Factorization of lacunary polynomials

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Factorization of a polynomial P

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Complexity

Polynomial in the **degree** of the polynomials

The case of lacunary polynomials

$$X^{102}Y^{101} + X^{101}Y^{102} - X^{101}Y^{101} - X - Y + 1$$

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- Restriction to **some** factors only

Integral roots of integral polynomials

Gap Theorem (Cucker-Koiran-Smale'98)

Let

$$P(X) = \underbrace{\sum_{j=1}^{\ell} a_j X^{\alpha_j}}_{P_0} + \underbrace{\sum_{j=\ell+1}^{k} a_j X^{\alpha_j}}_{P_1} \in \mathbb{Z}[X]$$

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$(Y - uX - v)$ divides $P(X, Y) \iff P(X, uX + v) \equiv 0$

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with $uv \neq 0$, $\alpha_1 \leq \cdots \leq \alpha_k$. If ℓ is the smallest index s.t.

$$\alpha_{\ell+1} > \alpha_1 + \binom{\ell}{2},$$

then $P \equiv 0$ iff both $P_0 \equiv 0$ and $P_1 \equiv 0$.

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then every linear factor of P divides both P_0 and P_1 if $uv \neq 0$.

Example

$$P = X^{31}Y^6 - 2X^{30}Y^7 + X^{29}Y^8 - X^{29}Y^6 + X^{18}Y^{13} - X^{16}Y^{15} + X^{17}Y^{13} + X^{16}Y^{14} + X^{10}Y^2 - X^9Y^3 + X^9Y^2 - X^5Y^6 + X^3Y^8 - 2X^3Y^7 + X^3Y^6$$

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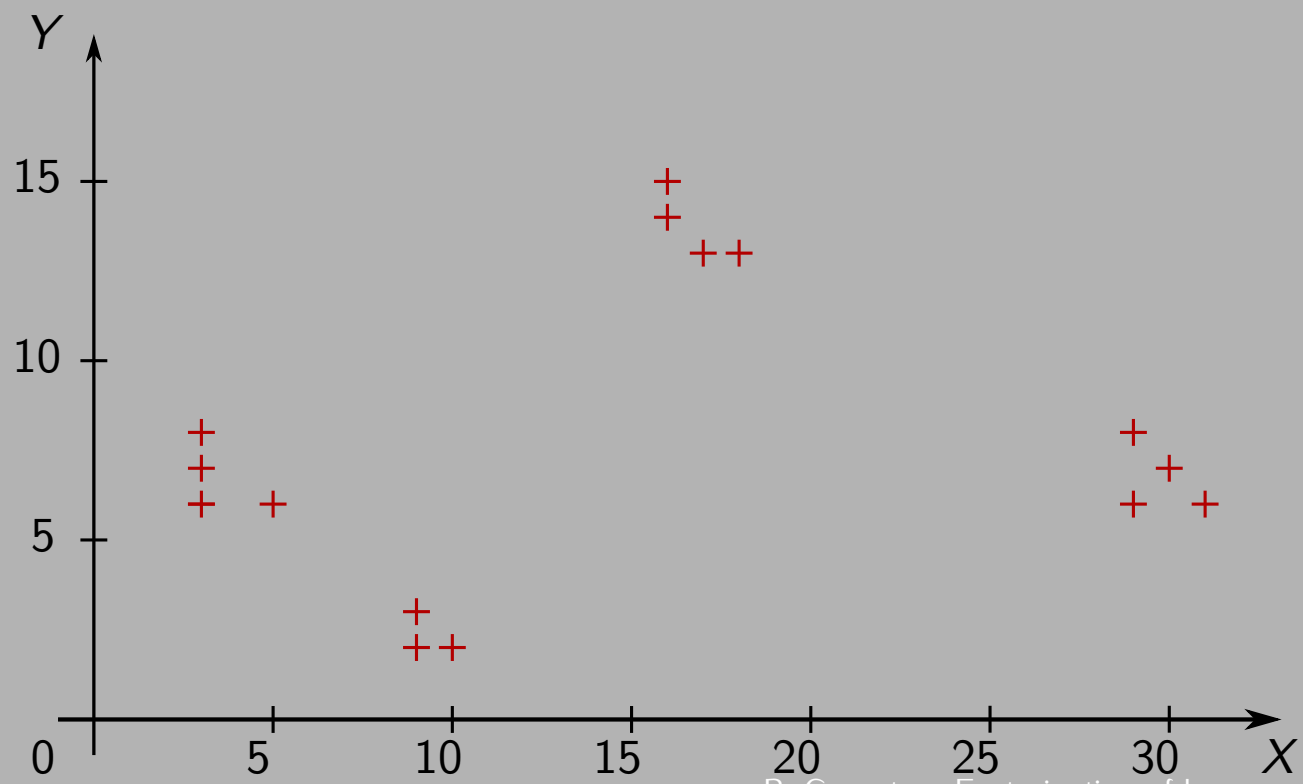

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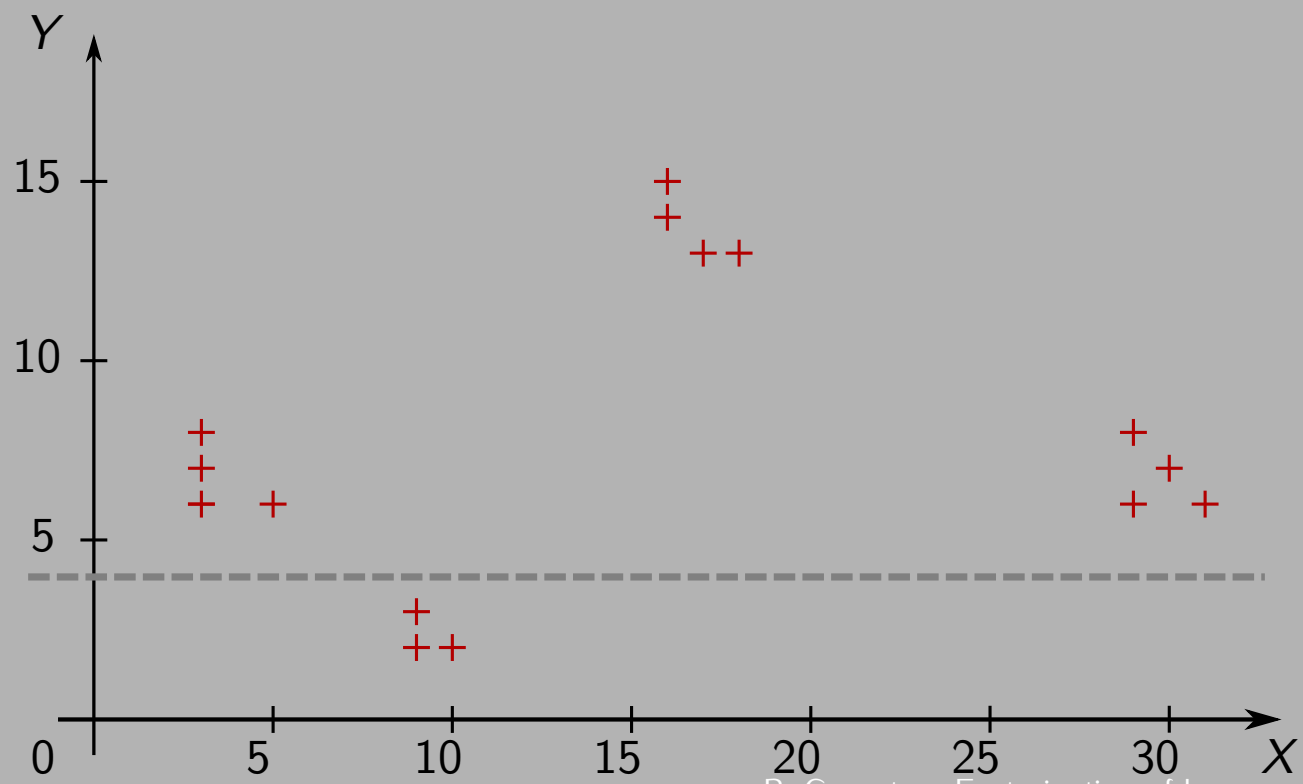

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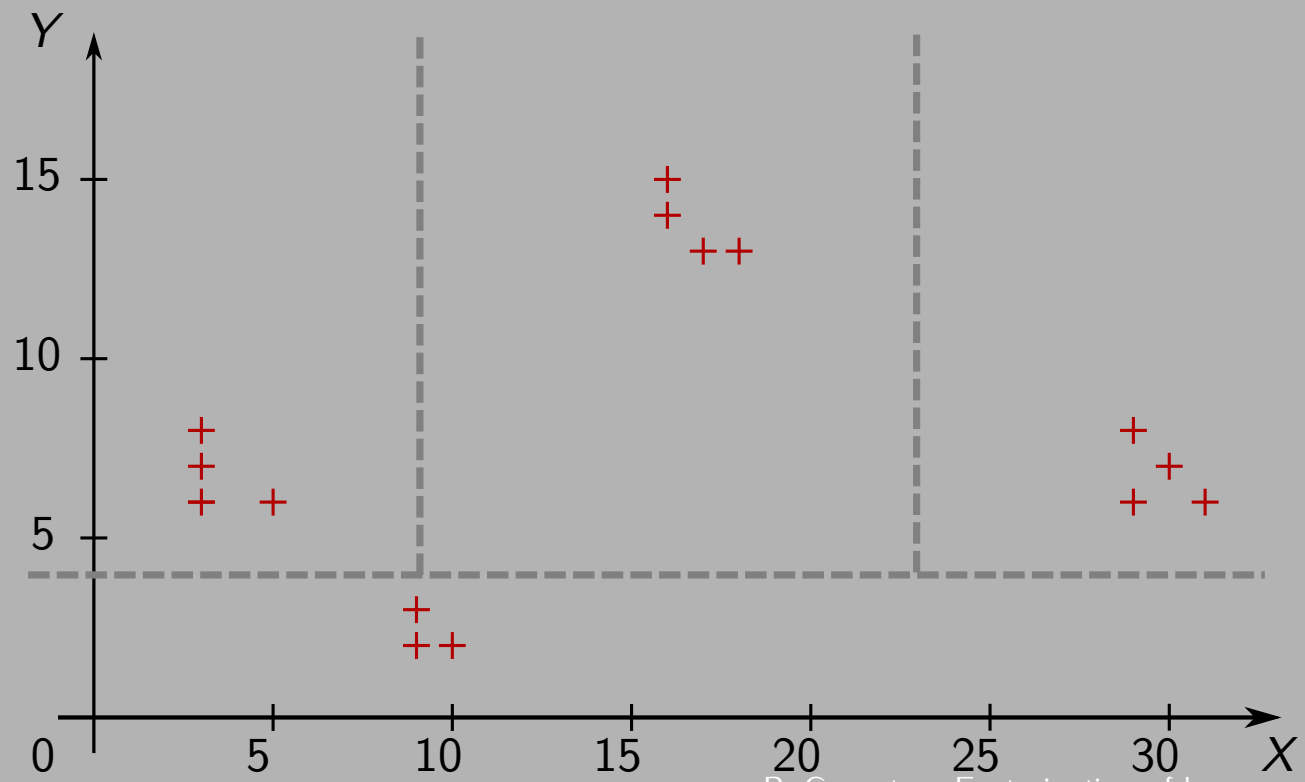

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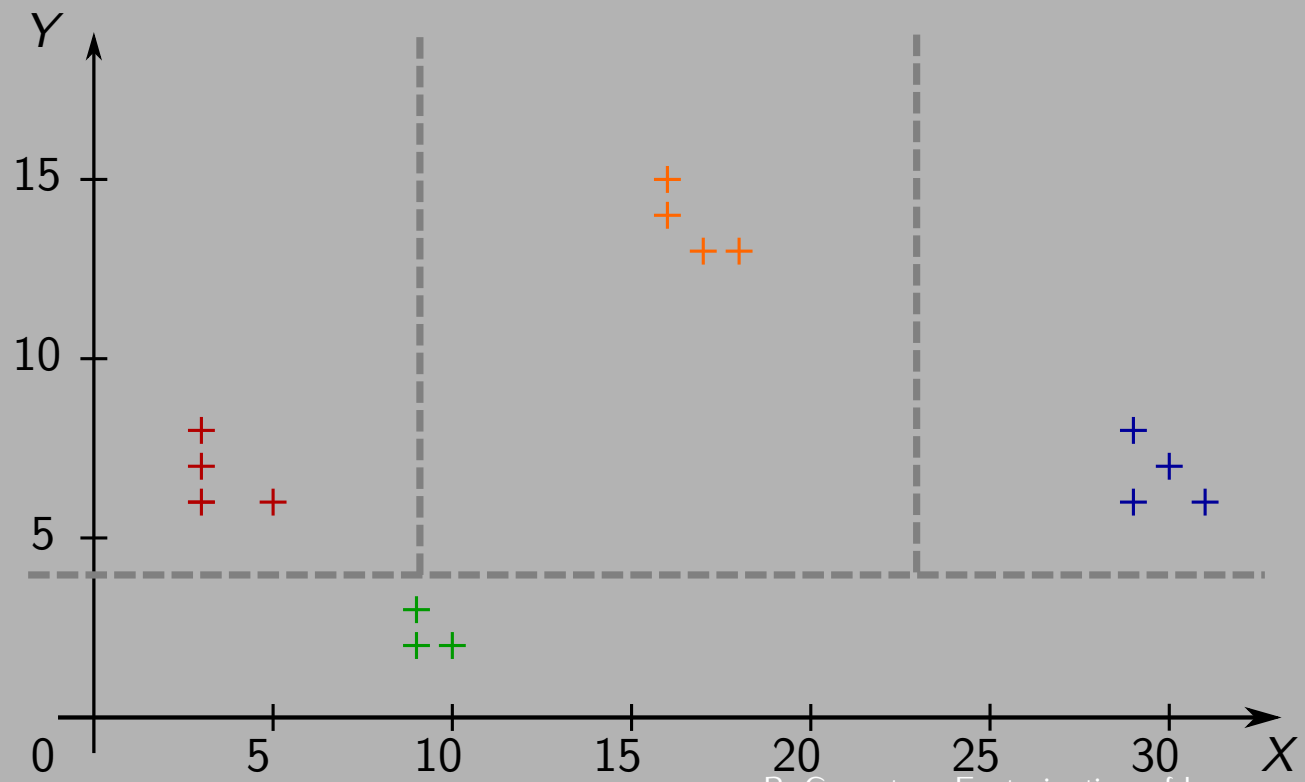

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$\Longrightarrow$ Linear factors of P: $(X - Y + 1, 1)$, $(X, 3)$, $(Y, 2)$

Bound on the valuation

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- Hajós' Lemma: if $\alpha_1 = \cdots = \alpha_\ell$, $\mathrm{val}(P) \leq \alpha_1 + (\ell - 1)$

Gap Theorem

Theorem

Let

$$P = \underbrace{\sum_{j=1}^{\ell} a_j X^{\alpha_j}(uX+v)^{\beta_j}}_{P_0} + \underbrace{\sum_{j=\ell+1}^{k} a_j X^{\alpha_j}(uX+v)^{\beta_j}}_{P_1}$$

with $uv \neq 0$, $\alpha_1 \leq \cdots \leq \alpha_k$. If

$$\alpha_{\ell+1} > \max_{1\leq j\leq \ell} \left(\alpha_j + \binom{\ell+1-j}{2}\right),$$

then $P \equiv 0$ iff both $P_0 \equiv 0$ and $P_1 \equiv 0$.

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The Wronskian

Definition

Let $f_1, \ldots, f_\ell \in \mathbb{K}[X]$. Then

$$\mathrm{wr}(f_1, \ldots, f_\ell) = \det \begin{bmatrix} f_1 & f_2 & \cdots & f_\ell \\ f_1' & f_2' & \cdots & f_\ell' \\ \vdots & \vdots & & \vdots \\ f_1^{(\ell-1)} & f_2^{(\ell-1)} & \cdots & f_\ell^{(\ell-1)} \end{bmatrix}.$$

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Proposition (Bôcher, 1900)

$\mathrm{wr}(f_1, \ldots, f_\ell) \neq 0 \iff$ the f_j's are linearly independent.

Wronskian & valuation

Lemma

$$\mathrm{val}(\mathrm{wr}(f_1,\ldots,f_\ell)) \geq \sum_{j=1}^{\ell} \mathrm{val}(f_j) - \binom{\ell}{2}$$

$$\begin{array}{c} \\ 0 \\ -1 \\ \vdots \\ -(k-1) \end{array} \begin{array}{c} \begin{array}{cccc} \mathrm{val}(f_1) & \mathrm{val}(f_2) & \ldots & \mathrm{val}(f_k) \end{array} \\ \begin{bmatrix} f_1 & f_2 & \ldots & f_k \\ f_1' & f_2' & \ldots & f_k' \\ \vdots & \vdots & & \vdots \\ f_1^{(k-1)} & f_2^{(k-1)} & \ldots & f_k^{(k-1)} \end{bmatrix} \end{array}$$

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$$\sum_{j=1}^{\ell} \alpha_j \geq \mathrm{val}(\mathrm{wr}(f_1, \ldots, f_\ell)) \geq \mathrm{val}(P) + \sum_{j=2}^{\ell} \alpha_j - \binom{\ell}{2}$$

Final algorithm

Find linear factors $(Y - uX - v)$ of $P(X, Y) = \sum_{j=1}^{k} a_j X^{\alpha_j} Y^{\beta_j}$

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   - 3.1 Compute $P = P_1 + \cdots + P_s$ where $P_t = \sum_j a_j X^{\alpha_j} Y^{\beta_j}$ with $\alpha_{\max} \leq \alpha_{\min} + \binom{k}{2}$ and $\beta_{\max} \leq \beta_{\min} + \binom{k}{2}$

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   - 3.3 Apply some dense factorization algorithm [Kaltofen'82, ..., Lecerf'07]

Code Sage

```
def SplitList(listOfCouples,comp,cst):
    listOfCouples.sort(key=lambda c:c[comp])
    gap=listOfCouples[0][comp];
    imin=0
    res=[]

    for i in range(1,len(listOfCouples)):
        if listOfCouples[i][comp] > cst*gap:
            res+=[listOfCouples[imin:i]];
            gap=listOfCouples[i][comp];
            imin=i;
        else:
            gap+=i-imin;
    res+=[listOfCouples[imin:]];

    return res

def Clusterize(listOfCouples,cst):

    firstSplit=SplitList(listOfCouples,0,cst)
    secondSplit=[]

    for l in firstSplit:
        secondSplit+=SplitList(l,1,cst)

    if len(secondSplit) == 1:
        return secondSplit
    else:
        res=[]
        for l in secondSplit:
            res+=Clusterize(l,cst)
        return res
```

```
def TrulyBivariateLinearFactors(poly):

    exponents=poly.exponents();
    clustersExp=Clusterize(exponents,1);

    pvar=poly.variables();
    x=pvar[0]
    y=pvar[1]

    clustersPoly=[]

    for cluster in clustersExp:
        xmin=min(cluster,key=lambda c:c[0])[0]
        ymin=min(cluster,key=lambda c:c[1])[1]
        p=sum([poly[e]*x^(e[0]-xmin)*y^(e[1]-
              ymin) for e in cluster])
        if p.nvariables()<2:
            return {}
        else:
            clustersPoly+=[p]

    d = dict(gcd(clustersPoly).factor())
    return dict((k,d[k]) for k in d.keys() \
                if k.nvariables()==2 and k.
                    degree()==1)
```

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- NP**-hard** under randomized reductions **otherwise**.

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Extend to low-degree factors of multivariate polynomials

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Thank you!