Factoring lacunary polynomials: the easy way

Bruno Grenet
LIX — École Polytechnique

Joint work with

Arkadev Chattopadhyay
TIFR, Mumbai

Pascal Koiran
ÉNS Lyon

Natacha Portier
ÉNS Lyon

Yann Strozecki
U. Versailles

Rencontres du GT CoA — Paris

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Classical factorization algorithms

Factorization of a polynomial P

Find $F_1, \ldots, F_t$, irreducible, s.t. $P = F_1 \times \cdots \times F_t$.

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Complexity

Polynomial in the **degree** of the polynomials

$\downarrow$

$\mathbb{Q}(\alpha)[X_1, \ldots, X_n]$

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Lacunary polynomials

$$X^{102}Y^{101} + X^{101}Y^{102} - X^{101}Y^{101} - X - Y + 1$$

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$$P(X_1, \ldots, X_n) = \sum_{j=1}^{k} a_j X_1^{\alpha_{1j}} \cdots X_n^{\alpha_{nj}}$$

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$$P(X_1, \ldots, X_n) = \sum_{j=1}^{k} a_j X_1^{\alpha_{1j}} \cdots X_n^{\alpha_{nj}}$$

- Lacunary representation: list of nonzero monomials
- $\text{size}(P) \simeq \sum_j \text{size}(a_j) + \log(\alpha_{1j}) + \cdots + \log(\alpha_{nj})$

Gap Theorem

[Cucker-Koiran-Smale'98]

Let

$$P(X) = \underbrace{\sum_{j=1}^{\ell} a_j X^{\alpha_j}}_{Q} + \underbrace{\sum_{j=\ell+1}^{k} a_j X^{\alpha_j}}_{R} \in \mathbb{Z}[X]$$

with $\alpha_1 \leqslant \alpha_2 \leqslant \cdots \leqslant \alpha_k$. Suppose that

$$\alpha_{\ell+1} - \alpha_\ell > 1 + \log\left(\max_{j \leqslant \ell} |a_j|\right),$$

then for all $x \in \mathbb{Z}$, $|x| \geqslant 2$, $P(x) = 0 \implies Q(x) = R(x) = 0$.

Integral roots of integral polynomials

Gap Theorem [Cucker-Koiran-Smale'98]

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$$-9 + X^2 + 6X^7 + 2X^8 = -9 + X^2 + X^7(6 + 2X)$$

Factorization of lacunary polynomials

Theorems

There exist deterministic polynomial-time algorithms computing

- **linear** factors of **univariate** polynomials over $\mathbb{Z}$;

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- **low-degree** factors of **multivariate** polynomials over $\mathbb{Q}(\alpha)$.

  [Kaltofen-Koiran'05 & '06]

Linear factors of bivariate polynomials

Observation

$(Y - uX - v)$ divides $P(X, Y) \iff P(X, uX + v) \equiv 0$

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$$P = \underbrace{\sum_{j=1}^{\ell} a_j X^{\alpha_j} (uX + v)^{\beta_j}}_{Q} + \underbrace{\sum_{j=\ell+1}^{k} a_j X^{\alpha_j} (uX + v)^{\beta_j}}_{R}$$

with $v \neq 0$, $\alpha_1 \leqslant \cdots \leqslant \alpha_k$. If ℓ is the smallest index s.t.

$$\alpha_{\ell+1} > \alpha_1 + \binom{\ell}{2},$$

then $P \equiv 0$ iff both $Q \equiv 0$ and $R \equiv 0$.

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$\mathbb{K}$: any field of characteristic 0

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Let $P = \sum_{j=1}^{\ell} a_j X^{\alpha_j}(uX+\nu)^{\beta_j} \not\equiv 0$, with $\nu \neq 0$ and $\alpha_1 \leqslant \cdots \leqslant \alpha_\ell$.

Then, if the family $(X^{\alpha_j}(uX+\nu)^{\beta_j})_j$ is linearly independent,

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- Hajós' Lemma: if $\alpha_1 = \cdots = \alpha_\ell$, $\mathrm{val}(P) \leqslant \alpha_1 + (\ell - 1)$

Definition

Let $f_1, \ldots, f_\ell \in \mathbb{K}[X]$. Then

$$\mathrm{wr}(f_1, \ldots, f_\ell) = \det \begin{bmatrix} f_1 & f_2 & \cdots & f_\ell \\ f_1' & f_2' & \cdots & f_\ell' \\ \vdots & \vdots & & \vdots \\ f_1^{(\ell-1)} & f_2^{(\ell-1)} & \cdots & f_\ell^{(\ell-1)} \end{bmatrix}.$$

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Proposition [Bôcher, 1900]

$\mathrm{wr}(f_1, \ldots, f_\ell) \neq 0 \iff$ the f_j's are linearly independent.

Lemma

$$\mathrm{val}(\mathrm{wr}(f_1,\dots,f_\ell)) \geqslant \sum_{j=1}^{\ell} \mathrm{val}(f_j) - \binom{\ell}{2}$$

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$$\sum_{j=1}^{\ell} \alpha_j \geqslant \operatorname{val}(\operatorname{wr}(f_1,\dots,f_\ell)) \geqslant \operatorname{val}(P) + \sum_{j=2}^{\ell} \alpha_j - \binom{\ell}{2}$$

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- Lemmas: bounds attained, but not simultaneously
- $(1+X)^{2\ell-3} - 1 - \sum_{j=3}^{\ell} \frac{2\ell-3}{2j-5}\binom{\ell+j-5}{2j-6} X^{2j-5}(1+X)^{\ell-1-j} = X^{2\ell-3}$

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- Independent from u and v
- X does not play a special role

Example

$$\begin{aligned} P = {} & X^{31}Y^6 - 2X^{30}Y^7 + X^{29}Y^8 - X^{29}Y^6 + X^{18}Y^{13} \\ & - X^{16}Y^{15} + X^{17}Y^{13} + X^{16}Y^{14} + X^{10}Y^2 - X^9Y^3 \\ & + X^9Y^2 - X^5Y^6 + X^3Y^8 - 2X^3Y^7 + X^3Y^6 \end{aligned}$$

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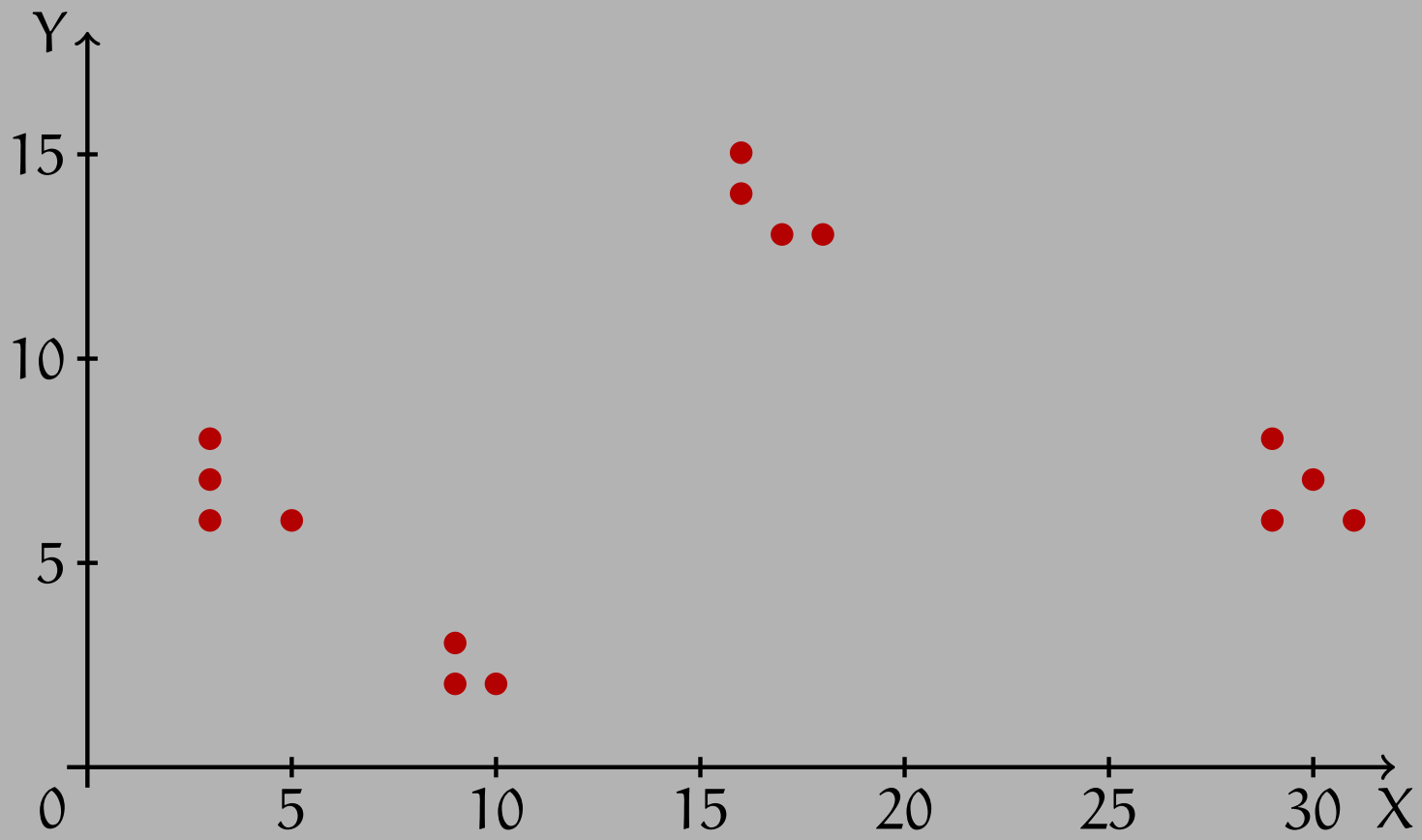

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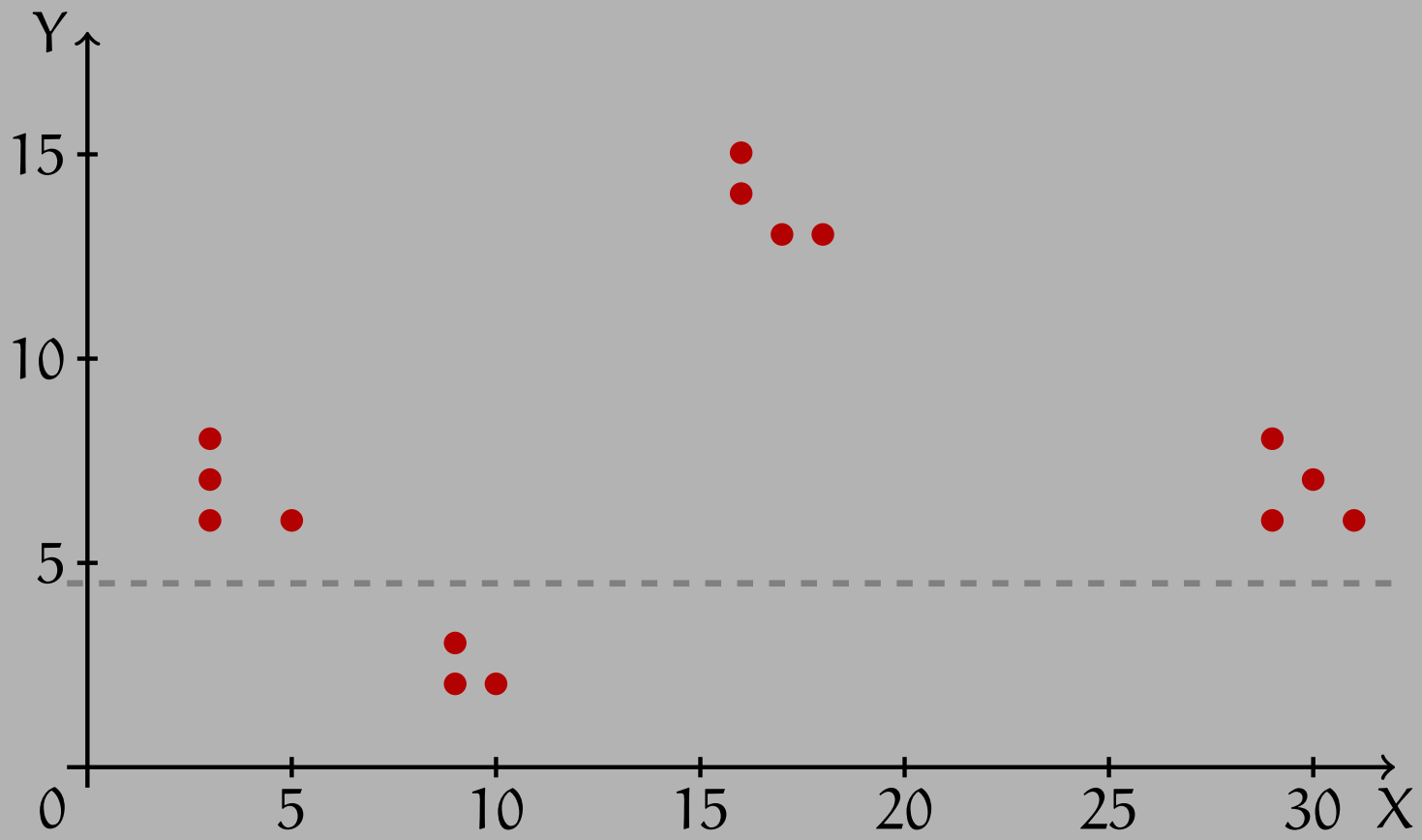

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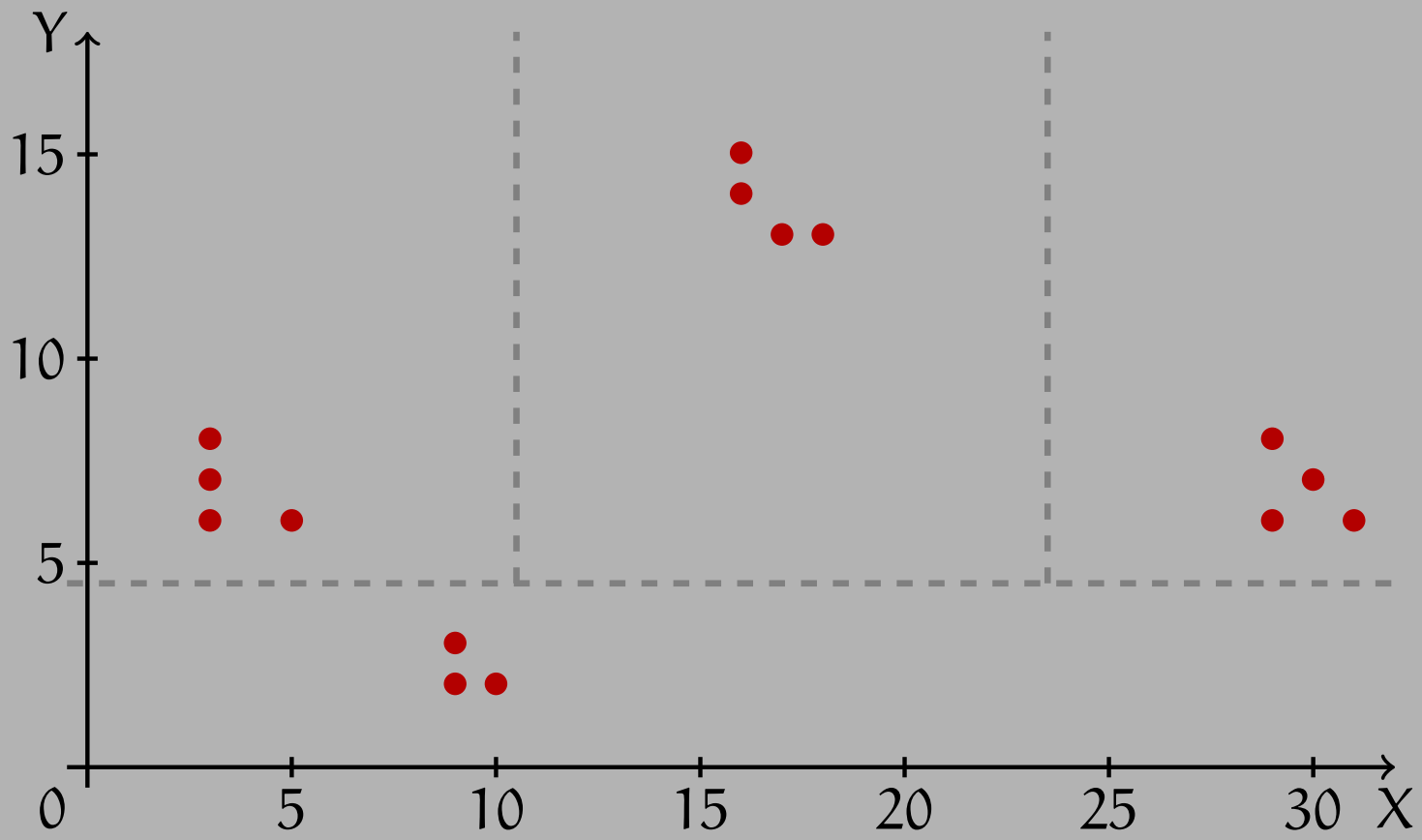

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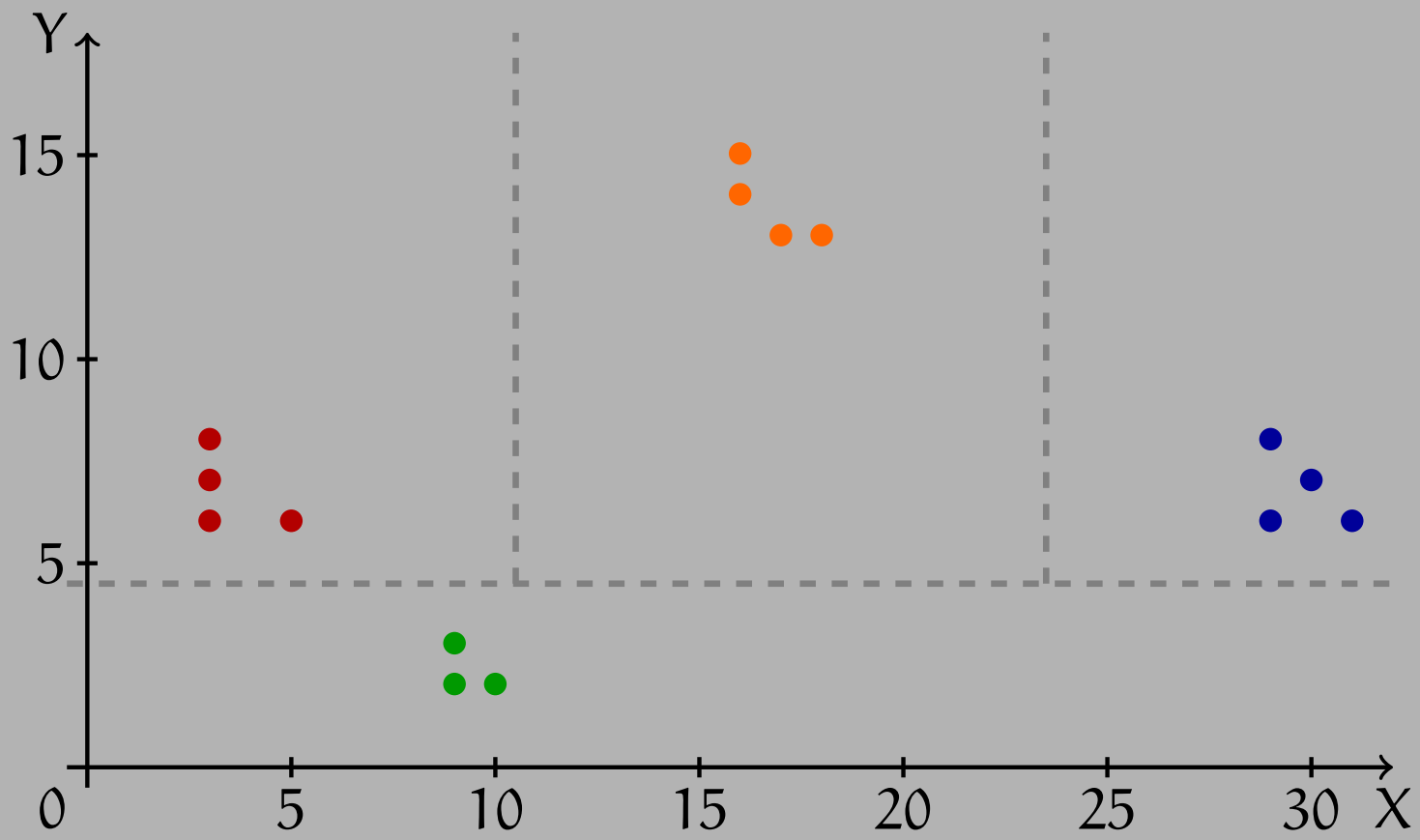

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$\Longrightarrow$ linear factors of P: $(X - Y + 1, 1)$, $(X, 3)$, $(Y, 2)$

Complete algorithm

Find linear factors of $P(X,Y) = \sum_{j=1}^{k} a_j X^{\alpha_j} Y^{\beta_j}$

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Common factors of
$P_t = \sum_{j=j_t}^{j_t+\ell_t-1} a_j X^{\alpha_j} Y^{\beta_j}$
$(\deg(P_t) \leqslant \mathcal{O}(\ell_t^2))$

Low-degree factorization
[Kaltofen'82, ..., Lecerf'07]

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monomials | binomials | trinomials

$(X, \min_j \alpha_j)$
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$$\frac{(X-a)}{\text{Factors of } \sum_j a_j X^{\alpha_j}} \qquad \frac{(Y-uX)}{\text{Roots of } u \mapsto \sum_j a_j u^{\beta_j}}$$

Univariate lacunary factorization
[H. Lenstra'99]

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Complete algorithm

Let $P = \sum_{j=1}^{k} a_j X^{\alpha_j} Y^{\beta_j} \in \mathbb{Q}(\alpha)[X, Y]$ be given in lacunary representation. There exists a **deterministic polynomial-time** algorithm to compute its linear factors, with multiplicities.

monomials | binomials | trinomials

$(X, \min_j \alpha_j)$
$(Y, \min_j \beta_j)$

$$\frac{(X - a)}{(Y - uX)}$$
Factors of $\sum_j a_j X^{\alpha_j}$
Roots of $u \mapsto \sum_j a_j u^{\beta_j}$

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$\Updownarrow$

Roots of univariate lacunary polynomials

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NP-complete under BPP reductions

[Kipnis-Shamir'99,Bi-Cheng-Rojas'13]

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Thank you!