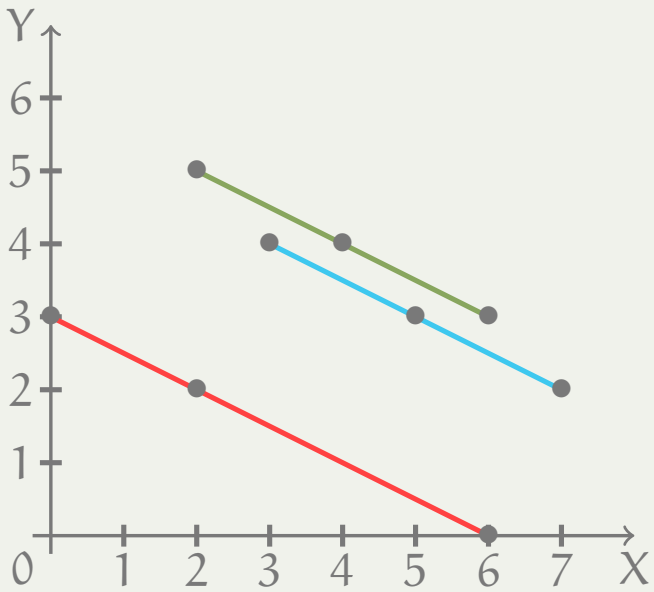


Multidimensional factors

⇓

Multivariate low-degree factorization

Unidimensional factors



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- Directions given by the support
- Each direction δ independently

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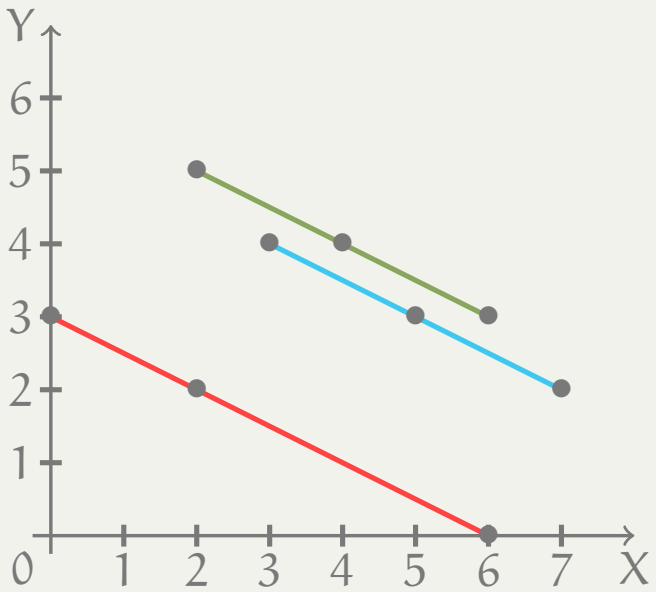

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1. Write f as a sum of **δ-components**
2. **Project** each component to a univariate lacunary polynomial
3. Compute the common **bounded-degree factors** of the projections
   Available for number field only. NP-hard in positive characteristic.
4. **Lift** the univariate factors to unidimensional factors

Linear factors of bivariate polynomials

[Chattopadhyay-G.-Koiran-Portier-Strozecki'13]

Observation

$(Y - uX - v)$ divides $f(X, Y) \iff f(X, uX + v) \equiv 0$

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An example

$$\begin{aligned} f = {} & X^{31}Y^6 - 2X^{30}Y^7 + X^{29}Y^8 - X^{29}Y^6 + X^{18}Y^{13} \\ & - X^{16}Y^{15} + X^{17}Y^{13} + X^{16}Y^{14} + X^{10}Y^2 - X^9Y^3 \\ & + X^9Y^2 - X^5Y^6 + X^3Y^8 - 2X^3Y^7 + X^3Y^6 \end{aligned}$$

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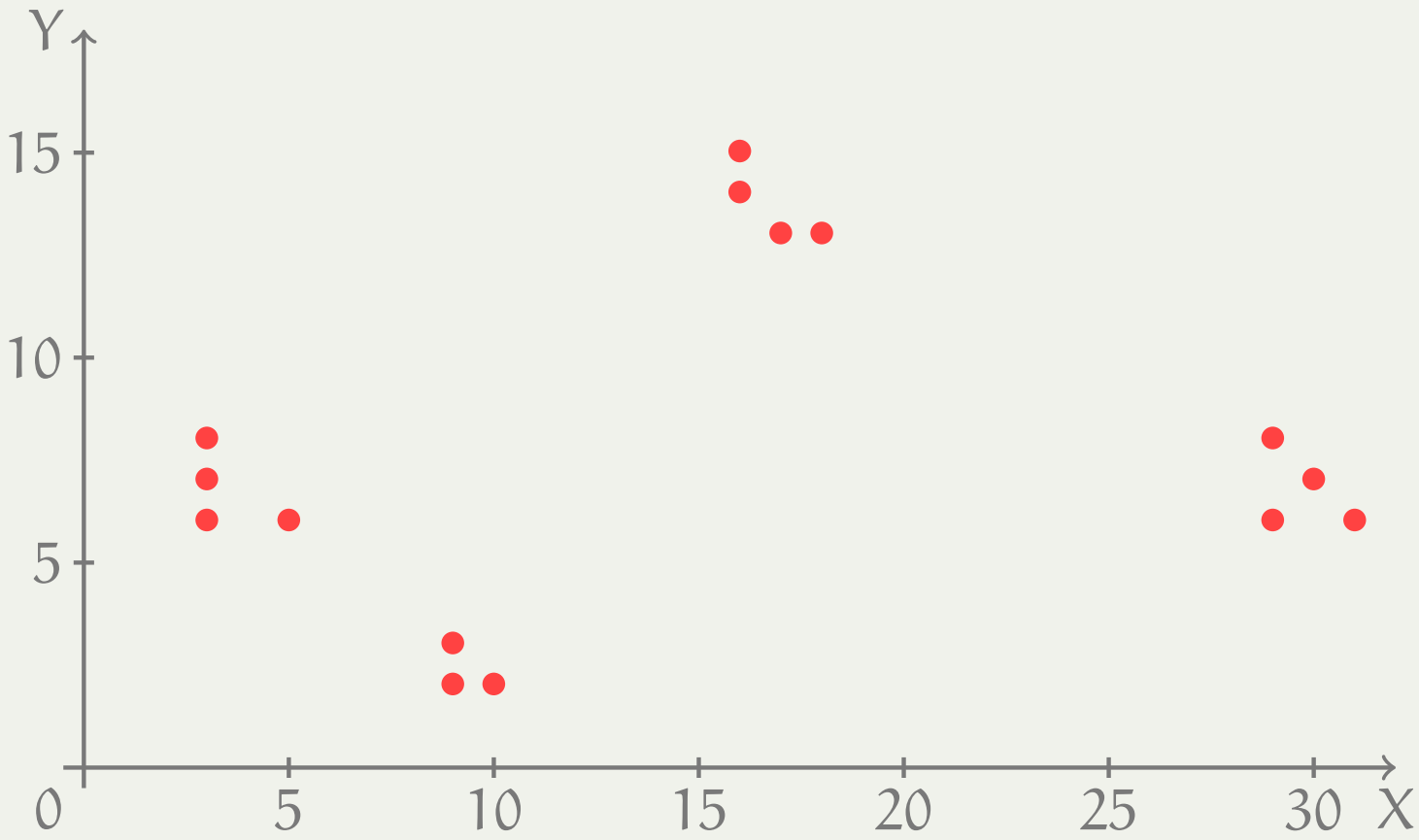

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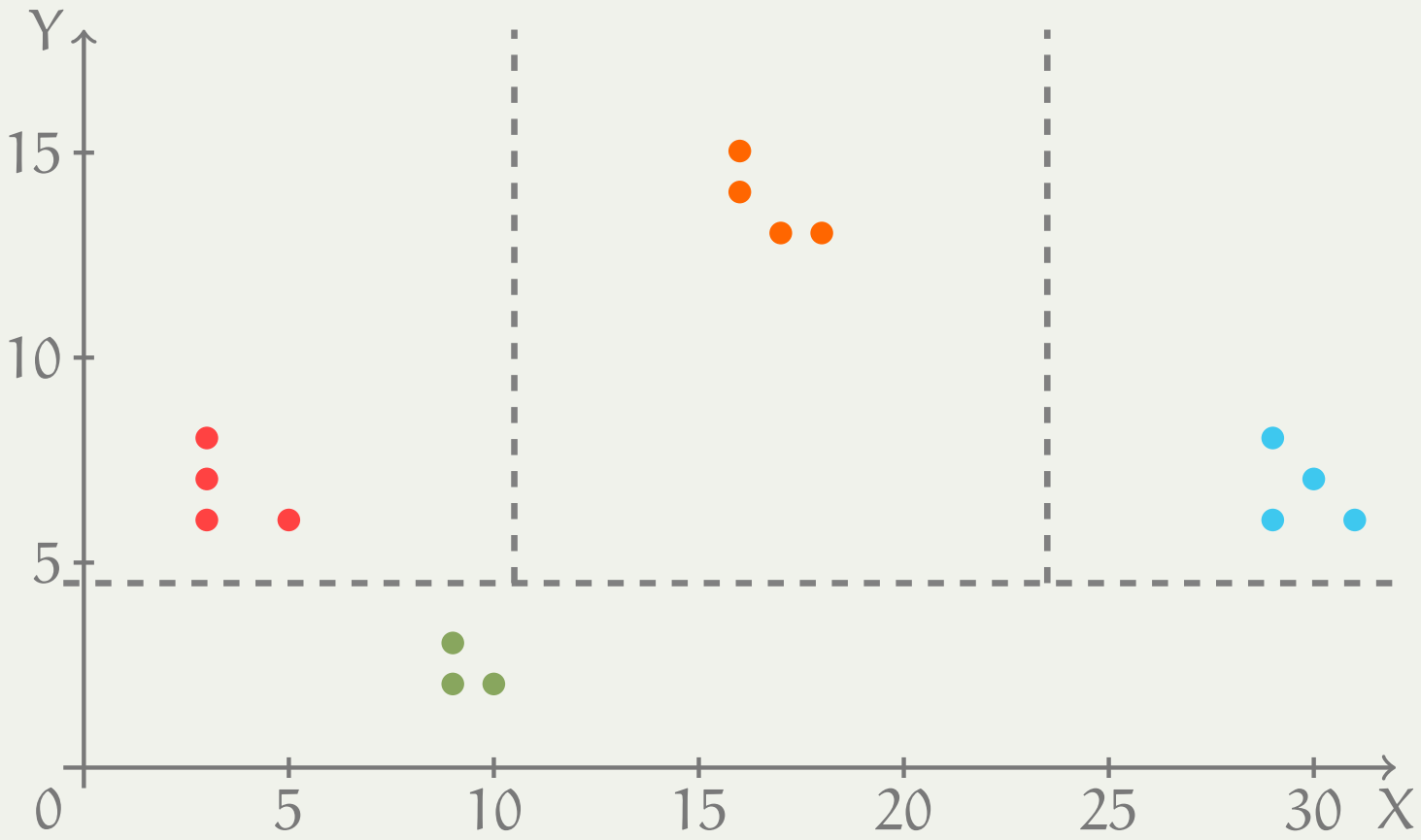

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$\Longrightarrow$ linear factors of f: $(X - Y + 1, 1)$, $(X, 3)$, $(Y, 2)$

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- $g_0 \in \mathbb{K}[X]$
- $\phi_1, \ldots, \phi_d \in \overline{\mathbb{K}(X)} \subset \overline{\mathbb{K}}\langle\langle X \rangle\rangle$ are **Puiseux series**:

$$\phi(X) = \sum_{t \geqslant t_0} a_t X^{t/n} \text{ with } a_t \in \overline{\mathbb{K}},\ a_{t_0} \neq 0. \qquad (\mathrm{val}(\phi) = t_0/n)$$

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- If g is irreducible,
  g divides $f \iff \exists i,\ f(X,\phi_i) = 0 \iff \forall i,\ f(X,\phi_i) = 0$

Theorem

$$\operatorname{val}\left(\sum_{j=1}^{\ell} c_j X^{\alpha_j} \phi(X)^{\beta_j}\right) \leqslant \min_j(\alpha_j + \nu\beta_j) + (8d^2 - \nu)\binom{\ell}{2},$$

where

- $\phi \in \overline{\mathbb{K}}\langle\langle X\rangle\rangle$ of valuation ν and degree-d minimal polynomial,
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  "If $f = f_1 + f_2$ with a gap, g divides f iff it divides f_1 and f_2"
- Bounds $\alpha_j + \nu\beta_j$, not the degree $\rightsquigarrow$ use two different ν's

Corollary

Inputs: $f \in \mathbb{K}[X_1, \ldots, X_n]$ in lacunary representation bound d

Output: Degree-$O(d^4 s(f)^2)$ polynomial f_{ld} s.t for all multidimensional degree-d polynomial g, $\mathrm{mult}_g(f) = \mathrm{mult}_g(f_{ld})$

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Complexity: Deterministic polynomial time

1. Write $f = f_1 + \cdots + f_s$ where $\deg_{X_i}(f_t) - \mathrm{val}_{X_i}(f_t) \leqslant (4d^4 - 2d^2)\binom{s(f_t)}{2}$ for all i;
2. Return $\gcd(f_1, \ldots, f_s)$.
3. (Factor the gcd using a low-degree factorization algorithm.)

Complete algorithm

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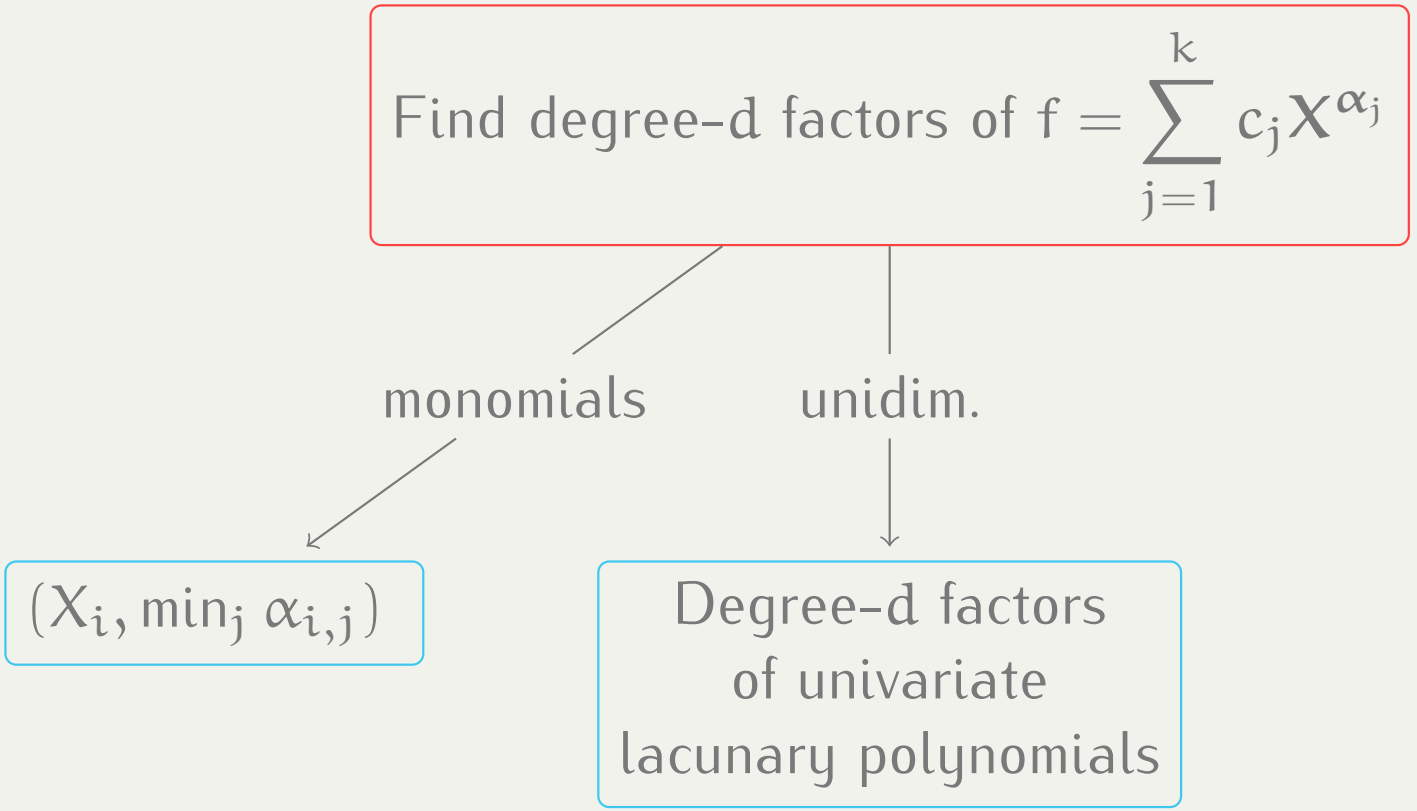
monomials

$(X_i, \min_j \alpha_{i,j})$

Complete algorithm



Available for $\mathbb{Q}(\alpha)$ only
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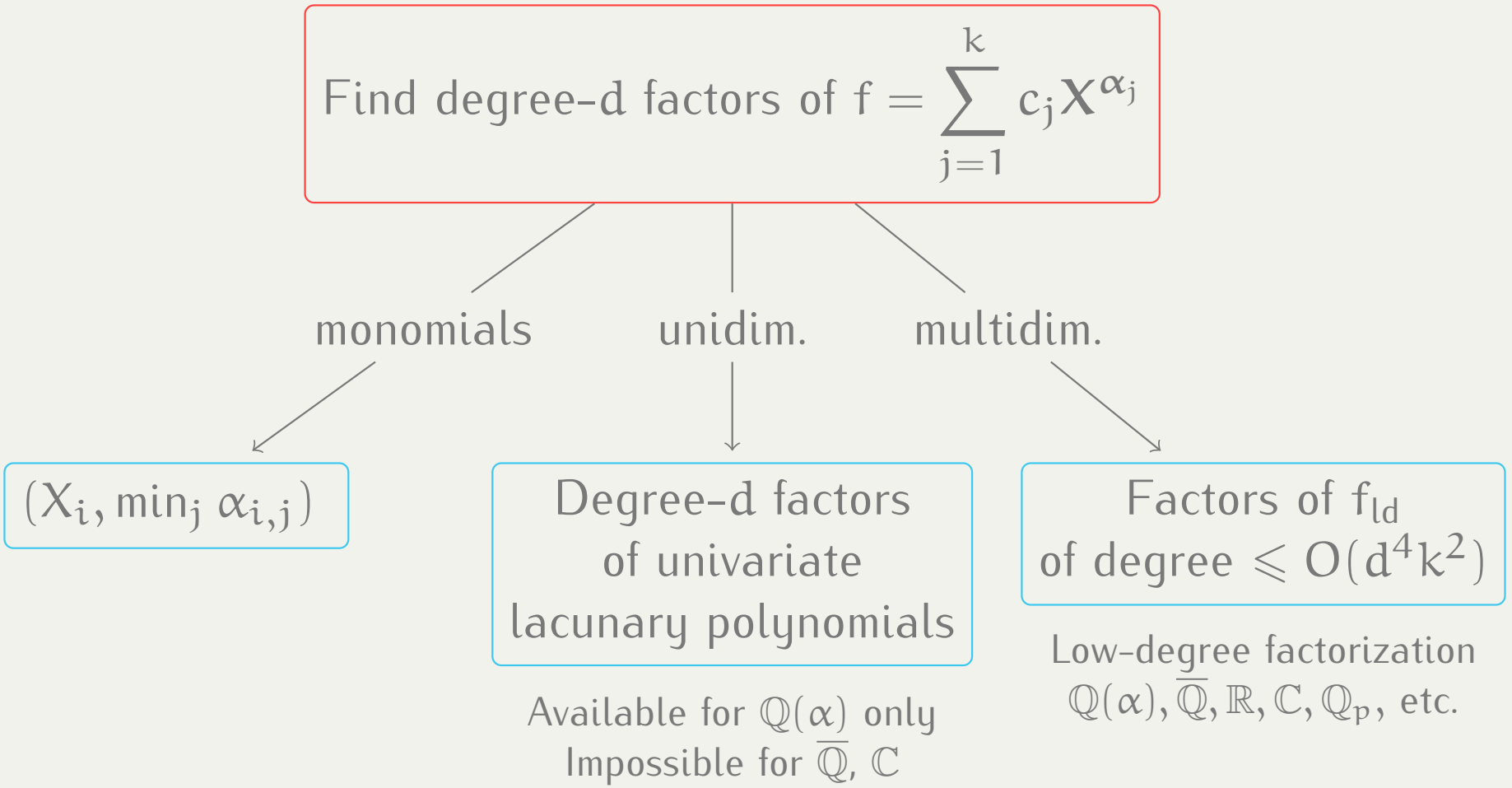

- $\sum_{j=1}^{k} a_j X^{\alpha_j} (uX^d + v)^{\beta_j} \in \mathbb{Q}(\alpha)[X]$
  - Deterministic PIT algorithm in $\mathrm{poly}(k, \log(\alpha_j), \log(\beta_j), \log d)$
  - Gap Theorem for linear factors + write $\alpha_j = q_j d + r_j$

Application to PIT

- $\sum_{j=1}^{k} a_j X^{\alpha_j} (uX^d + v)^{\beta_j} \in \mathbb{Q}(\alpha)[X]$
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  - Gap Theorem for linear factors + write $\alpha_j = q_j d + r_j$
- $\sum_{j=1}^{k} a_j \prod_{i=1}^{m} f_i^{\alpha_{ij}} \in \mathbb{Q}(\alpha)[X]$ with monic f_i's
  - Det. PIT alg. in $\mathrm{poly}(k, \log(\alpha_{ij}), \deg(f_i))$
  - Modified Gap Theorem (for multiplicity of factors)
  - If f_i's not monic: needs to test $\sum_j \prod_i \lambda_i^{\alpha_{ij}} = 0$?
  - Maybe valid with multivariate f_i's...

Conclusion

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  - Implementation available: package Lacunaryx of Mathemagix

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  - More general settings: SLP/arithmetic circuits
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