Lecture 9. RSA public-key encryption and signatures

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https://membres-ljk.imag.fr/Bruno.Grenet/IntroCrypto.html

Introduction to cryptology
Université Grenoble Alpes – IM²AG
M1 INFO, MOSIG & AM

Programming Techniques — S.L. Graham, R.L. Rivest* Editors

A Method for Obtaining Digital Signatures and Public-Key Cryptosystems

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An encryption method is presented with the novel property that publicly revealing an encryption key does not thereby reveal the corresponding decryption key. This has two important consequences:
(1) Couriers or other secure means are not needed to transmit keys, since a message can be enciphered using an encryption key publicly revealed by the intended recipient. Only he can decipher the message, since only he knows the corresponding decryption key.
(2) A message can be "signed" using a privately held decryption key. Anyone can verify this signature using the corresponding publicly revealed encryption key. Signatures cannot be forged, and a signer cannot later

I. Introduction

The era of "electronic mail" [10] may soon be upon us; we must ensure that two important properties of the current "paper mail" system are preserved: (a) messages are *private*, and (b) messages can be *signed*. We demonstrate in this paper how to build these capabilities into an electronic mail system.

At the heart of our proposal is a new encryption method. This method provides an implementation of a "public-key cryptosystem", an elegant concept invented by Diffie and Hellman [1]. Their article motivated our research, since they presented the concept but not any practical implementation of such a system. Readers familiar with [1] may wish to skip directly to Section V for a description of our method.

II. Public-Key Cryptosystems

In a "public-key cryptosystem" each user places in a public file an encryption procedure E. That is, the public file is a directory giving the encryption procedure of each user. The user keeps secret the details of his corresponding decryption procedure D. These procedures have the following four properties:

(a) Deciphering the enciphered form of a message M

A Method for Obtaining Digital Signatures and Public-Key Cryptosystems

R. Rivest, A. Shamir & L. Adleman (1978)

- Basics of RSA encryption scheme
- Signature using the encryption scheme in *reverse mode*

Pros

- First proposal of a public-key encryption scheme
- Use of computational difficulty as security

Cons

- As presented, the encryption scheme is completely unsafe!
- The signature is not a good idea!

Remark

- Already known to GHCQ (UK) in 1973, declassified only in 1997 Clifford Cocks

Contents of this lecture

1. The maths of RSA: the trapdoor permutation
   - $\mathbb{Z}/N\mathbb{Z}$ where $N = p \times q$
   - Designing a *trapdoor permutation*

   $\rightarrow \pm$ the content of the original paper

2. RSA encryption scheme
   - What should be added to obtain a proper encryption scheme?

3. RSA signatures
   - How to obtain a proper signature scheme?

Contents

Representation and ring operations

General context

$N = p \times q$ where p, q are prime numbers; computations *modulo* N

Representation and modular operations

- $\mathbb{Z}/N\mathbb{Z} = \{0, 1, \ldots, N-1\}$ with *modular* addition, subtraction and multiplication:
  1. Perform the operation in the integers
  2. Reduce the result *modulo* N
- Complexity: $O(\log^2 N)$ or $O(\log N \log \log N)$

Example: $\mathbb{Z}/35\mathbb{Z}$

$21 + 18 = 39 = 4$ $\qquad -12 = 23$

$5 \times 10 = 50 = 15$

Detour by a fundamental algorithm

The extended Euclidean Algorithm (xGCD)

Input: $a, b \in \mathbb{Z}, a > b \geq 0$

Output: $g, u, v \in \mathbb{Z}$ s.t. $g = au + bv$ and $g = \gcd(a, b)$

1. $(r_0, u_0, v_0) \leftarrow (a, 1, 0)$
2. $(r_1, u_1, v_1) \leftarrow (b, 0, 1)$
3. $i \leftarrow 2$
4. while $r_{i-1} \neq 0$:
5. $\quad (q_i, r_i) \leftarrow \textsc{QuoRem}(r_{i-2}, r_{i-1})$
6. $\quad (u_i, v_i) \leftarrow (u_{i-2} - q_i u_{i-1}, v_{i-2} - q_i v_{i-1})$
7. $\quad i \leftarrow i + 1$
8. return $(r_{i-2}, u_{i-2}, v_{i-2})$

21 15

i	r_i		u_i	v_i	q_i
0	21	=	1 × 21 +	0 × 15	✓
1	15	=	0 × 21 +	1 × 15	✓
2	6	=	1 × 21 +	−1 × 15	1
3	3	=	−2 × 21 +	3 × 15	2
4	0	=	5 × 21 +	−7 × 15	2

$\hookrightarrow 3 = -2 \times 21 + 3 \times 15$

$3 = \gcd(21, 15)$

Detour by a fundamental algorithm

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7. $\quad i \leftarrow i + 1$
8. return $(r_{i-2}, u_{i-2}, v_{i-2})$

Correction

- For all i, $\gcd(a, b) = \gcd(r_i, r_{i+1})$
- For all i, $r_i = a \cdot u_i + b \cdot v_i$

Consequence

$\gcd(a, b) = 1 \iff$
there exists $u, v \in \mathbb{Z}$ s.t. $1 = a \cdot u + b \cdot v$

Complexity

The bit complexity of the extended Euclidean Algorithm is $O(\log(a) \log(b))$

Inversion and division in $\mathbb{Z}/N\mathbb{Z}$

Definition

$a \in \mathbb{Z}/N\mathbb{Z}$ is invertible if there exists $b \in \mathbb{Z}/N\mathbb{Z}$ s.t. $a \times b = 1$ *modular* $\times$

- one can *divide by a in* $\mathbb{Z}/N\mathbb{Z}$

Theorem

$a \in \mathbb{Z}/N\mathbb{Z}$ is invertible modulo N $\iff$ $\gcd(a, N) = 1$ $\iff$ $\exists u, v$ s.t. $1 = au + Nv$

(previous slide)

Proof

$\exists u, v,\ 1 = au + Nv \iff \exists u,\ 1 = au \bmod N \iff u$ is the inverse of $a \bmod N$

Algorithms

Inverse: Use the extended Euclidean Algorithm
Running time: $O(\log^2 N)$ or $O(\log N \log^2 \log N)$

Division: Use multiplication and inverse
Same running time

Invertible elements of $\mathbb{Z}/N\mathbb{Z}$

Definition

- The multiplicative group $\mathbb{Z}/N\mathbb{Z}^\times$ is the set of invertible elements of $\mathbb{Z}/N\mathbb{Z}$
- Its number of elements is denoted $\varphi(N)$

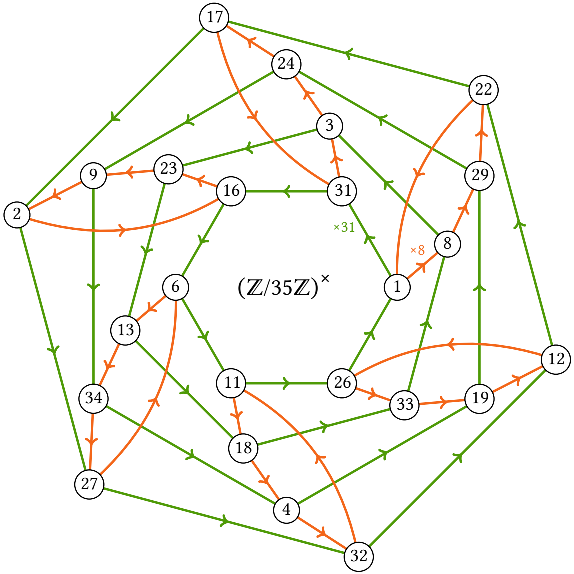
Proposition

If $N = p \times q$ with primes $p \neq q$, $\varphi(N) = (p-1)(q-1)$

Proof

- a invertible $\iff \gcd(a,N) = 1 = \gcd(a, pq) \iff$ neither p nor q divides a
- Multiples of p: $0, p, 2p, 3p, \dots, (q-1)p$
  q: $0, q, 2q, 3q, \dots, (p-1)q$ } total: $q + p - 1$
- $\varphi(N) = pq - (p+q-1) = (p-1)(q-1)$

The multiplicative group is **not** cyclic!



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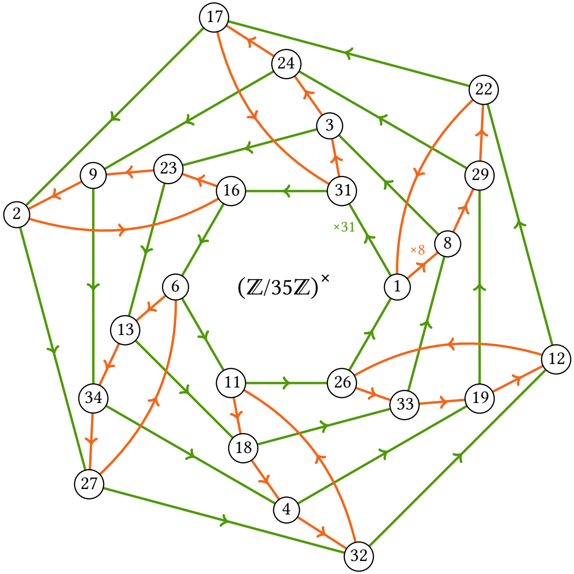
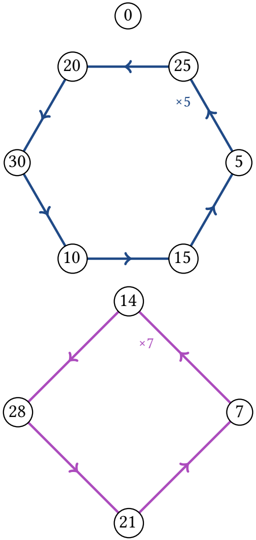




Non-invertible elements

The "RSA theorem"

Theorem

Let $N = p \times q$ with primes $p \neq q$. Then for all $a \in \mathbb{Z}/N\mathbb{Z}$, $a^{1+\varphi(N)} = a$.

Proof

(1) Fermat's little theorem: $\forall a \in \{1, \dots, p-1\}$, $a^{p-1} \bmod p = 1$

$$\{ax \bmod p : 1 \le x \le p-1\} = \{y : 1 \le y \le p-1\}$$

$$\Rightarrow \prod_{x=1}^{p-1} (ax) = \prod_{y=1}^{p-1} y \Rightarrow a^{p-1}(p-1)! = (p-1)! \Rightarrow a^{p-1} = 1 \quad (\text{all mod } p)$$

(2) $a^{1+\varphi(N)} = a^{1+(p-1)(q-1)}$:

$+ \; a^{p-1} = 1 \bmod p \Rightarrow a^{1+\varphi(N)} \bmod p = a \Rightarrow p$ divides $a^{1+\varphi(N)} - a$

$+ \; a^{q-1} = 1 \bmod q \Rightarrow a^{1+\varphi(N)} \bmod q = a \Rightarrow q$ divides $a^{1+\varphi(N)} - a$

$\Rightarrow pq$ divides $a^{1+\varphi(N)} - a \Rightarrow a^{1+\varphi(N)} \bmod N = a$

The RSA trapdoor permutation

The original (unsafe!) RSA encryption scheme

Definition as an encryption scheme

Public key: (N, e) where $N = p \times q$ with primes $p \neq q$ and $\gcd(e, \varphi(N)) = 1$

Private key: (N, d) where $d \times e \bmod \varphi(N) = 1$

Encryption: Given $m \in \mathbb{Z}/N\mathbb{Z}$, compute $c = m^e \bmod N$

Decryption: Given $c \in \mathbb{Z}/N\mathbb{Z}$, compute $m = c^d \bmod N$

Correction

$c^d = m^{ed}$. $ed \bmod \varphi(N) = 1 \Leftrightarrow \exists\, k$ s.t. $ed = 1 + k \cdot \varphi(N)$

$\hookrightarrow c^d = m^{1+k\varphi(N)} = m^{1+\varphi(N)} \cdot m^{(k-1)\varphi(N)} = m^{1+(k-1)\varphi(N)} = \cdots = m^{1+\varphi(N)} = m$.

The algorithms and complexities

Key generation

1. Generate two random primes $p \neq q$ — $O(\log^4 N)$
   - Sample random (odd) integers — $O(\log N)$ samples
   - Test their primality — $O(\log^3 N)$
2. Compute $N = p \times q$ and $\varphi(N) = (p-1) \times (q-1)$ — $O(\log^2 N)$
3. Generate e, d such that $e \times d \bmod \varphi(N) = 1$ — $O(\log^2 N)$
   - Sample random integers e — $1 + O(1/\sqrt{N})$ samples
   - Apply xGCD$(e, \varphi(N))$ to test invertibility and get d — $O(\log^2 N)$

Encryption and decryption

- Modular exponentiation in $\mathbb{Z}/N\mathbb{Z}$ — $O(\log^3 N)$
  - *Binary powering*, using $a^n = \begin{cases} a^{\lfloor n/2 \rfloor} \cdot a^{\lfloor n/2 \rfloor} & \text{for even } n \\ a \cdot a^{\lfloor n/2 \rfloor} \cdot a^{\lfloor n/2 \rfloor} & \text{for odd } n \end{cases}$

Remark

- All complexities can be slightly improved

Attacks on the trapdoor

Possible goals

Key recovery: Given (N, e), compute d s.t. $d \times e \bmod \varphi(N) = 1$

Plaintext recovery: Given (N, e) and c, compute m s.t. $m^e \bmod N = c$

Computational problems

Modular e-th root: Given N, c, e, compute m s.t. $m^e \bmod N = c$

Computation of φ: Given $N = p \times q$ (for unknown p, q), compute $\varphi(N) = (p-1)(q-1)$

Factorization: Given $N = p \times q$, compute p and q

Reductions between problems

- Plaintext recovery $\iff$ modular e-th root
- Computation of φ $\implies$ Key recovery $\implies$ plaintext recovery
- Computation of φ $\iff$ Factorization of N: ($\Leftarrow$) Compute p and q and return $(p-1)(q-1)$

($\Rightarrow$) $\varphi(N) = (p-1)(q-1) = N - (p+q) + 1$
↳ We know $N = pq$ and $S = p+q$

Consider $(X-p)(X-q) = X^2 - (p+q)X + pq = X^2 - SX + N$
↳ Compute the roots of the poly. to get p and q.

Integer factorization

Complexity of integer factorization

- Brute force algorithm: $O(\sqrt{N}) = O(2^{\frac{\log N}{2}})$
- ...
- General Number Field Sieve: $2^{O(\log^{\frac{1}{3}} N \log^{\frac{2}{3}} \log N)}$ *Lenstra, Lenstra (1993) and others...*
- Quantum algorithms:
  - $O(\log^3 N) = O(2^{3 \log \log N})$ *Shor (1994)*
  - $O(\log^{2.5} N) = O(2^{2.5 \log \log N})$ *Regev (2025)*

(Remark: no known NP-hardness result $\rightarrow$ could be polynomial in $\log N$)

Current record: 829-bit (250-digit) integer factorization

- Boudot, Gaudry, Guillevic, Heninger, Thomé, Zimmermann (Feb. 2020)
- Software: CADO-NFS
- Hardware: (mainly) academic clusters
- Approx. 2,700 core-years in a few months

Contents

The original RSA scheme is unsafe!

Deterministic encryption

- Two ciphertexts are equal iff the corresponding messages are equal
- The scheme cannot be IND-CPA/CCA secure

Examples of other difficulties

Small exponent: If e and m are small: $m^e \bmod N = m^e$ in $\mathbb{Z} \to \sqrt[e]{c}$ in $\mathbb{Z}$

Related messages: Given the ciphertexts of m and $m + \delta$ with small $\delta \to m$

Multiple receivers: Given the ciphertexts of m with several distinct keys $\to m$

> The original RSA encryption scheme is severely flawed and **should never be used!**

- Solution: use (random) padding

The padded RSA encryption scheme: overview

Construction

Parameters: n: number of bits of N; ℓ: length of the messages

$\mathrm{Gen}_n()$:
1. $p, q \twoheadleftarrow$ two random primes s.t. $p \times q$ has bit-length n
2. $N \leftarrow p \times q$, $\varphi(N) \leftarrow (p-1) \times (q-1)$
3. $e \twoheadleftarrow$ random integer invertible modulo $\varphi(N)$, $d \leftarrow e^{-1} \bmod \varphi(N)$
4. return $pk = (N, e)$, $sk = (N, d)$

$\mathrm{Enc}_{pk}(m)$: $m \in \{0,1\}^{\ell}$
1. $r \twoheadleftarrow \{0,1\}^{n-\ell}$
2. while $\hat{m} = r \| m \notin \mathbb{Z}/N\mathbb{Z}$: $r \twoheadleftarrow \{0,1\}^{n-\ell}$
3. return $c = \hat{m}^e \bmod N$

$\mathrm{Dec}_{sk}(c)$:
1. $\hat{m} \leftarrow c^d \bmod N$
2. Return $m = \hat{m}_{[n-\ell..n[}$

Correction

- As for the original RSA

Security of padded RSA

The security depends on $n - \ell$ — number of padding bits

Small values of $n - \ell$

- $2^{n-\ell}$ possible paddings
- Sufficient to break $2^{n-\ell}$ original RSA instances

$\rightarrow$ Not secure!

Very large value of $n - \ell$: $\ell = 1$

- If computing e-th root in $\mathbb{Z}/N\mathbb{Z}$ is hard, IND-CPA secure encryption scheme
- Very inefficient secure encryption scheme, one bit at a time
- Slightly better if used as a KEM — *still useless!*

Medium values of $n - \ell$

- Open problem!

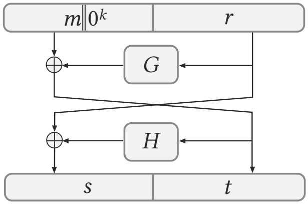
Padded RSA in practice

RSA PKCS1

- Standardized by RSA laboratories
- Padding: $m \rightarrow \texttt{0x00} \| \texttt{0x02} \| r \| \texttt{0x00} \| m$ where r is random
- Attack using failure of the unpadding procedure *Bleichenbacher (1998)*
  - Used against SSL 3.0
  - Workaround: in case of failure, return a random value
  - Prevents IND-CCA security

RSA Optimal Asymmetric Encryption Padding (OAEP) *Bellare, Rogaway (1994)*

- Padding: $m \rightarrow s \| t$ where
  - G, H: hash functions
  - r: random bits
- Standardized as PKCS1 v2
- IND-CCA secure under two assumptions
  - RSA trapdoor is *one-way*
  - G and H are random oracles



Source : J. Katz, Y. Lindell. Introduction to modern cryptography. 3rd ed, CRC Press, 2021. (modif.)

Contents

Original (broken…) version

Signatures are not "encryption with the private key"

Construction

$\mathsf{Gen}_n()$:
1. $p, q \twoheadleftarrow$ two random primes s.t. $p \times q$ has bit-length n
2. $N \leftarrow p \times q$, $\varphi(N) \leftarrow (p-1) \times (q-1)$
3. $e \twoheadleftarrow$ random integer invertible modulo $\varphi(N)$, $d \leftarrow e^{-1} \bmod \varphi(N)$
4. return $pk = (N, e)$, $sk = (N, d)$

$\mathsf{Sign}_{sk}(m)$:
1. return $m^d \bmod N$ $\qquad m \in \mathbb{Z}/N\mathbb{Z}$

$\mathsf{Vrfy}_{pk}(m, \sigma)$:
1. test whether $m = \sigma^e \bmod N$

Correction

- As for the original RSA encryption scheme

Attacks

existential forgeries

1. The adversary chooses σ and computes $m = \sigma^e \bmod N$
2. The adversary sees (m_1, σ_1) and (m_2, σ_2) and computes $m = m_1 \cdot m_2$ and $\sigma = \sigma_1 \cdot \sigma_2$

RSA FDH (*Full Domain Hash*)

Construction

$\text{Gen}_n()$:
1. Compute $pk = (N, e)$, $sk = (N, d)$ as previously
2. Choose a hash function $H : \{0,1\}^* \to \mathbb{Z}/N\mathbb{Z}$

$\text{Sign}_{sk}(m)$:
1. return $H(m)^d \bmod N$ — $m \in \{0,1\}^*$

$\text{Vrfy}_{pk}(m, \sigma)$:
1. test whether $H(m) = \sigma^e \bmod N$

What should H satisfy to avoid attacks?

1. $\sigma \to h = \sigma^e \to H(m) = h$ — *first preimage resistance*
2. $m_1, m_2 \to H(m) = H(m_1) \cdot H(m_2) \bmod N$ — *"non-multiplicative"*
3. If $H(m_1) = H(m_2)$, $\sigma_1 = \sigma_2$ — *collision resistance*
4. The image of H should be the full $\mathbb{Z}/N\mathbb{Z}$ — *full domain*

Bad and good news

- We *do not know* how to build a satisfying H — *no security proof*
- Security proof if RSA trapdoor is *one-way* and H is a random oracle

Proof sketch of RSA FDH

(Informal) Theorem

If e-th roots in $\mathbb{Z}/N\mathbb{Z}$ are hard to compute and H is random, RSA FDH is secure

Proof sketch : Prove that if $\mathcal{A}$ attacks RSA-FDH, then $\mathcal{A}$ is able to compute eth roots in $\mathbb{Z}/N\mathbb{Z}$

- What is an adversary $\mathcal{A}$?
  - → Requests m_i, gets $\sigma_i = H(m_i)^d \bmod N$
  - → Hyp.: for each m_i, $\mathcal{A}$ also requests $h_i = H(m_i)$
- Build $\mathcal{A}'$ that given c, computes m s.t. $c = m^e \bmod N$, by playing the EUF-CMA game
  - → Let t be the total number of requests to H made by $\mathcal{A}$.

  1. $\mathcal{A}'$ guesses a value $j \in \{1, \ldots, t\}$
  2. $\mathcal{A}'$ simulates $\mathcal{A}$:
     - for a request $H(m_i)$ $(i \neq j)$, $\mathcal{A}'$ samples σ_i and returns $h_i = \sigma_i^e \bmod N$
     - for $H(m_j)$, $\mathcal{A}'$ returns c. [if $\mathcal{A}$ returns (m_j, σ) then $\sigma = H(m_j)^d = c^d$]

Proof sketch of RSA FDH

(Informal) Theorem

If e-th roots in $\mathbb{Z}/N\mathbb{Z}$ are hard to compute and H is random, RSA FDH is secure

Proof sketch

<u>Analysis</u>:

1. If $\mathcal{A}'$ makes a correct guess j, then $\mathcal{A}$ returns a pair (m_j, σ) where $H(m_j)^d = \sigma$. Since $\mathcal{A}'$ decided to set $H(m_j) = c$, $\mathcal{A}'$ knows σ s.t. $c^d = \sigma$ $(\Leftrightarrow c = \sigma^e)$

   ↳ $\mathcal{A}'$ has computed the eth root of c.

2. If $\mathcal{A}'$ makes an incorrect guess, it fails.

$$\Pr[\text{Success}(\mathcal{A}')] = \frac{1}{t} \Pr[\text{Success}(\mathcal{A})].$$

Conclusion

RSA is a *one-way* trapdoor function

- One direction is easy to compute: $(m, e) \to m^e \bmod N$
- The other direction is (hopefully!) hard to compute: $(c, e) \to \sqrt[e]{c} \bmod N$
- But there is a trapdoor: given $d = e^{-1} \bmod \varphi(N)$, easy to compute $m = c^d \bmod N$

Use of RSA trapdoor function

- No direct use!
- Public-key encryption scheme $\to$ RSA OAEP
- Digital signatures $\to$ RSA FDH

Security

- No formal proof that RSA is *one-way* *assumption*
- Related but not equivalent to the difficulty of integer factorization
- Typical key sizes: N with $\geq$ 2048 bits
- Many other pitfalls: implementation, randomness quality, dependent keys, …