Comparisons of mean hitting times for tabu random walks on finite graphs

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Abstract

A tabu random walk on a graph is a partially self-avoiding random walk to nearest neighbors with finite memory. The walker is endowed with a finite word, called tabu list, whose letters are vertices he has already visited. The policy to insert or remove occurrences of vertices in the tabu list is called update rule. First, we enunciate a necessary and sufficient condition on the update rule that ensures, on all simple, finite and connected graphs, the finiteness of mean hitting times of each vertex. Then, we describe, on large classes of graph, the update rules having the smallest mean hitting times. Finally, we compare mean hitting times for three particular collections update rules.

1 Introduction

A tabu random walk on a graph is a partially self-avoiding random walk to nearest neighbors with finite memory. The walker is endowed with a finite word, called tabu list, whose letters are vertices he has already visited, sorted in chronological order. The tabu list evolves with time and the policy to insert or remove occurrences of vertices in the tabu list is the only parameter of the tabu random walk, called update rule. The walker never revisits a vertex with at least one occurrence in the tabu list, unless all neighbors of the current vertex have at least one occurrence in the tabu list.

We postpone the precise definition of tabu random walks and the statements of our results to Section 2. Here follows an informal description. The successive ordered pairs formed by the vertex currently visited by the walker and its current tabu list is a Markov chain. At each step, the next vertex visited follows the uniform law on the set formed by the neighbors of the current vertex without occurrence in the tabu list. Otherwise, the next vertex visited follows the uniform law on the neighborhood of the current vertex. Afterwards, the next tabu list is formed by concatenating the current vertex and the current tabu list, then by possibly removing a letter of the concatenated word. The position of the letter removed is a random variable whose law, conditionally on the current vertex and the current tabu list, is given by the update rule. In particular, the length of the tabu list is bounded by an integer that depends on the update rule, called length of the update rule.

As an example, we will describe a collection of update rules denoted by $(FIFO_m)_{m\geqslant 0}$. For every nonnegative integer m, the update rule $FIFO_m$ the following update rule of length m: the new tabu list is the concatenation of the current vertex and the current tabu list, except if the current tabu list has length m. In the latter case, moreover, the last letter of the concatenated word is removed in order to keep the length of the new tabu list equal to m. The update rule $FIFO_0$ is the only update rule of zero length. It yields a simple random walk, because the tabu list is always the empty word. The update rule $FIFO_1$ yields a non-backtracking random walk: the next vertex is uniform on the neighborhood of the current vertex, minus the previous vertex if the current vertex has degree at least two. Thus, the walker does not backtrack, unless it visits a vertex of degree one.

We focus on mean hitting times associated to tabu random walks. The hitting time of each vertex is the random number of steps needed by the walker to reach it. More specifically, we examine mean hitting times for a walker that starts almost surely at a given vertex with empty tabu list.

As an application, in a probabilistic routing protocol, the hitting time is the number of steps needed by a routed message before being delivered to its destination. Consequently, it might be relevant to compare mean hitting times associated for distinct routing protocols. Actually, tabu random walks have been introduced in the article [1] in order to improve the routing protocol by simple random walk by reducing mean hitting times. Being non-deterministic, the latter protocol can be used to deliver messages in a environment that faces failures or intruders [4]. Besides, it requires few memories and computing time, therefore is well suited for networks with limited resources, for example wireless sensors networks. Adding a small tabu list to the message, namely considering a tabu random walk with tabu lists of small length, generate a non-deterministic routing protocol that still needs few memories and computing time.

To begin with, we give a necessary and sufficient condition on the update rule that ensures the finiteness of associated mean hitting times on all graphs. This guarantees that the routing protocol by tabu random walk delivers every messages in almost surely finite time, provided that the update rule satisfies the latter condition. This assertion is stated in Theorem 5.

Next, we compare hitting times for distinct update rules. For each positive integer m, we introduce the class of m-free graphs, which includes all $(m+1)$-regular graphs. Theorem 7 claims that on every m-free graph, the $FIFO_m$ update rule has the smallest mean hitting times among all update rules of length less than or equal to m. We infer that, on every graph, the mean hitting times for the non-backtracking random walk are always smaller than those associated to each update rule of length zero or one.

Then, we study hitting times on particular graphs. First, Proposition 10 expresses the mean hitting times on the cliques for each update rule. As a consequence, consider the update rules such that, if the length of the current tabu list is strictly less than the length of the update rule, then the next tabu list is the concatenation of the current vertex and the current tabu list. Thus, the larger the length of the tabu list, the smaller the mean hitting times. Second, we exhibit graphs on which mean hitting times increase when the length of the update rule increases, for three collections of update rules: $(FIFO_m)_{m\geqslant 0}$, $(LRU_m)_{m\geqslant 0}$ and $(RAND_m)_{m\geqslant 0}$, defined in Section 3. We consider three infinite collections of graphs: paddles, lines and flowers, whose definitions are postponed at Section 4. All these graphs own 0 and 1 as vertices. We express the mean hitting time of the vertex 0 for a walker that starts at vertex 1 with empty tabu list:

- On paddles, for $(LRU_m)_{m\geqslant 0}$ and $(RAND_m)_{m\geqslant 0}$, in Proposition 13.
- On lines, for $(FIFO_m)_{m\geqslant 0}$ and $(LRU_m)_{m\geqslant 0}$, in Proposition 14.
- On flowers for $(FIFO_m)_{m\geqslant 0}$, $(LRU_m)_{m\geqslant 0}$ and $(RAND_m)_{m\geqslant 0}$, in Proposition 15.

Besides, Proposition 16 states that for every update rule distinct from $FIFO_1$, of length at most one, there exist a flower graph and two vertices x and y such that the mean hitting time to y, when the walker starts at x with empty tabu list, is larger than that associated to the simple random walk.

The paper is organized as follows. The formal definition of a tabu random walk is postponed in Section 2 and examples of update rules are described in Section 3. Our results are precisely stated in Section 4. The remainder sections contain the proofs of our results: Section 5 for Theorem 5, Section 6 for Theorem 7, Section 7 for Proposition 16, Section 8 for Proposition 10, Section 9 for Proposition 13, Section 10 for Proposition 14 and Section 11 for Proposition 15.

2 Tabu random walks

We restrict ourselves to simple graphs: every edge is a subset formed by two distinct vertices. In particular, edges are undirected and there are no self-loops nor parallel edges. Besides, we assume that the vertex set is finite and neither empty nor reduced to a singleton. A walk of length ℓ from a vertex x to a vertex y is a finite sequence $(x_n)_{n=0}^{\ell}$ of vertices such that $x_0 = x$, $x_\ell = y$ and for each integer n in $\{0, \dots, \ell - 1\}$, $\{x_n, x_{n+1}\}$ is an edge. In this note, every graph is connected: for every two distinct vertices x and y, there exists a walk from x to y.

We introduce some notations that keep easier the definition of a tabu random walk and the mathematical statement of our results.

The vertex set of every graph G is denoted by $\mathcal{V}$. For each vertex x of G, $\mathcal{V}_x$ is the neighborhood of x: $\mathcal{V}_x = \{y \in \mathcal{V} : \{x, y\} \text{ is an edge of } G\}$. The degree of a vertex is its number of neighbors. A vertex with a unique neighbor, or equivalently of degree one, is called pendant. A vertex with at least two neighbors, or equivalently of degree at least two, is called non-pendant.

For every discrete set $\mathcal{U}$, and every positive integer m, $\mathcal{U}^m$ is the set of words of length m with alphabet $\mathcal{U}$:

$$\mathcal{U}^m = \left\{ t = t^1 \cdots t^m \,:\, \forall k \in \{1, \dots, m\},\, t^k \in \mathcal{U} \right\} .$$

The set $\mathcal{U}^0$ of words of zero length is the singleton formed by the empty word ε. With a slight abuse, the same notation ε holds to designate the empty word for all discrete sets $\mathcal{U}$. Generally, $\mathcal{U}$ will be either a vertex set $\mathcal{V}$ or $\{0, 1\}$. The word in $\{0, 1\}^m$ with each letter equals 0 is denoted by 0^m. Moreover, we set $0^0 = \varepsilon$. For every nonnegative integer m, the set of words of length less than or equal to m will be denoted by $\mathcal{U}^{(m)}$.

For each word t in $\mathcal{U}^{(m)}$, let $|t|$ denote the length of t and let $\mathcal{L}(t)$ denote the set formed by its letters. For example, $|0^m| = m$ and $\mathcal{L}(0^m) = \{0\}$. Generally, the length $|t|$ of the word t and the cardinality $|\mathcal{L}(t)|$ of the associated set $\mathcal{L}(t)$ are distinct.

The concatenation of two words t and s is denoted by $t \cdot s$. For every nonempty word t and every integer k in $\{1, \dots, |t|\}$, $t[k]$ denotes the word of length $|t| - 1$ obtained by removing the k-th letter of t: if $|t| = 1$, then $t[1] = \varepsilon$; if $|t| \geqslant 2$, then for all integers j in $\{1, \dots, |t| - 1\}$,

$$(t[k])^j = \begin{cases} t^j & \text{if } j < k, \\ t^{j+1} & \text{if } j \geqslant k. \end{cases}$$

Besides, we set $t[0] = t$.

Now, we define tabu random walks. For every nonnegative integer n, let X_n denote the position of the walker at step n and let T_n represent its memory: a finite word whose letters are some vertices already visited by the walker. The word T_n is called tabu list and the successive ordered pairs $(X_n, T_n)_{n \geqslant 0}$ formed by the position of the walker and its current tabu list is a Markov chain called tabu chain, defined below. The tabu random walk is the trajectory $(X_n)_{n \geqslant 0}$ of the walker.

As already mentioned, the policy to insert or remove occurrences of vertices in the tabu list is the only parameter of the tabu random walk, called update rule. In particular, each update rule determines the maximal length of the successive tabu lists, called length of the update rule.

For every nonnegative integer n, the next tabu list T_{n+1} is obtained by concatenation of X_n and T_n, then by possibly removing a letter of the concatenated word $X_n \cdot T_n$. If the removed letter of $X_n \cdot T_n$ is not the first one, then we say that the current vertex is inserted in the tabu list. Formally, there exists a random integer C_{n+1} such that $T_{n+1} = (X_n \cdot T_n)[C_{n+1}]$: C_{n+1} is either 0, in which case $T_{n+1} = X_n \cdot T_n$, or the position of the removed letter. Conditionally on (X_n, T_n), the law of the random variable C_{n+1} is determined by the update rule. We say that a tabu list is full if its length equals the length of the update rule. The random integer C_{n+1} must fulfill the following requirements:

- If the current tabu list T_n is not full, then all letters of T_n different from the current vertex X_n are kept, and one of the three disjoint cases occurs:
  - The current vertex is not inserted and all its occurrences in the tabu list are kept: $C_{n+1} = 1$ and $T_{n+1} = T_n$.
  - The current vertex is inserted and all its occurrences in the tabu list are kept: $C_{n+1} = 0$ and $T_{n+1} = X_n \cdot T_n$.
  - The current vertex is inserted and one of its occurrences in the tabu list is removed: C_{n+1} belongs to $\{i \in \{2, \ldots, |T_n| + 1\} : T_n^{i-1} = X_n\}$ and $T_{n+1} = X_n \cdot T_n^1 \cdots T_n^{C_{n+1}-2} \cdot T_n^{C_{n+1}} \cdots T_n^{|T_n|}$.
- If the current tabu list T_n is full, then in order to ensure $|T_{n+1}| \leqslant m$, the current vertex X_n cannot be inserted in the tabu list without removing another letter of T_n. Thus, $C_{n+1} \neq 0$ almost surely.

We infer that the occurrences of vertices in the tabu list are sorted in chronological order of their insertions.

If R is a transition kernel from a measurable set $(\mathcal{A}, \mathfrak{A})$ to a measurable set $(\mathcal{B}, \mathfrak{B})$, then for each element a of $\mathcal{A}$, he probability law $\mathcal{U} \mapsto R(a, \mathcal{U})$ on $(\mathcal{B}, \mathfrak{B})$ is denoted by R^a. Here is the formal definition of an update rule.

Definition 1. *Let m be a nonnegative integer. An update rule of length m is a probability kernel R from $\{0,1\}^{(m)}$ to $\{0, \ldots, m+1\}$ which satisfies the following conditions. Let k be an integer in $\{0, \ldots, m\}$ and let b be a word in $\{0,1\}^k$.*

1. *If $k \leqslant m-1$ and $b \neq 0^k$, then the support of the probability law R^b is included in the set $\{0,1\} \cup \{i \in \{2, \ldots, |b|+1\} \, : \, b_{i-1} = 1\}$.*
2. *If $k = m$, then $R^b(\{0\}) = 0$.*

The word b with alphabet $\{0,1\}$ in the definition above will stand for the successive words formed by replacing each letter in $X_n \cdot T_n$ by 1 if it is an occurrence of X_n, 0 otherwise. Formally, for every vertex x and every word t in $\mathcal{V}^{(m)}$, we introduce the word $\psi(x,t)$ in $\{0,1\}^{|t|}$ given for every integer j in $\{1, \ldots, |t|\}$ by

$$(\psi(x,t))^j = \begin{cases} 1 & \text{if } t^j = x, \\ 0 & \text{if } t^j \neq x. \end{cases}$$

Hence, $\psi(X_n, T_n) = 0^{|T_n|}$ if and only if the current vertex does not belong to $\mathcal{L}(T_n)$.

On every graph with vertex set $\mathcal{V}$, the states space of each tabu chain associated to an update rule of length m is $\mathcal{V} \times \mathcal{V}^{(m)}$, where $\mathcal{V}^{(m)}$ is the set of words with alphabet $\mathcal{V}$ and length less than or equal to m. At each step, the walker avoids, if possible, to revisit the vertices in its tabu list. For every nonnegative integer n, conditionally on (X_n, T_n), the law of (X_{n+1}, T_{n+1}) is defined as follows. The next vertex visited X_{n+1} is uniform on the set $\mathcal{V}_{X_n} \setminus \mathcal{L}(T_n)$ formed by the neighbors of the current vertex X_n that does not have any occurrence in the current tabu list T_n. Otherwise, the next vertex visited is uniform on the set $\mathcal{V}_{X_n}$ of neighbors of the current vertex. The next tabu list T_{n+1} is $(X_n \cdot T_n)[C_{n+1}]$, where C_{n+1} is a random integer, independent of the past process $(X_k, T_k)_{k=0}^{n-1}$ conditionally on (X_n, T_n), that follows the probability law $R^{\psi(X_n, T_n)}$.

To sum up, we introduce the transition kernel Q given for every subset $\mathcal{U}$ of $\mathcal{V}$ and every subset $\mathcal{W}$ of $\mathcal{V}^{(m)}$ by:

$$Q_{(x,t)}(\mathcal{U} \times \mathcal{W}) = \begin{cases} \frac{|\mathcal{U} \cap (\mathcal{V}_x \setminus \mathcal{L}(t))|}{|\mathcal{V}_x \setminus \mathcal{L}(t)|} \times R^{\psi(x,t)}(\phi_{x,t}^{-1}(\mathcal{W})) & \text{if } \mathcal{V}_x \not\subset \mathcal{L}(t), \\ \frac{|\mathcal{U}|}{|\mathcal{V}_x|} \times R^{\psi(x,t)}(\phi_{x,t}^{-1}(\mathcal{W})) & \text{if } \mathcal{V}_x \subset \mathcal{L}(t), \end{cases}$$

where, for every vertex x and for every word t in $\mathcal{V}^{(m)}$, $\phi_{u,t}$ is the function from $\{0,\ldots,|t|+1\}$ to $\mathcal{V}^{|t|} \cup \mathcal{V}^{|t|-1}$ defined by $\phi_{u,t}(c) = (u \cdot t)[c]$,

Now, we can state the formal definition of a tabu chain and a tabu random walk.

Definition 2. *A tabu chain associated to an update rule R of length m, on a graph with vertex set $\mathcal{V}$, is an homogeneous Markov chain $(X_n, T_n)_{n\geqslant 0}$ with states space $\mathcal{V} \times \mathcal{V}^{(m)}$ and transition kernel Q. The stochastic process $(X_n)_{n\geqslant 0}$ is called tabu random walk.*

Since we will often consider tabu chains on the same graph associated to the same update rule, but with different starting distributions, we introduce the following notations. For every element u of every set $\mathcal{U}$, the Dirac measure on $\mathcal{U}$ concentrated at u is denoted by δ_u. The walker will always start at a given vertex and, most often, with empty tabu list. Therefore, for each update rule of length m on a graph with vertex set $\mathcal{V}$, and each ordered pair (x,t) of $\mathcal{V} \times \mathcal{V}^{(m)}$, $P_{(x,t)}$ will denote the probability law of the tabu chain started at (x,t), namely such that (X_0, T_0) follows the law $\delta_{(x,\varepsilon)}$. The related expectation will be denoted by $E_{(x,t)}$.

3 Examples of update rules

In this section, we describe some particular update rules, associated tabu chains and related tabu random walks.

We start with some general observations. For all update rules, the sequence $(|T_n|)_{n\geqslant 0}$ formed by the successive sizes of the tabu lists is almost surely nondecreasing. Hence, when the tabu list becomes full, it remains full for all later steps. Besides, for all positive integers n:

- While the tabu list is not full, the number of distinct vertices that have an occurrence in it is non decreasing: $|\mathcal{L}(T_n)| \geqslant |\mathcal{L}(T_{n-1})|$ almost surely on the event $\{|T_{n-1}| < m\}$, where m is the length of the update rule.

- Each vertex that has an occurrence in the tabu list must have been already visited by the walker: $\mathcal{L}(T_n) \subset \{X_k,\, k \in \{0,\ldots,n-1\}\}$.

- The occurrences of vertices in the tabu list are sorted in chronological order; in particular, the last position in the tabu list is hold by the oldest visited vertex:

$$T_n^{|T_n|} = \operatorname{argmax}\{k \in \{0,\ldots,n-1\} \,:\, X_k \in \mathcal{L}(T_n)\}\ .$$

The following update rules yield simple random walks. An update rule R is said trivial if $R^\varepsilon = \delta_1$, namely if the current vertex is never inserted in the tabu list. For example, the unique update rule of zero length is trivial. If $T_0 = \varepsilon$ and if the update rule is trivial, then every tabu list is always the empty word and every tabu random walk is a simple random walk: at each step, the next vertex visited is chosen uniformly at random among the neighbor of the current vertex.

Let m be a positive integer. We introduce four non trivial update rules R of length m and we describe associated tabu chains. Let k be an integer in $\{0, \dots, m+1\}$ and let b be a word in $\{0,1\}^k$.

The FIX_m update rule is given by:

$$R^b = \begin{cases} \delta_0 & \text{if } 0 \leqslant k \leqslant m-1, \\ \delta_1 & \text{if } k = m. \end{cases}$$

the current vertex is inserted while the current tabu list has length less that m; the tabu list is left unchanged otherwise. Therefore, for every integer n greater than or equal to m, the tabu list T_n is $X_{m-1} \cdots X_0$. Thus, the first m vertices visited by the walker are kept forever in the tabu list.

The $FIFO_m$ update rule is given by:

$$R^b = \begin{cases} \delta_0 & \text{if } 0 \leqslant k \leqslant m-1, \\ \delta_{m+1} & \text{if } k = m. \end{cases}$$

The current vertex is inserted and, if the current tabu list has length m, then its last letter is removed. Hence, for every integer n greater than or equal to m, the tabu list T_n is $X_{n-1} \cdots X_{n-m}$. Consequently, the tabu random walk $(X_n)_{n \geqslant 0}$ is a Markov chain of order $m+1$ on $\mathcal{V}$. The update rule $FIFO_1$ yields a non-backtracking random walk: for every positive integer n, $T_n = X_{n-1}$ and the next vertex X_{n+1} is uniform on $\mathcal{V}_{X_n} \setminus \{X_{n-1}\}$ if X_n is non-pendant, $X_{n+1} = X_{n-1}$ if X_n is pendant. Therefore, $(X_n)_{n \geqslant 0}$ is a random walk that does not backtrack, unless the walker visits a pendant vertex. The acronym $FIFO$ stands for "First In First Out".

The LRU_m update rule is given by:

$$R^b = \begin{cases} \delta_0 & \text{if } 0 \leqslant k \leqslant m-1 \text{ and } b = 0^k, \\ \delta_{m+1} & \text{if } k = m \text{ and } b = 0^m, \\ \delta_{1+\max\{i \,:\, b_i = 1\}} & \text{if } b \neq 0^k. \end{cases}$$

The current vertex is inserted and every other occurrence of this vertex is removed. If the current tabu list has length m without any occurrence of the current vertex, then its last letter is removed. Therefore, for every nonnegative integer n, all letters of T_n are distinct and formed by the last $|\mathcal{L}(T_n)|$ visited states by the tabu random walk $(X_n)_{n\geqslant 0}$, counted without multiplicity. The acronym LRU stands for "Least Recently Used".

The $RAND_m$ update rule is given by:

$$R^b = \begin{cases} \delta_0 & \text{if } 0 \leqslant k \leqslant m-1 \text{ and } b = 0^k, \\ \text{uniform}(\{1,\dots,m+1\}) & \text{if } k = m \text{ and } b = 0^m, \\ \delta_1 & \text{if } b \neq 0^k. \end{cases}$$

If the current vertex has an occurrence in the tabu list, then the tabu list is unchanged. If the current vertex does not have any occurrence in the current tabu list and if the tabu list has length m, then the new tabu list is formed by inserting the current vertex and removing one of the $m+1$ letters, uniformly at random. Therefore, for every positive integer n, all letters of T_n are distinct elements of $\{X_k : k \in \{0,\dots,n-1\}\}$.

To keep the notations easier, we consider that FIX_0, $FIFO_0$, LRU_0 and $RAND_0$ refer to the trivial update rule of zero length. We remark that when m equals 1 or 2, then the update rules $FIFO_m$ and LRU_m coincide and are both distinct from FIX_m and $RAND_m$. However, for every integer m greater than or equal to 3, the update rules FIX_m, $FIFO_m$, LRU_m and $RAND_m$ are distinct.

4 Statement of results

In this section, we state the results we have obtained. We start with some notations. Consider an update rule of length m and an associated tabu chain $(X_n, T_n)_{n\geqslant 0}$ on a graph with vertex set $\mathcal{V}$. For every vertex y and every word s in $\mathcal{V}^{(m)}$, the hitting time $H_{(y,s)}$ of (y,s) is the first instant n when the tabu chain reaches (y,s):

$$H_{(y,s)} = \inf\{n \geqslant 0 : (X_n, T_n) = (y,s)\} .$$

The hitting time H_y of y is the first instant when the tabu random walk $(X_n)_{n\geqslant 0}$ reaches y:

$$H_y = \inf\{n \geqslant 0 : X_n = y\} .$$

In particular, $H_y = \inf_{s\in\mathcal{V}^{(m)}} H_{(y,s)}$. Likewise, the first return times are defined by

$$H^+_{(y,s)} = \inf\{n \geqslant 1 : (X_n, T_n) = (y,s)\}$$

and

$$H_y^+ = \inf\{n \geqslant 1 \, : \, X_n = y\} \ .$$

Similarly, $H_y^+ = \inf_{s\in\mathcal{V}^{(m)}} H_{(y,s)}^+$.

Definition 3. *A tabu random walk terminates if for every two vertices x and y, the mean hitting time $E_{(x,\varepsilon)}H_y$ is finite.*

Introduce the following condition:

Definition 4. *An update rule R of length m satisfies Condition* $(\mathbf{C})$ *if and only if:*

- *For all integers k in $\{0,\dots,m-1\}$, the probability $R^{0^k}(\{0\})$ is positive.*
- *The probability $R^{0^m}(\{m+1\})$ is positive.*

In words, at each step:

- The current vertex is possibly inserted in the tabu list when it is not already in.
- If the tabu list is full, without any occurrence of the current vertex, then its last letter is possibly removed.

The trivial update rule of zero length satisfies Condition $(\mathbf{C})$. For every positive integer m, the update rules $FIFO_m$, LRU_m and $RAND_m$ fulfill Condition $(\mathbf{C})$, whereas the update rule FIX_m does not.

Our first result states that Condition $(\mathbf{C})$ is necessary and sufficient to ensure that all tabu random walks terminate, whatever the graph might be.

Theorem 5. *All tabu random walks associated to a given update rule terminate if and only if that update rule either is trivial or satisfies* $(\mathbf{C})$.

The remainders results compare hitting times for distinct update rules. We emphasize the dependence on an update rule R by writing every associated tabu chain as $(X_n(R), T_n(R))_{n\geqslant 0}$.

Before introducing our second result, we define the class of m-free graphs.

Definition 6. *Let m be a positive integer m. A graph is m-free if there does not exit any walk $x_0,\dots,x_k$ of positive length k such that:*

- *The vertex x_k is non-pendant.*
- *For all integers j in $\{0,\dots,k-1\}$, the vertices x_j and x_k are distinct.*

- *All non-pendant neighbors of x_k belong to $\{x_0, \ldots, x_{k-1}\}$.*
- *The integer $k + 2d$ is less than or equal to m, where d is the number of pendant neighbors of x_k.*

We illustrate the definition above:

- Let m be a positive integer and G an m-free graph. Then, for every integer k in $\{1, \ldots, m\}$, G is k-free.
- Every graph is 1-free.
- A graph is 2-free if and only if it does not possess any vertex x such that $\mathcal{V}_x = \{y, z\}$ and $\{x, z\} \subset \mathcal{V}_y$. Otherwise, the walk $x_0 = y$, $x_1 = z$, $x_2 = x$, with $k = 2$ and $d = 0$ implies that the graph is not m-free. In words: a graph is 2-free if and only if it does not contain any triangle with one vertex of degree exactly two. This condition is necessary and sufficient.
- Every $(m + 1)$-regular graph, namely with every vertices of degree $m+1$, is m-free. Indeed, for every walk $(x_i)_{i=0}^k$, the set $\{x_0, \ldots, x_{k-1}\}$ has cardinality less than or equal to k, while x_k has $m + 1$ neighbors of degree $m + 1$. Whence, the non-pendant neighbors of x_k cannot all belong to $\{x_0, \ldots, x_{k-1}\}$.
- A cycle is a walk from a vertex to itself. Every graph of girth greater than or equal to $m + 2$, that is to say with all cycles having at least $m + 2$ edges, is m-free. Assume that $(x_i)_{i=0}^k$ is a walk that contradicts the m-freeness. Since the girth is strictly greater than two, all vertices are non-pendant. Let x_j denote a neighbor of x_k distinct from x_{k-1}. Then $(x_k, x_j, x_{j+1}, \ldots, x_{k-1}, x_k)$ is a cycle of length strictly less than $m + 2$.

The previous definition has been motivated by the following fact. Let m be an integer m greater than or equal to 2 and consider a m-free graph. Then, in every tabu random walk associated to the update rule $FIFO_m$, the walker will never visit any vertex having an occurrence in its current tabu list if the vertex currently visited is non-pendant: for every nonnegative integer n, $\mathcal{V}_{X_n(FIFO_m)}$ is not included in $\mathcal{L}(T_n(FIFO_m))$.

Our second results asserts that, for every positive integer m and for every m-free graph, the update rule $FIFO_m$ yields minimal mean hitting times among all update rules of length less than or equal to m.

Theorem 7. *Let m be a positive integer and let R be an update rule of length k less than or equal to m. Consider a m-free graph. Then, for all vertices x and y,*

$$E_{(x,\varepsilon)}H_y^+(FIFO_m) \leqslant E_{(x,\varepsilon)}H_y^+(R) \ .$$

Besides, the update rules R and $FIFO_k$ yield the same transition kernel if and only if:

- *For every integer j in $\{0, \ldots, k-1\}$, $R^{0^j} = \delta_0$.*
- *$R^{0^k} = \delta_{k+1}$.*

Being m-free is not a necessary condition to have, for all two vertices x and y, $E_{(x,\varepsilon)}H_y^+(FIFO_m) \leqslant E_{(x,\varepsilon)}H_y^+(R)$. Indeed, consider the clique with vertex set $\{0,1,2\}$ and edge set $\{\{0,1\},\{0,2\},\{1,2\}\}$. The walk $(0,1,2)$ ensures that the graph is not 2-free. Yet, for every two vertices x and y and for every update rule R: $E_{(x,\varepsilon)}H_y(R) \geqslant 3/2$, while $E_{(x,\varepsilon)}H_y(FIFO_2) = 3/2$.

The first corollary of Theorem 7 is a direct application of its second assertion.

Corollary 8. *Let m be a positive integer. On every m-free graph, for every integer k in $\{0, \ldots, m\}$, LRU_k and $FIFO_k$ yield the same transition kernel.*

The second corollary claims that the non-backtracking random walk is the fastest among all update rules of length zero or one.

Corollary 9. *For every update rule R of length zero or one, and for every ordered pair (x,y) of vertices of every graph,*

$$E_{(x,\varepsilon)}H_y^+(FIFO_1) \leqslant E_{(x,\varepsilon)}H_y^+(R) \ .$$

The proof results from the fact that every graph is 1-free,

From now on, we focus on expression of mean hitting times on four particular collections of graphs: the cliques $(K_r)_{r\geqslant 2}$, the paddles $(G_r)_{r\geqslant 3}$, the lines $(L_r)_{\geqslant 2}$ and the flowers $(F_\ell)_{\ell\geqslant 1}$. Their definitions are postponed before each related result. The graphs K_6, G_6, L_4 and F_3 are drawn on Figure 4.

For all integers r greater than or equal to 2, the clique K_r is the complete graph with vertex set $\{0, \ldots, r-1\}$: each vertex is neighbor to all other vertices. In a first place, we express, for every integer r greater than or equal to 2 and for every update rule R, the mean hitting time $E_{(1,\varepsilon)}H_0(R)$

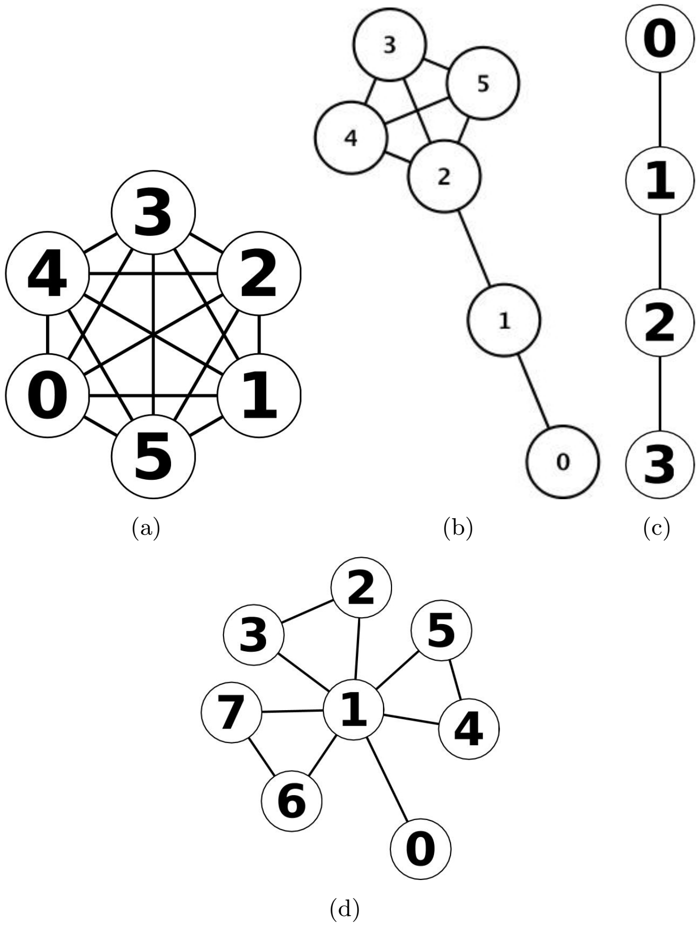

Figure 1: Figure (a) represents the graph K_6, Figure (b) represents the graph G_6, Figure (c) represents the graph L_4 and Figure (d) represents the graph F_3.

on K_r. For every nonnegative integer k, every nonnegative real number x and every real number p in $[0,1]$, we set

$$g_k(p,x) = \frac{r-1-k+(r-2-k)px}{1+(r-2-k)p} \ .$$

Proposition 10. *Consider an update rule R of length m and an integer r greater than or equal to 2. On the clique K_r, the mean hitting time $E_{(1,\varepsilon)}H_0(R)$ is the last term of the finite sequence $(u_k)_{k=0}^{\min\{m,r-1\}}$ defined by $u_0 = \max\{r-1-m, 1\}$ and for all integers k in $\{1,\ldots,\min\{m,r-1\}\}$, $u_k = g_k\left(R^{0^{\min\{m,r-1\}-k}}(\{0\}), u_{k-1}\right)$.*

Proposition 10 induces the two following corollaries. Let r be an integer greater than or equal to 2. Consider the clique K_r. We will restrict ourselves to update rules of length less than or equal to $r-2$.

On one hand, consider the set of update rules with length less than or equal to $r-2$ and such that, if the tabu list is not full and if the current vertex does not have any occurrence in the tabu list, then this latter is always inserted. That is, we consider update rule R of length m such that for every integer k in $\{0,\ldots,m-1\}$, $R^{0^k}(\{0\}) = 1$. Corollary 11 implies that the mean hitting time of the vertex 0, starting at vertex 1 with empty tabu list, increases with the length of the tabu list. In particular, for the collection of update rules $(FIFO_m)_{m\geqslant 0}$, $(LRU_m)_{m\geqslant 0}$ and $(RAND_m)_{m\geqslant 0}$, the larger the length of the update rule, the smaller the mean hitting times.

Corollary 11. *Let (ℓ,m,r) a triple of integers such that $0 \leqslant \ell < m \leqslant r-2$. Let S be an update rule of length ℓ and let R be an update rule of length m such that for all integers k in $\{0,\ldots,\ell-1\}$, $S^{0^k}(\{0\}) \leqslant R^{0^k}(\{0\})$. On the clique K_r,*

$$E_{(1,\varepsilon)}H_0(R) < E_{(1,\varepsilon)}H_0(S) \ .$$

On the other hand, Corollary 12 claims that, generally, having a larger length does not imply having smaller mean hitting times. Informally, if R is an update rule of length two with $R^0(\{0\})$ small enough, then the mean hitting time $E_{(1,\varepsilon)}H_0(FIFO_1)$ is smaller than the mean hitting time $E_{(1,\varepsilon)}H_0(R)$. Indeed, if the current vertex has small probability to be inserted, then every tabu random walk associated to R is a simple random walk during a large number of steps.

Corollary 12. *Let (ℓ,m,r) a triple of integers such that $1 \leqslant \ell < m \leqslant r-2$. There exist an update rule R with length m and an update rule S with length*

ℓ *such that for all integers* k *in* $\{0,\ldots,\ell-2\}$, $S^{0^k}(\{0\}) = R^{0^k}(\{0\})$ *and, on the clique* K_r,

$$E_{(1,\varepsilon)}H_0(S) < E_{(1,\varepsilon)}H_0(R) \ .$$

Henceforth, we restrict ourselves to the three collections of update rules $(FIFO_m)_{m\geqslant 0}$, $(LRU_m)_{m\geqslant 0}$ and $(RAND_m)_{m\geqslant 0}$. We deal with tabu random walks on paddles $(G_r)_{r\geqslant 3}$, lines $(L_r)_{\geqslant 2}$ and flowers $(F_\ell)_{\ell\geqslant 1}$. All these graphs own 0 and 1 as vertices, and we study the mean hitting time $E_{(1,\varepsilon)}H_0$ of the vertex 0 for a walker that starts at vertex 1 with empty tabu list.

We begin with paddle graphs. For every integer r greater than or equal to 3, the paddle G_r is the graph with vertex set $\{0,\ldots,r-1\}$ such that the vertices $2,\ldots,r-1$ form a clique with $r-1$ elements and the vertex 1 has 0 and 2 as neighbors. As an example, the graph G_6 is drawn in Figure 4. Proposition 13 deals with tabu random walks on paddle graphs for the collections of update rules $(LRU_m)_{m\geqslant 0}$ and $(RAND_m)_{m\geqslant 0}$. A walker on the paddle G_r that starts at the vertex 1 and does not hit the vertex 0 at its first move, must stay on the set of vertices $\{3,\ldots,r-1\}$ until its tabu list is full. Thus, the length of the update rule may raise the mean hitting time $E_{(1,\varepsilon)}H_0$.

Proposition 13. *Let* m *and* m' *denote two integers:*

- *If* $3 \leqslant m < m'$, *then on* G_{m+2},

  $$E_{(1,\varepsilon)}H_0(LRU_m) < E_{(1,\varepsilon)}H_0(LRU_{m'})$$

  and

  $$E_{(1,\varepsilon)}H_0(RAND_m) < E_{(1,\varepsilon)}H_0(RAND_{m'}) \ .$$

- *If* $1 \leqslant m \leqslant 2 < m'$, *then on* G_4,

  $$E_{(1,\varepsilon)}H_0(LRU_m) < E_{(1,\varepsilon)}H_0(LRU_{m'}) \ .$$

- *If* $1 \leqslant m \leqslant 2 < m'$ *and* $(m,m') \neq (1,3)$, *then on* G_5,

  $$E_{(1,\varepsilon)}H_0(RAND_m) < E_{(1,\varepsilon)}H_0(RAND_{m'}) \ .$$

- *For every integer* r *greater than or equal to* 3,

  $$E_{(1,\varepsilon)}H_0(LRU_m) \leqslant E_{(1,\varepsilon)}H_0(RAND_m)$$

  on G_r, *with strict inequality if and only if* $1 \leqslant m \leqslant r-2$.

Afterwards, we handle the case of line graphs. For every integer r greater than or equal to 2, the line L_r is the graph with vertex set $\{0, \dots, r-1\}$ and edge set $\{\{i, i+1\}, i \in \{0, \dots r-2\}\}$. As an example, the graph L_4 is drawn in Figure 4. Proposition 14 deals with tabu random walks on lines for the collections of update rules $(FIFO_m)_{m \geqslant 0}$ and $(LRU_m)_{m \geqslant 0}$. On lines, the length of the update rule raises the mean hitting time $E_{(1,\varepsilon)} H_0$. Indeed, assume that m is a positive integer and consider a walker on the line L_r with update rule $FIFO_m$ or LRU_m that starts at the vertex 1 and does not hit the vertex 0 at its first move. First, the walker must go to the end of the line, the vertex $r-1$, without backtracking. Then, its tabu list is almost surely $(r-2) \cdots (r-m-1)$. Thus, the walker performs a simple random walk, until it reaches a vertex with a neighbor not included in the tabu list. The duration of the simple random walk behavior increases with the length of the update rule. Next, the walker goes to the vertex 0 without backtracking.

Proposition 14. *Let m, m' and r denote three integers. Consider a tabu random walk on L_r:*

- *If $1 \leqslant m < m'$, $(m, m') \neq (1, 2)$ and $r \geqslant m+2$, then*
$$E_{(1,\varepsilon)} H_0(LRU_m) < E_{(1,\varepsilon)} H_0(LRU_{m'}) \, .$$
- *If $m \geqslant 1$ and $r \geqslant 4$, then*
$$E_{(1,\varepsilon)} H_0(FIFO_{2m-1}) = E_{(1,\varepsilon)} H_0(FIFO_{2m}) < E_{(1,\varepsilon)} H_0(FIFO_{2m+1}) \, .$$
- *If $m \geqslant 3$ and $r \geqslant 4$, then*
$$E_{(1,\varepsilon)} H_0(FIFO_m) < E_{(1,\varepsilon)} H_0(LRU_m) \, .$$
- *If $r \geqslant 2$, then*
$$\lim_{m \to +\infty} E_{(1,\varepsilon)} H_0(FIFO_m) = E_{(1,\varepsilon)} H_0(LRU_r) \, .$$

Finally, we deal with the flower graphs. For all positive integers ℓ, the flower F_ℓ is the graph with vertex set $\{0, \dots, 2\ell+1\}$ as follows. Initially, the vertices 0 and 1 are isolated and for each integer k in $\{1, \dots, \ell\}$, the vertices $2k$ and $2k+1$ are neighbor. Then, each vertex is linked to the vertex 1, except the vertex 1 itself. As an example, the graph F_3 is drawn in Figure 4. Proposition 15 deals with tabu random walks on paddles for

$RAND_0$, $FIFO_1$, $RAND_1$ and $RAND_2$. We recall that $RAND_0$, $FIFO_0$ and LRU_0 yield the same tabu chain. Likewise, $FIFO_1$ and LRU_1 yield the same tabu chain. We aim to show that the larger the length of the update rule, the larger the mean hitting time $E_{(1,\varepsilon)}H_0$. Without loss of symmetry, we may assume that a walker on F_ℓ that starts at the vertex 1 and does not hit the vertex 0 at its first move, moves to vertex 2. Now, the mean return time to the vertex 1 increases when the length of the update rule increases. After having returned to the vertex 1, the walker either hits 0 or finds itself in a similar situation, namely at vertex 2 and forced to return to the vertex 1.

Proposition 15. *The following inequalities hold:*

- *On the graph F_5,*

$$E_{(1,\varepsilon)}H_0(RAND_0) < E_{(1,\varepsilon)}H_0(RAND_1) < E_{(1,\varepsilon)}H_0(RAND_2)\ .$$

- *On the graph F_6,*

$$E_{(1,\varepsilon)}H_0(RAND_1) < E_{(1,\varepsilon)}H_0(RAND_2)\ .$$

- *On the graph F_4,*

$$E_{(1,\varepsilon)}H_0(FIFO_1) < E_{(1,\varepsilon)}H_0(FIFO_2)\ .$$

- *On the graph F_7,*

$$E_{(1,\varepsilon)}H_0(FIFO_0) < E_{(1,\varepsilon)}H_0(FIFO_2)\ .$$

Proposition 16 completes Corollary 9. It states that $FIFO_1$ is the only update rule of length one such that, on all graphs, all hitting times are smaller than those associated to the simple random walk.

Proposition 16. *If R is an update rule of length one distinct from $FIFO_1$, then there exists a positive integer ℓ such that on the graph F_ℓ,*

$$E_{(1,\varepsilon)}H_0(FIFO_0) < E_{(1,\varepsilon)}H_0(R)\ .$$

The three following tables sum up the results for mean hitting times obtained for the three collections of update rules $(FIFO_m)_{m\geqslant 0}$, $(LRU_m)_{m\geqslant 0}$ and $(RAND_m)_{m\geqslant 0}$.

The first table deals with the collection of update rules $(FIFO_m)_{m \geqslant 0}$. On the graph written at k-th row and m-th column, $E_{(1,\varepsilon)}H_0(FIFO_m) < E_{(1,\varepsilon)}H_0(FIFO_k)$. The symbol "$\times$" means that for all graphs G and all vertices x and y of G, $E_{(1,\varepsilon)}H_0(FIFO_m) \geqslant E_{(1,\varepsilon)}H_0(FIFO_k)$:

	0	1	2	3	4	$m \geqslant 5$
0	$\times$	K_3	K_3	K_3	K_3	K_3
1	$\times$	$\times$	K_3	K_3	K_3	K_3
2	F_7	F_4	$\times$	K_4	K_4	K_4
3	L_4	L_4	L_4	$\times$	K_5	K_5
4	L_4	L_4	L_4	L_5	$\times$	K_6
$k \geqslant 5$	L_4	L_4	L_4	L_5	L_6	$\begin{cases} L_{m+2} & \text{if } m < k, \\ \times & \text{if } m = k, \\ K_{k+2} & \text{if } m > k. \end{cases}$

This table treats the collection of update rules $(LRU_m)_{m \geqslant 0}$:

	0	1	2	3	4	$m \geqslant 5$
0	$\times$	K_3	K_3	K_3	K_3	K_3
1	$\times$	$\times$	K_3	K_3	K_3	K_3
2	F_7	F_4	$\times$	K_4	K_4	K_4
3	L_4	G_4	G_4	$\times$	K_5	K_5
4	L_4	G_4	G_4	G_5	$\times$	K_6
$k \geqslant 5$	L_4	G_4	G_4	G_5	G_6	$\begin{cases} G_{m+2} & \text{if } m < k, \\ \times & \text{if } m = k, \\ K_{k+2} & \text{if } m > k. \end{cases}$

This table treats the collection of update rules $(RAND_m)_{m \geqslant 0}$:

	0	1	2	3	4	$m \geqslant 5$
0	$\times$	K_3	K_3	K_3	K_3	K_3
1	F_5	$\times$	K_3	K_3	K_3	K_3
2	F_5	F_6	$\times$	K_4	K_4	K_4
3	L_4	L_4	G_5	$\times$	K_5	K_5
4	L_4	G_5	G_5	G_5	$\times$	K_6
$k \geqslant 5$	L_4	G_5	G_5	G_5	G_6	$\begin{cases} G_{m+2} & \text{if } m < k, \\ \times & \text{if } m = k, \\ K_{k+2} & \text{if } m > k. \end{cases}$

The rest of the technical report is organized as follows. Theorem 5 is proved in Section 5 and Proposition 7 in Section 6, Proposition 10, Corollary 11 and Corollary 12 are proved in Section 8. Section 9 is devoted to the proof of Proposition 13 and Section 10 to the proof of Proposition 14. Finally, Section 11 deals with proofs of Proposition 15 and Proposition 16.

5 Proof of Theorem 5

First of all, we prove that Condition (**C**) is necessary. We proceed by establishing the contrapositive. Let R be an update rule of length m that does not fulfill Condition (**C**). Consider the graph with vertex set $\{0, \ldots, m+2\}$ and edge set

$$\{0,1\} \cup \{\{i,j\} : 1 \leqslant i < j \leqslant m+2\}\ .$$

In words, the vertices $1, \ldots, m+2$ form a clique and the vertex 0 has the vertex 1 as unique neighbor. Let $(X_n, T_n)_{n \geqslant 0}$ denote the associated tabu chain on this graph that starts at $(1, \varepsilon)$. We claim that the tabu random walk $(X_n)_{n \geqslant 0}$ does not terminate. Indeed, we will show that the probability $P_{(1,\varepsilon)}(H_0 = +\infty)$ is positive, which implies $E_{(1,\varepsilon)} H_0 = +\infty$.

Here is the idea of the proof: since the update rule is not trivial, with $P_{(1,\varepsilon)}$-positive probability, the walker does not hit the vertex 0 at its first step and the vertex 1 is inserted in the tabu list. Assume that this latter event is realized. Then, the walker needs to return to the vertex 1 before hitting the vertex 0. Since all neighbors of the vertex 1 distinct form 0 have degree $m+1$, the removal of the vertex 1 is needed in order to hit the vertex 0. Nevertheless, we will show that the vertex 1 stays forever in the tabu list.

First, assume that the set $\{k \in \{0, \ldots, m-1\} : R^{0^k} = \delta_1\}$ is nonempty and let j denote its minimum. As the update rule is not trivial, $j \neq 0$. We set $\mathcal{A}_j = \{(X_j, T_j) = (j+1, j \cdots 2 \cdot 1)\}$. The event $\mathcal{A}_j$ has $P_{(1,\varepsilon)}$-positive probability. Indeed:

- If $j = 1$, then
$$P_{(1,\varepsilon)}(\mathcal{A}_j) = \frac{R^{\varepsilon}(\{0\})}{m+2}\ .$$
- If $j \geqslant 2$, then
$$P_{(1,\varepsilon)}(\mathcal{A}_j) \geqslant \frac{R^{\varepsilon}(\{0\})}{m+2} \times \prod_{i=1}^{j-1} \frac{R^{0^i}(\{0\})}{m-i+1}\ .$$

By minimality of j, for every integer i in $\{0, \ldots, j-1\}$, the probability $R^{0^i}(\{0\})$ is positive. Whence, $P_{(1,\varepsilon)}(\mathcal{A}_j)$ is positive.

Now, since $R^{0^j} = \delta_1$, the tabu list remains $P_{(1,\varepsilon)}$-almost surely constant on the event $\mathcal{A}_j$: for all integers $n \geqslant j$, $P_{(1,\varepsilon)}(T_n = (j+1, j \cdots 1) \,|\, \mathcal{A}_j) = 1$. Thus, $P_{(1,\varepsilon)}(H_1^+ = +\infty) > 0$ and we infer that $P_{(1,\varepsilon)}(H_0 = +\infty) > 0$.

Second, we suppose that $R^{0^m}(\{m+1\}) = 0$. Let $\mathcal{A}_m$ denote the event $\{(X_m, T_m) = (m+1, m \cdots 1)\}$. As already shown, $P_{(1,\varepsilon)}(\mathcal{A}_m) > 0$. Yet, $P_{(1,\varepsilon)}(\forall n \geqslant m, T_n^m = 1 \,|\, \mathcal{A}_m) = 1$. Thus, $P_{(1,\varepsilon)}(H_1^+ = +\infty) > 0$ and $P_{(1,\varepsilon)}(H_0 = +\infty) > 0$.

Now, we prove that Condition (**C**) is sufficient. If the update rule is trivial, then every associated tabu random walk is a simple random walks. A simple random walk on a finite and connected graph is positive recurrent, hence terminates. Whence, we consider a non trivial update rule R of length m which satisfies Condition (**C**). Let G be a graph with vertex set $\mathcal{V}$ and let Q denote the associated transition kernel. We will show that all tabu random walks on G terminate.

As stated for example in the book [2, p. 16], an essential communicating class of Q is a subset $\mathcal{C}$ of $\mathcal{V} \times \mathcal{V}^{(m)}$ such that, for every ordered pair $((x,t),(y,s))$ of elements of $\mathcal{C}$:

- The probability $P_{(x,t)}(H_{(y,s)} < +\infty)$ is positive.
- The probability $P_{(y,s)}(H_{(x,t)} < +\infty)$ is positive.
- If (z,u) is an element of $\mathcal{V} \times \mathcal{V}^{(m)}$ such that the probability $P_{(x,t)}(H_{(z,u)} < +\infty)$ is positive, then (z,u) belongs to $\mathcal{C}$.

In particular, a tabu chain cannot leave an essential communicating class. Besides, since $\mathcal{V} \times \mathcal{V}^{(m)}$ is finite, the set $\mathcal{C}$ is finite and the restriction of the transition kernel Q to the finite set $\mathcal{C}$ is positive recurrent. Consequently, if (x,t) and (y,s) belong to $\mathcal{C}$, then the mean hitting time $E_{(x,t)}H_{(y,s)}$ is finite.

Moreover, each tabu chain eventually reaches an essential communicating class in mean finite time. Hence, in order to show that all tabu random walks terminate, it is enough to show that for every essential communicating class $\mathcal{C}$, every element (x,t) of $\mathcal{C}$ and every vertex y of G, the mean hitting time $E_{(x,t)}H_y$ is finite. Yet, $E_{(x,t)}H_y \leqslant \inf_{s \in \mathcal{V}^{(m)}} E_{(x,t)}H_{(y,s)}$ and $E_{(x,t)}H_{(y,s)}$ is finite if (y,s) belongs to $\mathcal{C}$. Consequently, it suffices to show that the set $\mathcal{V}(\mathcal{C})$ formed by the first coordinates of all elements of $\mathcal{C}$ is the vertex set $\mathcal{V}$. Formally, $\mathcal{V}(\mathcal{C}) = \mathcal{V}$, where

$$\mathcal{V}(\mathcal{C}) = \{z \in \mathcal{V} \,:\, \exists u \in \mathcal{V}^{(m)},\, (z,u) \in \mathcal{C}\}\ .$$

Since (x, t) belongs to $\mathcal{C}$, we already know that x belongs to $\mathcal{V}(\mathcal{C})$. The graph G being connected, the proof is complete if we show that the neighborhood $\mathcal{V}_x$ of x is included in $\mathcal{V}(\mathcal{C})$. We will proceed by contradiction: let $\mathcal{W}$ denote the set $\mathcal{V}_x \setminus \mathcal{V}(\mathcal{C})$ of neighbors of x that do not belong to $\mathcal{V}(\mathcal{C})$ and assume that $\mathcal{W}$ is nonempty.

If either $\mathcal{V}_x$ is included in $\mathcal{L}(t)$ or $\mathcal{W}$ is not included in $\mathcal{L}(t)$, then $Q_{(x,t)}(\mathcal{W} \times \mathcal{V}^{(m)}) > 0$ and we reach a contradiction. Necessarily, $\mathcal{V}_x$ is not included in $\mathcal{L}(t)$ while $\mathcal{W}$ is. Hence, there exists an ordered pair (z, u) in $\mathcal{C}$ such that z is a neighbor of x and z does not belong to $\mathcal{L}(t)$.

As already remarked at the beginning of Section 3, the sequence $(|T_n|)_{n\geqslant 0}$ formed by the successive sizes of the tabu lists is almost surely nondecreasing. The ordered pair (z, u) belongs to the essential communicating class $\mathcal{C}$, the size of the tabu list is constant. Assume that u is not full. Since $R^{0^{|t|}}(\{0\}) > 0$ and $\phi(z, u) = 0^{|u|}$, we obtain $P_{(z,u)}(|T_1| = |u| + 1)) > 0$. Whence, we have reached a contradiction and the tabu list u must be full: $|u| = m$. Yet, $R^{0^m}(\{m + 1\}) > 0$ and we infer that $P_{(z,u)}(T_1 = (z \cdot u)[m + 1])$. Thus, the last element of the tabu list is infinitely removed from the tabu list. Yet, the tabu random walk never visits the set $\mathcal{W}$ and, consequently, occurrences of vertices in the set $\mathcal{W}$ are never inserted in the tabu list. We infer that, eventually, the tabu list does not contain any occurrences of vertices in $\mathcal{W}$. It contradicts the assumption that $\mathcal{W}$ is included in $\mathcal{L}(t)$ and so $\mathcal{V}(\mathcal{C}) = \mathcal{V}$. We conclude that every tabu random walk associated to R on G terminates.

6 Proof of Theorem 7

Let m be a positive integer and let G be a m-free graph. For all integers j, we set $j^+ = \max\{j, 0\}$. For each integer ℓ greater than or equal to $m+1$, and for each integer i in $\{2, \ldots, \ell\}$, we say that a walk $(x_n)_{n=0}^{\ell}$ retracts at step i if x_i belongs to $\{x_{(i-2)^+}, \ldots, x_{(i-m-1)^+}\}$ whereas $\mathcal{V}_{x_{i-1}}$ is not included in $\{x_{(i-2)^+}, \ldots, x_{(i-m-1)^+}\}$. We remark that a walk cannot retract at step i if x_{i-1} is pendant, because, necessarily, $x_i = x_{i-2} = x_{(i-2)^+}$, $\mathcal{V}_{x_{i-1}} = \{x_{(i-2)^+}\}$ and consequently $\mathcal{V}_{x_{i-1}}$ is included in $\{x_{(i-2)^+}, \ldots, x_{(i-m-1)^+}\}$.

Fix a vertex y of G and consider a tabu random walk $(X_n)_{n\geqslant 0}$ on G associated to $FIFO_m$ update rule. We will show that the walk $(X_n)_{n=0}^{H_y^+}$ does not retract. On the event $\{H_y^+ \geqslant 2\}$, for every integer i in the set $\{2, \ldots, H_y^+\}$, the vertex X_i belongs to $\mathcal{L}(T_{i-1})$ if and only if $\mathcal{V}_{X_{i-1}}$ is included in $\mathcal{L}(T_{i-1})$. Yet, for every nonnegative integer i,

$$\mathcal{L}(T_{i-1}) = \{X_{(i-2)^+}, \ldots, X_{(i-m-1)^+}\} .$$

Hence, $(X_n)_{n=0}^{H_y^+}$ retracts at step i if and only if $\mathcal{V}_{X_{i-1}}$ is included in $\mathcal{L}(T_{i-1})$. The proof is by contradiction: we denote by i the first retract of $(X_n)_{n=0}^{H_y^+}$. We set $x = X_{i-1}$ and we denote by $\mathcal{V}_{x,1}$ the set of pendant neighbors of x. As remarked above, x is non-pendant and $\mathcal{V}_x \setminus \mathcal{V}_{x,1}$ is nonempty. Necessarily, this last set is included in $\mathcal{L}(T_{i-1})$. Let j be the last visit before step $i-1$ of non-pendant neighbors of x:

$$j = \min\{k \geqslant 2 \,:\, X_{(i-k)^+} \in \mathcal{V}_x \setminus \mathcal{V}_{x,1}\} \,.$$

Hence, j is even and, if $j \geqslant 4$, then for all integers r in $\{1, \ldots, j/2 - 1\}$, $X_{(i-2r+1)^+} = X_{i-1}$ and $X_{(i-2r)^+}$ belongs to $\mathcal{V}_{x,1}$. Thus, $\mathcal{V}_x \setminus \mathcal{V}_{x,1}$ is included in $\mathcal{L}(T_{(i-j+1)^+})$. By minimality of i, the walk $(X_n)_{n=(i-j-m)^+}^{(i-j+1)^+}$ does not retract. Hence, all the vertices it visits are distinct, except if there exists an integer k in $\{(i-j-m)^+, \ldots, (i-j-1)^+\}$ such that $X_k = X_{k+2}$ and X_{k+1} is pendant. Consider the increasing sequence J of integers formed from $\{(i-j-m)^+, \ldots, (i-j+1)^+\}$ by removing all such integers k. The sequence $(X_m)_{m \in J}$ is a walk that contradicts the m-freeness of G.

Now, consider an update rule R of length k, where the integer k belongs to $\{0, \ldots, m\}$ and fix two vertices x and y of G. We will show that $E_{(x,\varepsilon)} H_y^+(R) \leqslant E_{(x,\varepsilon)} H_y^+(FIFO_m)$, due to possible retracts of the tabu chain $(X_n(R), T_n(R))_{n \geqslant 0}$.

We associate to each walk w from x to y a walk $\widetilde{w}$ from x to y without any retract. The walk $\widetilde{w}$ is the last term of the sequence of walks $(w^r)_{r=0}^{s}$, inductively defined as follows. First, set $w^0 = w$. Suppose that $w^r = (x_n^r)_{n=0}^{\ell_r}$ has been defined for some nonnegative integer r. Let I the set of integers i in $\{0, \ldots, \ell_r - 1\}$ such that w^r has a retract at step i; besides, if $x = y$, then we require that $x_i^r \neq x$. If I is empty, then $\widetilde{w} = w^r$ and $r = s$. Otherwise, let $i = \max I$ and $j = \min\{k \geqslant 2 \,:\, x_i^r = x_{(i-k)^+}^r\}$. We remark that $i > j$ and we define w^{r+1} by removing all steps of w^r from $i - j + 1$ to i:

$$x_n^{r+1} = \begin{cases} x_n^r & \text{if } n \leqslant i - j, \\ x_{n+j}^r & \text{if } n > i - j. \end{cases}$$

The resulting walk $\widetilde{w}$ does not retract.

We denote by $(\widetilde{X_n})_{n=0}^{\widetilde{H}_y^+}$ the random walk from x to y of length $\widetilde{H}_y^+$, obtained by erasing all retracts of the random walk $(X_n(R))_{n=0}^{H_y^+(R)}$ by following the procedure defined in the previous paragraph. By construction, $\widetilde{H}_y^+ \leqslant H_y^+$. We will show by induction that $(\widetilde{X}_n)_{n=0}^{\widetilde{H}_y^+}$ follows the same law than the first $H_y^+(FIFO_m)$ steps of the walk $(X_n(FIFO_m))_{n=0}^{H_y^+(FIFO_m)}$ starting from x with empty tabu list. Therefore, we will derive the equality

$$E_{(x,\varepsilon)}\widetilde{H}_y^+(R) = E_{(x,\varepsilon)}H_y^+(FIFO_m) \ ,$$

which implies $E_{(x,\varepsilon)}H_y^+(R) \leqslant E_{(x,\varepsilon)}H_y^+(FIFO_m)$.

First, both walks start at x with empty tabu list. Second, assume that n is a nonnegative integer such that $\widetilde{H}_y^+ > n$ and that $(\widetilde{X}_k)_{k=0}^n$ has the same law that $(X_k(FIFO_m))_{k=0}^n$. By construction, $\widetilde{X_{n+1}}$ follows the uniform law on $\mathcal{V}_{\widetilde{X_n}} \setminus \{\widetilde{X_{(n-1)^+}}, \dots, \widetilde{X_{(n-m)^+}}\}$ and $X_{n+1}(FIFO_m)$ follows the uniform law on $\mathcal{V}_{X_n} \setminus \{X_{(n-1)^+}, \dots, X_{(n-m)^+}\}$.

Finally, R and $FIFO_k$ yield the same transition kernel if and only if $R^{0^j} = \delta_0$ for every integer j in $\{0, \dots, k-1\}$ and $R^{0^k} = \delta_{k+1}$. Indeed, on every m-free graph, the current vertex never has any occurrence in the current tabu list. Hence, the transition kernel of the tabu chain is given by the probability laws R^{0^j}, where j runs through $\{0, \dots, k\}$.

Reciprocally, consider the clique with vertex set $\{0, \dots, m+2\}$. This graph is $(m+2)$-regular, thus m-free. First, assume that the set

$$\{j \in \{0, \dots, k-1\} \ : \ R^{0^j}(\{0\}) < 1\}$$

is nonempty and let i denote its minimum. We remark that

$$P_{(0,\varepsilon)}(T_{i+1}(R) = (i-1)\cdots 0, \, \forall n \in \{0, \dots, i+1\}, \, X_n(R) = n) > 0 \ .$$

Hence

$$P_{(0,\varepsilon)}(X_{i+2}(R) = i, \, \forall n \in \{0, \dots, i+1\}, \, X_n(R) = n) > 0 \ ,$$

whereas

$$P_{(0,\varepsilon)}(X_{i+2}(FIFO_k) = i, \, \forall n \in \{0, \dots, i+1\}, \, X_n(FIFO_k) = n) = 0 \ .$$

Second, assume that $R^{0^k}(\{k+1\}) < 1\}$ and let j be an integer in $\{1, \dots, k\}$ such that $R^{0^k}(\{j\}) > 0$. Hence

$$P_{(0,\varepsilon)}(X_{k+2}(R) = k+1-j, \, \forall n \in \{0, \dots, k+1\}, \, X_n(R) = n) > 0 \ ,$$

whereas

$$P_{(0,\varepsilon)}(X_{k+2}(FIFO_k) = k+1-j, \, \forall n \in \{0, \dots, k+1\}, \, X_n(FIFO_k) = n) = 0.$$

We have shown that, in both cases, the tabu random walk associated to R that starts at x with empty tabu list does not follow the same law than that associated to $FIFO_k$.

7 Proof of Proposition 16

Assume that R is an update rule of length one distinct from $FIFO_1$ and consider an associated tabu random walk on the flower F_ℓ that starts at the vertex 1 with empty tabu list. Let $a = R^\varepsilon(\{1\})$ and $b = R^0(\{1\})$. The vectors

$$h = \begin{pmatrix} E_{(1,\varepsilon)}H_0(1,r) \\ E_{(2,\varepsilon)}H_0(1,r) \\ E_{(2,1)}H_0(1,r) \\ E_{(3,2)}H_0(1,r) \\ E_{(1,2)}H_0(1,r) \\ E_{(4,2)}H_0(1,r) \end{pmatrix} \quad \text{and} \quad v = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

satisfy $h = Ah + v$, where A is the (6×6)-matrix given by

$$A = \begin{pmatrix} 0 & \frac{2a\ell}{2\ell+1} & \frac{2(1-a)\ell}{2\ell+1} & 0 & 0 & 0 \\ \frac{a}{2} & \frac{a}{2} & 0 & \frac{1-a}{2} & \frac{1-a}{2} & 0 \\ 0 & 0 & b & 1-b & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{(2\ell-1)(1-b)}{2\ell} & \frac{b}{2\ell} & 0 & \frac{2b(\ell-1)}{2\ell} \\ 0 & 0 & 0 & \frac{1-b}{2} & \frac{1}{2} & \frac{b}{2} \end{pmatrix} .$$

hence,

$$E_{(1,\varepsilon)}H_0(R) = \frac{g(a,b,\ell)}{(2-b)(1-b)(2a^2\ell + 2a\ell - 4\ell + a - 2)} ,$$

where $g(a,b,\ell)$ is the following polynomial in (a,b,ℓ):

$$\begin{aligned} &8a^2b^2\ell^2 + 8ab^2\ell^2 - 16b^2\ell^2 - 32a^2b\ell^2 - 32ab\ell^2 + 64b\ell^2 + \\ &+24a^2\ell^2 + 24a\ell^2 - 48\ell^2 + 2ab^2\ell - 8b^2\ell + 10a^2b\ell - 4ab\ell + \\ &+12b\ell - 12a^2\ell + 8a\ell - 8\ell + ab^2 - 2b^2 - 3ab + 6b + 2a - 4 . \end{aligned}$$

therefore,

$$E_{(1,\varepsilon)}H_0(R) \sim_{\ell\to+\infty} \frac{4\ell(3-b)}{2-b} ,$$

while $E_{(1,\varepsilon)}H_0(FIFO_0) = 6\ell + 1$. Since $0 < b < 1$, if ℓ is great enough, then $E_{(1,\varepsilon)}H_0(R) > E_{(1,\varepsilon)}H_0(FIFO_0)$ on F_ℓ.

8 Mean hitting times on cliques

Proposition 10, Corollary 11 and Corollary 12 are proved in this section.

We begin with Proposition 10.

Proof of Proposition 10. Let r be a integer greater than or equal to 2. Consider an update rule R of length m and an associated tabu chain $(X_n, T_n)_{n\geqslant 0}$ on the clique formed by the r vertices $\{0, \dots, r-1\}$. Set $H_0 = E_{(1,\varepsilon)}H_0$, $H_{m+1} = 0$ and for every integer k in $\{1, \dots, m\}$, $H_k = E_{(k+1,k\cdots 1)}H_0$. Hence, for every nonnegative integer k in $\{0, \dots, \min\{m, r-2\}\}$,

$$H_k = 1 + \frac{r-2-k}{r-1-k}\Big((1 - R^{0^k}(\{0\}))H_k + R^{0^k}(\{0\})H_{k+1}\Big) \ .$$

Thus, for every integer k in $\{0, \dots, \min\{m, r-2\} - 1\}$, the hitting time H_k equals $g_k(R^{0^k}(\{0\}), H_{k+1})$. First, assume that $m \leqslant r-2$. Consequently, $H_m = r - 1 - m$ and $E_{(1,\varepsilon)}H_0 = (g_0 \circ \cdots \circ g_{m-1})(r-1-m)$. Second, assume that $m > r-2$. Thus, $H_{r-2} = 1$ and $E_{(1,\varepsilon)}H_0 = (g_0 \circ \cdots \circ g_{r-2})(1)$. □

Proof of Corollary 11. For all integers k in $\{0, \dots, \min\{m, r-2\} - 1\}$ and for all real numbers x, the function $p \mapsto g_k(p, x)$ decreases. □

Proof of Corollary 12. According to Proposition 10, the inequality of mean hitting times $E_{(1,\varepsilon)}H_0(S) < E_{(1,\varepsilon)}H_0(R)$ holds if and only if

$$g_{\ell-1}(S^{0^{\ell-1}}(\{0\}), r-\ell-1) < g_{\ell-1}(R^{0^{\ell-1}}(\{0\}), E_{(\ell+1,\ell\cdots 1)}H_0(R)) \ .$$

Yet,

$$g_{\ell-1}(1, r-\ell-1) = 1 + \frac{(r-\ell-1)^2}{r-\ell}$$

and

$$g_{\ell-1}(0, E_{(\ell+1,\ell\cdots 1)}(R)) = r - \ell \ .$$

By continuity of the function $p \mapsto g_{\ell-1}(p, x)$, if $(S^{0^{\ell-1}}(\{0\}), R^{0^{\ell-1}}(\{0\}))$ is close enough to $(1, 0)$, then the result follows. □

9 Mean hitting times on paddles

Proposition 13 is devoted to the expressions of the mean hitting times $E_{(1,\varepsilon)}H_0(LRU_m)$ and $E_{(1,\varepsilon)}H_0(RAND_m)$ on the paddle G_r, where r is an integer greater than or equal to 3 and m is a nonnegative integer. To obtain such expressions is not an easy task, and we divide the proof of Proposition 13 in several lemmas.

The first lemma deals with the case $m = 0$, corresponding to the simple random walk on the paddle G_r.

Lemma 17. *For every integer $r \geqslant 3$,*

$$E_{(1,\varepsilon)}H_0(LRU_0) = E_{(1,\varepsilon)}H_0(RAND_0) = \begin{cases} 2 & \textit{if } r = 3, \\ r^2 - 5r + 9 & \textit{if } r \geqslant 4. \end{cases}$$

Proof. A direct application of potential theory, see for example the book [3, p. 55], yields

$$E_{(1,\varepsilon)}H_0 = \sum_{x=0}^{r-1} v(x) \deg(x) ,$$

where v is an harmonic function on $\{0, \ldots, r-1\} \setminus \{0, 1\}$ such that $v(0) = 0$ and $\sum_{x \in \mathcal{V}_1}(v(1) - v(x)) = 1$. Since for all integers x in $\{1, \ldots, r-1\}$, $v(x) = 1$, the result follows. □

If m is greater than or equal to $r - 1$, then the mean hitting times $E_{(1,\varepsilon)}H_0(LRU_m)$ and $E_{(1,\varepsilon)}H_0(RAND_m)$ share the following expression.

Lemma 18. *For all integers r and m such that $r \geqslant 3$ and $m \geqslant r - 1$,*

$$E_{(1,\varepsilon)}H_0(LRU_m) = E_{(1,\varepsilon)}H_0(RAND_m) = \frac{r^2 - 3r + 4}{2} .$$

Proof. A tabu random walk on the paddle G_r starting at the vertex 1, that does not hit the vertex 0 at its first move, has the same law for both update rules LRU_m or $RAND_m$ and follows almost surely the three successive steps:

1. The walker loads in its tabu list one occurrence of each vertex in $\{1, \ldots, r-1\}$, having the letter 1 at $(r-1)$-th position and the letter 2 at $(r-2)$-th position.

2. The walker performs a simple random walk starting at the vertex 2 if $r = 3$ or at an uniformly random vertex in $\{3, \ldots, r-1\}$ if $r \geq 4$, until he returns to the vertex 1.

3. The walker goes almost surely to the vertex 0.

By symmetry, the first step reads:

$$E_{(1,\varepsilon)}H_0 = \frac{1}{2}\left(r-1+E_{(r-1,(r-2)\cdots 1)}H_0\right)\ .$$

If $r=3$, then $E_{(2,1)}H_0 = 1+E_{(1,2\cdot 1)}H_0 = 2$ and $E_{(1,\varepsilon)} = 2$. If $r \geqslant 4$, then starting from an uniformly random vertex x in $\{3,\dots,r-1\}$ with tabu set $\{1,\dots,r-1\}\setminus\{x\}$, the hitting time of the vertex 2 follows the geometric law of parameter $1/(r-3)$:

$$E_{(r-1,(r-2)\cdots 1)}H_0 = r-3+E_{(2,(r-1)\cdots 1)}H_0\ .$$

Then,

$$E_{(2,(r-1)\cdots 1)}H_0 = 1+\frac{1}{r-2}E_{(1,(r-1)\cdots 1)}H_0+\frac{r-3}{r-2}E_{(r-1,(r-1)\cdots 1)}H_0\ .$$

Since $\mathcal{V}_1\setminus\{r-1,\dots,1\}=\{0\}$, we get $E_{(1,(r-1)\cdots 1)}H_0 = 1$. Furthermore, $E_{(r-1,(r-2)\cdots 1)}H_0 = E_{(r-1,(r-1)\cdots 1)}H_0$. Consequently $E_{(2,(r-1)\cdots 1)}H_0 = r^2 - 5r+8$, $E_{(r-1,(r-2)\cdots 1)}H_0 = r^2-4r+5$ and $E_{(1,\varepsilon)}H_0 = (r^2-3r+4)/2$. $\square$

We assume that $r \geqslant 3$ and $1 \leqslant m \leqslant r-2$. A tabu random walk on G_r starting at the vertex 1, that does not hit the vertex 0 at its first move, goes almost surely to the vertex 2. Next, if $m \geqslant 2$, the walker stays on the vertices in $\{3,\dots,r-1\}$, until its tabu list is full. By symmetry,

$$E_{(1,\varepsilon)}H_0 = \frac{1}{2}\left(m+1+E_{(m+1,m\cdots 1)}H_0\right)\ .$$

Afterwards, the behavior of the walker differs according to its update rule.

First we deal with LRU_m. We distinguish between the three cases $1 \leqslant m \leqslant r-4$, $m=r-3$ and $m=r-2$.

Lemma 19. *For all integers r and m such that r is greater than or equal to 5 and $1 \leqslant m \leqslant r-4$,*

$$E_{(1,\varepsilon)}H_0(LRU_m) = \frac{1}{2}\left(r^2-4r-mr+2m+8\right)\ .$$

Proof. Starting from $m+1$ with tabu list $m\cdots 1$, the vertices 1 and 2 are almost surely removed from the tabu list in two steps:

$$E_{(m+1,m\cdots 1)}H_0(LRU_m) = 2+E_{(m+3,(m+2)\cdots 3)}H_0(LRU_m)\ .$$

Then, the hitting time of the vertex 2 follows the geometric law of parameter $1/(r-m-3)$:

$$E_{(m+3,(m+2)\cdots 3)}H_0(LRU_m) = r-m-3+E_{(2,(m+2)\cdots 3)}H_0(LRU_m)\ .$$

Next, the walker goes to the vertex 0 in two steps with probability $1/(r-m-2)$ and to a vertex in $\{3,\ldots,r-1\}$ otherwise. In the latter case, the walker must removes the vertex 2 from its tabu list before being able to return to 2 again: the mean hitting time $E_{(2,(m+2)\cdots 3)}H_0(LRU_m)$ equals

$$\frac{2}{r-m-2}+\frac{r-m-3}{r-m-2}\left(m+1+E_{(m+3,(m+2)\cdots 3)}H_0(LRU_m)\right) .$$

We infer that:

$$\begin{aligned}
E_{(2,(m+2)\cdots 3)}H_0(LRU_m) =&2+(r-m-3)(r-2) ,\\
E_{(m+3,(m+2)\cdots 3)}H_0(LRU_m) =&2+(r-m-3)(r-1) ,\\
E_{(m+1,m\cdots 1)}H_0(LRU_m) =&4+(r-m-3)(r-1) ,\\
E_{(1,\varepsilon)}H_0(LRU_m) =&\frac{1}{2}\left(r^2-4r-mr+2m+8\right) .
\end{aligned}$$

□

Lemma 20. *For all integers r greater than or equal to 4,*

$$E_{(1,\varepsilon)}H_0(LRU_{r-3})=\frac{r^2-10}{2(r-3)} .$$

Proof. Starting from $r-2$ with tabu list $(r-3)\cdots 1$, the vertex 1 is almost surely removed from the tabu list in one step:

$$E_{(r-2,(r-3)\cdots 1)}H_0(LRU_{r-3})=1+E_{(r-1,(r-2)\cdots 2)}H_0(LRU_{r-3}) .$$

Each neighbor of $r-1$ has an occurrence in the tabu list, thus

$$\begin{aligned}
E_{(r-1,(r-2)\cdots 2)}H_0(LRU_{r-3}) =&1+\frac{1}{r-3}E_{(2,(r-1)\cdots 3)}H_0(LRU_{r-3})+\\
&+\frac{r-4}{r-3}E_{(r-1,(r-1)\cdots 3)}H_0(LRU_{r-3}) .
\end{aligned}$$

Since $E_{(2,(r-1)\cdots 3)}H_0(LRU_{r-3})=2$ and $E_{(r-1,(r-1)\cdots 3)}H_0(LRU_{r-3})=1+E_{(2,(r-1)\cdots 3)}H_0(LRU_{r-3})=3$, we derive:

$$\begin{aligned}
E_{(r-1,(r-2)\cdots 2)}H_0(LRU_{r-3}) =&\frac{4r-13}{r-3} ,\\
E_{(r-2,(r-3)\cdots 1)}H_0(LRU_{r-3}) =&\frac{5r-16}{r-3} ,\\
E_{(1,\varepsilon)}H_0(LRU_{r-3}) =&\frac{r^2-10}{2(r-3)} .
\end{aligned}$$

□

Finally, we consider the update rule LRU_{r-2}.

Lemma 21. *For all integers r greater than or equal to 3,*

$$E_{(1,\varepsilon)}H_0(LRU_{r-2}) = r - 1 \ .$$

Proof. On the graph G_3, $E_{(1,\varepsilon)}(LRU_{r-2}) = 2$. From now on, assume that $r \geqslant 4$. Hence, the mean hitting time $E_{(r-1,(r-2)\cdots 1)}H_0(LRU_{r-2})$ equals

$$1 + \frac{1}{r-3}E_{(2,(r-1)\cdots 2)}H_0(LRU_{r-2}) + \frac{r-4}{r-3}E_{(r-1,(r-1)\cdots 2)}H_0(LRU_{r-2}) \ .$$

Starting at $r-1$ with tabu list $(r-1)\cdots 2$, the walker performs a simple random walk on the subgraph induced by $\{2, \ldots, r-1\}$ until he reaches the vertex 2. Consequently,

$$E_{(r-1,(r-1)\cdots 2)}H_0(LRU_{r-2}) = r - 3 + E_{(2,(r-1)\cdots 2)}H_0(LRU_{r-2}) \ .$$

It follows $E_{(2,(r-1)\cdots 2)}H_0(LRU_{r-2}) = 2$ and $E_{(r-1,(r-2)\cdots 1)}H_0(LRU_{r-2}) = r - 1$. Thus,

$$E_{(1,\varepsilon)}H_0(LRU_{r-2}) = \frac{1}{2}\left(r - 1 + r - 1\right) = r - 1 \ .$$

□

Second, we deal with the $RAND_m$ update rule. In a first time, we assume that $m = 1$.

Lemma 22. *For all integers r greater than or equal to 3,*

$$E_{(1,\varepsilon)}H_0(RAND_1) = \begin{cases} \frac{7}{3} & \text{if } r = 3, \\ \frac{8r^3-61r^2+184r-195}{6(2r-5)} & \text{if } r \geqslant 4. \end{cases}$$

Proof. If $r = 3$, then $E_{(1,\varepsilon)}H_0(RAND_1) = 7/3$. From now on, assume that $r \geqslant 4$. The vectors

$$h = \begin{pmatrix} E_{(2,1)}H_0(RAND_1) \\ E_{(3,1)}H_0(RAND_1) \\ E_{(3,2)}H_0(RAND_1) \\ E_{(4,3)}H_0(RAND_1) \\ E_{(2,3)}H_0(RAND_1) \\ E_{(1,3)}H_0(RAND_1) \end{pmatrix} \quad \text{and} \quad v = \begin{pmatrix} 1 \\ 1 \\ 2 \\ r-4 \\ 1 + \frac{1}{2(r-3)} \\ 1 \end{pmatrix}$$

satisfy $h = Ah + v$, where A is the (6×6)-matrix given by

$$A = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2(r-3)} & \frac{r-4}{2(r-3)} & 0 & \frac{r-4}{2(r-3)} & \frac{1}{2(r-3)} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{r-4}{2(r-3)} & \frac{r-4}{2(r-3)} & 0 & \frac{1}{2(r-3)} \\ \frac{1}{4} & 0 & 0 & 0 & \frac{1}{4} & 0 \end{pmatrix} .$$

We get

$$h = \frac{1}{3(2r-5)} \begin{pmatrix} 8r^3 - 61r^2 + 172r - 165 \\ 8r^3 - 61r^2 + 163r - 135 \\ 8r^3 - 61r^2 + 169r - 165 \\ 8r^3 - 61r^2 + 157r - 135 \\ 8r^3 - 67r^2 + 196r - 195 \\ 8r^3 - 32r^2 + 98r - 105 \end{pmatrix} ,$$

which yields

$$E_{(1,\varepsilon)} H_0(RAND_1) = \frac{8r^3 - 61r^2 + 184r - 195}{6(2r-5)} .$$

□

Lemma 23. *For all integers r and m such that r is greater than or equal to 6 and $2 \leqslant m \leqslant r-4$, the mean hitting time $E_{(1,\varepsilon)} H_0(RAND_m)$ equals*

$$\frac{f(m,r)}{2(m+1)(2m+1)(2r-m-4)} ,$$

where $f(m,r)$ is the following polynomial in m and r:

$$4m^2r^3 + 8mr^3 + 4r^3 - 6m^3r^2 - 36m^2r^2 - 54mr^2 - 26r^2 + 2m^4r + 34m^3r +$$
$$+ 120m^2r + 146mr + 66r - 6m^4 - 51m^3 - 134m^2 - 140m - 59 .$$

Proof. The vectors

$$h = \begin{pmatrix} E_{(m+1,m\cdots 1)} H_0(RAND_m) \\ E_{(m+2,(m+1)\cdots 3\cdot 1)} H_0(RAND_m) \\ E_{(m+2,(m+1)\cdots 2)} H_0(RAND_m) \\ E_{(m+3,(m+2)\cdots 3)} H_0(RAND_m) \\ E_{(2,(m+1)\cdots 3\cdot 1)} H_0(RAND_m) \\ E_{(2,(m+2)\cdots 3)} H_0(RAND_m) \\ E_{(1,(m+2)\cdots 3)} H_0(RAND_m) \end{pmatrix} \quad \text{and} \quad v = \begin{pmatrix} 1 \\ 1 \\ m+1 \\ r-m-3 \\ 1 \\ 1 + \frac{1}{r-m-2}\frac{m}{m+1} \\ 1 \end{pmatrix}$$

satisfy $h = v + Ah$, where A is the (7×7)-matrix

$$\begin{pmatrix}
\frac{m-1}{m+1} & \frac{1}{m+1} & \frac{1}{m+1} & 0 & 0 & 0 & 0\\
0 & \frac{r-m-3}{r-m-2}\frac{m}{m+1} & 0 & \frac{r-m-3}{r-m-2}\frac{1}{m+1} & \frac{1}{r-m-2}\frac{m}{m+1} & \frac{1}{r-m-2}\frac{1}{m+1} & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 0\\
\frac{m-1}{m+1} & \frac{1}{m+1} & \frac{1}{m+1} & 0 & 0 & 0 & 0\\
0 & 0 & \frac{r-m-3}{r-m-2}\frac{m}{m+1} & \frac{r-m-3}{r-m-2}\frac{1}{m+1} & 0 & 0 & \frac{1}{r-m-2}\frac{1}{m+1}\\
0 & 0 & 0 & 0 & \frac{1}{2}\frac{m}{m+1} & \frac{1}{2}\frac{1}{m+1} & 0
\end{pmatrix}.$$

Hence,

$$\begin{aligned}
h_1 =& \frac{1}{(m+1)(2m+1)(2r-m-4)}\big(4m^2r^3+8mr^3+4r^3-6m^3r^2-\\
&-36m^2r^2-54mr^2-26r^2+2m^4r+30m^3r+110m^2r+138mr+\\
&+64r-4m^4-38m^3-110m^2-123m-55\big)\ ,
\end{aligned}$$

$$\begin{aligned}
h_2 =& \frac{1}{(m+1)(2m+1)(2r-m-4)}\big(4m^2r^3+8mr^3+4r^3-6m^3r^2-\\
&-36m^2r^2-54mr^2-26r^2+2m^4r+28m^3r+103m^2r+131mr+\\
&+62r-2m^4-27m^3-87m^2-104m-50\big)\ ,
\end{aligned}$$

$$\begin{aligned}
h_3 =& \frac{1}{(m+1)(2m+1)(2r-m-4)}\big(4m^2r^3+8mr^3+4r^3-6m^3r^2-\\
&-36m^2r^2-54mr^2-26r^2+2m^4r+28m^3r+107m^2r+137mr+\\
&+64r-4m^4-36m^3-109m^2-125m-56\big)\ ,
\end{aligned}$$

$$\begin{aligned}
h_4 =& \frac{1}{(m+1)(2m+1)(2r-m-4)}\big(4m^2r^3+8mr^3+4r^3-6m^3r^2-\\
&-36m^2r^2-54mr^2-26r^2+2m^4r+24m^3r+97m^2r+129mr+\\
&+62r-2m^4-23m^3-85m^2-108m-52\big)\ ,
\end{aligned}$$

$$\begin{aligned}
h_5 =& \frac{1}{(m+1)(2m+1)(2r-m-4)}\big(4m^2r^3+8mr^3+4r^3-6m^3r^2-\\
&-36m^2r^2-54mr^2-26r^2+2m^4r+30m^3r+110m^2r+138mr+\\
&+64r-4m^4-38m^3-110m^2-123m-55\big)\ ,
\end{aligned}$$

$$
\begin{aligned}
h_6 =&\frac{1}{(m+1)(2m+1)(2r-m-4)}\left(4m^2r^3+8mr^3+4r^3-6m^3r^2-\right.\\
&-40m^2r^2-60mr^2-28r^2+2m^4r+30m^3r+126m^2r+162mr+\\
&\left.+72r-4m^4-40m^3-131m^2-151m-64\right)\ ,
\end{aligned}
$$

$$
\begin{aligned}
h_7 =&\frac{1}{(2m+1)(2r-m-4)}\left(2mr^3+2r^3-3m^2r^2-15mr^2-14r^2+\right.\\
&\left.+m^3r+14m^2r+45mr+38r-2m^3-19m^2-48m-36\right)\ ,
\end{aligned}
$$

Since

$$E_{(1,\varepsilon)}H_0(RAND_m)=\frac{m+1+H_1}{2}\ ,$$

the result follows. $\square$

Then, we treat the case $r \geqslant 5$ and $m = r-3$.

Lemma 24. *For all integers r greater than or equal to 5,*

$$E_{(1,\varepsilon)}H_0(RAND_{r-3})=\frac{20r^3-104r^2+177r-100}{4(3r^2-10r+6)}\ .$$

Proof. The vectors

$$
h=\begin{pmatrix}
E_{(r-2,(r-3)\cdots 1)}H_0(RAND_{r-3})\\
E_{(r-2,(r-2)\cdots 2)}H_0(RAND_{r-3})\\
E_{(r-1,(r-2)\cdots 3\cdot 1)}H_0(RAND_{r-3})\\
E_{(r-1,(r-2)\cdots 2)}H_0(RAND_{r-3})\\
E_{(r-1,(r-1)\cdots 3)}H_0(RAND_{r-3})\\
E_{(2,(r-2)\cdots 3\cdot 1)}H_0(RAND_{r-3})\\
E_{(2,(r-2)\cdots 2)}H_0(RAND_{r-3})\\
E_{(2,(r-1)\cdots 3)}H_0(RAND_{r-3})\\
E_{(1,(r-1)\cdots 3)}H_0(RAND_{r-3})
\end{pmatrix}
\quad\text{and}\quad
v=\begin{pmatrix}
1\\ 1\\ 1\\ 1\\ 1\\ 1\\ \frac{3}{2}\\ 1+\frac{r-3}{r-2}\\ 1
\end{pmatrix}
$$

satisfy $h = v + Ah$, where A is the (9×9)-matrix

$$
\begin{pmatrix}
\frac{r-4}{r-2} & 0 & \frac{1}{r-2} & \frac{1}{r-2} & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & \frac{r-3}{r-2} & 0 & \frac{1}{r-2} & 0\\
0 & \frac{(r-4)^2}{(r-3)(r-2)} & 0 & \frac{r-4}{(r-3)(r-2)} & \frac{r-4}{(r-3)(r-2)} & 0 & \frac{1}{r-2} & \frac{1}{(r-3)(r-2)} & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\
\frac{r-4}{r-2} & 0 & \frac{1}{r-2} & \frac{1}{r-2} & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{r-2}\\
0 & 0 & 0 & 0 & 0 & \frac{r-3}{2(r-2)} & 0 & \frac{1}{2(r-2)} & 0
\end{pmatrix}.
$$

Hence,

$$h = \frac{1}{3r^2-10r+6}\begin{pmatrix} \frac{14r^3-72r^2+125r-76}{2} \\ 4r^3-13r^2+5r+8 \\ 7r^3-40r^2+84r-66 \\ (r-2)(4r^2-8r-1) \\ \frac{25r^2-89r+72}{2} \\ \frac{14r^3-72r^2+125r-76}{2} \\ \frac{4r^3-7r^2-15r+20}{2} \\ \frac{19r^2-69r+60}{2} \\ \frac{7r^3-37r^2+74r-60}{2} \end{pmatrix}$$

and we derive:

$$E_{(1,\varepsilon)}H_0(RAND_{r-3}) = \frac{20r^3-104r^2+177r-100}{4(3r^2-10r+6)}\ .$$

□

Finally, we treat the case $r \geqslant 4$ and $m = r-2$.

Lemma 25. *For all integers r greater than or equal to 4,*

$$E_{(1,\varepsilon)}H_0(RAND_{r-2}) = \begin{cases} 4 & \textit{if } r=4, \\ \frac{(r-1)(8r^2-28r+27)}{4(r-2)r} & \textit{if } r \geqslant 5. \end{cases}$$

Proof. If $r=4$, then the mean hitting time $E_{(1,\varepsilon)}H_0(RAND_2)$ equls 4. From now on, we assume that $r \geqslant 5$. We remark that $E_{(2,(r-1)\cdots 2)} = 2$ and

$$E_{(r-1,(r-1)\cdots 2)}H_0(RAND_{r-2}) = r-3+E_{(2,(r-1)\cdots 2)}H_0(RAND_{r-2})\ ,$$

hence $E_{(r-1,(r-1)\cdots 2)}H_0(RAND_{r-2}) = r-1$. The vectors

$$h = \begin{pmatrix} E_{(r-1,(r-2)\cdots 1)}H_0(RAND_{r-2}) \\ E_{(r-1,(r-1)\cdots 3\cdot 1)}H_0(RAND_{r-2}) \\ E_{(r-2,(r-2)\cdots 1)}H_0(RAND_{r-2}) \\ E_{(2,(r-2)\cdots 1)}H_0(RAND_{r-2}) \\ E_{(2,(r-1)\cdots 3\cdot 1)}H_0(RAND_{r-2}) \\ E_{(1,(r-1)\cdots 3\cdot 1)}H_0(RAND_{r-2}) \end{pmatrix} \quad \text{and} \quad v = \begin{pmatrix} 1+\frac{2}{(r-3)(r-1)}+\frac{r-4}{r-3} \\ 1 \\ 1 \\ 1 \\ 1+\frac{1}{r-1}+\frac{r-3}{r-2} \\ \frac{3}{2} \end{pmatrix}$$

satisfy $h = v + Ah$, where A is the (6×6)-matrix

$$\begin{pmatrix} \frac{r-4}{(r-3)(r-1)} & \frac{r-4}{(r-3)(r-1)} & \frac{(r-4)^2}{(r-3)(r-1)} & \frac{1}{r-1} & \frac{1}{(r-3)(r-1)} & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{r-3}{(r-2)(r-1)} & \frac{r-3}{(r-2)(r-1)} & \frac{(r-3)(r-4)}{(r-2)(r-1)} & 0 & 0 & \frac{1}{(r-2)(r-1)} \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \end{pmatrix}.$$

Consequently,

$$h = \begin{pmatrix} \frac{3(r-1)(2r^2-8r+9)}{2(r-2)r} \\ \frac{3r^3-17r^2+37r-27}{(r-2)r} \\ \frac{6r^3-28r^2+47r-27}{2(r-2)r} \\ \frac{6r^3-28r^2+47r-27}{2(r-2)r} \\ \frac{3(r^3-6r^2+13r-9)}{(r-2)r} \\ \frac{3(r^3-5r^2+11r-9)}{2(r-2)r} \end{pmatrix}$$

and we infer that

$$E_{(1,\varepsilon)} H_0(RAND_{r-2}) = \frac{(r-1)(8r^2-28r+27)}{4(r-2)r}.$$

□

10 Mean hitting times on lines

If the length m of the update rule is zero, then every tabu random walk is a simple random walk. Direct application of potential theory, see for example the book [3, p. 55], yields the equality $E_{(1,\varepsilon)} H_0 = 2r - 3$.

Now, we assume that m is a positive integer. A tabu random walk $(X_n)_{n \geqslant 0}$ with LRU_m or $FIFO_m$ update rule that starts at the vertex 1 and does not hit the vertex 0 at first move follows almost surely the three successive steps.

1. The walker hits the vertex $r-1$ without backtracking.
2. The walker performs a simple random walk starting at $r-1$, until he hits a vertex whose neighborhood is not included in the current tabu set.
3. The walker goes to the vertex 0 without backtracking.

Fix a positive integer m. Let $s = \max\{r-m, 1\}$ and $H = \inf\{n \geqslant 0 : X_n \leqslant \min \mathcal{L}(T_n)\}$. The random variable H is a stopping time for the natural filtration associated to the tabu random walk and the stochastic process $(\min \mathcal{L}(T_n))_{n=0}^{H}$ is $P_{(r-1,(r-2)\cdots s)}$-almost surely non decreasing. Thus, H is $P_{(r-1,(r-2)\cdots s)}$-almost surely finite and

$$E_{(1,\varepsilon)} H_0 = \frac{1}{2} + \frac{1}{2}\left(r - 2 + E_{(r-1,(r-2)\cdots s)}(H + X_H)\right) .$$

First, we treat the LRU_m update rule. We remark that

$$P_{(r-1,(r-2)\cdots s)}\left(\forall n \in \{0, \ldots, H\}, \min \mathcal{L}(T_n)(LRU_m) = s\right) = 1 .$$

Thus, $P_{(r-1,(r-2)\cdots s)}(X_H(LRU_m) = s) = 1$, $E_{(r-1,(r-2)\cdots s)} H(LRU_m) = (r - 1 - s)^2$ and

$$E_{(1,\varepsilon)} H_0(LRU_m) = \frac{1}{2}\left(r - 1 + (r-1-s)^2 + s\right) .$$

Second, we deal with the $FIFO_m$ update rule. We remark that

$$P_{(r-1,(r-2)\cdots s)}\left(\forall n \in \{0, \ldots, H\}, \min \mathcal{L}(T_n)(FIFO_m) = s + n\right) .$$

Hence, if $m > 2$, then $E_{(r-1,(r-2)\cdots s)} H(FIFO_m) < E_{(r-1,(r-2)\cdots s)} H(LRU_m)$, which yields

$$E_{(1,\varepsilon)} H_0(FIFO_m) < E_{(1,\varepsilon)} H_0(LRU_m) .$$

As far as we know, there is no convenient expression for the mean hitting time $E_{(r-1,(r-2)\cdots s)} H(FIFO_m)$. However, we will prove that for every positive integer k,

$$E_{(1,\varepsilon)} H_0(FIFO_{2k-1}) = E_{(1,\varepsilon)} H_0(FIFO_{2k}) < E_{(1,\varepsilon)} H_0(FIFO_{2k+1}) .$$

Indeed, assume that

$$(X_0(FIFO_m), T_0(FIFO_m)) = (r-1, (r-2)\cdots s)$$

and

$$(X_0(FIFO_{m+1}), T_0(FIFO_{m+1})) = (r-1, (r-2)\cdots \max\{s-1, 1\}) .$$

To keep the notations easier, we set $H(m) = H(FIFO_m)$. The hitting time $H(m)$ is stochastically dominated by the hitting time $H(m+1)$ and the random variable $X_{H(m)}(FIFO_m)$ is stochastically dominated by the random variable $X_{H(m+1)}(FIFO_{m+1})$. More precisely, the almost surely finite sequences $(X_n(FIFO_{m+1}))_{n=0}^{H(m)}$ and $(X_n(FIFO_m))_{n=0}^{H(m)}$ follow the same law and without loss of generality, we may suppose that

$$(X_n(FIFO_{m+1}))_{n=0}^{H(m)} = (X_n(FIFO_m))_{n=0}^{H(m)} .$$

If m is odd, then $H(m+1) = H(m)$ and

$$X_{H(m)}(FIFO_{m+1}) = X_{H(m)}(FIFO_m) .$$

In contrary, assume that m is even. The event $\mathcal{A}$ defined by

$$\mathcal{A} = \{r - 2 + H(m) > m\}$$

has positive probability. On the event $\mathcal{A}^c$, $H(m+1) = H(m)$ and

$$X_{H(m)}(FIFO_{m+1}) = X_{H(m)}(FIFO_m) ,$$

whereas on the event $\mathcal{A}$,

$$X_{H(m)}(FIFO_{m+1}) = \min \mathcal{L}(T_n(FIFO_{m+1})) + 1$$

and $H(m+1) = H(m) + \max\{r - 1 - H(m), Y\}$, where Y follows the geometric law with parameter $1/2$.

We summarize our results. On one hand,

$$E_{(1,\varepsilon)} H_0(LRU_m) = \begin{cases} 2r - 3 & \text{if } m = 0, \\ \frac{1}{2}\left(m^2 - 3m + 2r\right) & \text{if } 1 \leqslant m < r - 1, \\ \frac{1}{2}\left(r^2 - 3r + 4\right) & \text{if } m \geqslant r - 1. \end{cases}$$

On the other hand, for every positive integer k,

$$E_{(1,\varepsilon)} H_0(FIFO_{2k-1}) = E_{(1,\varepsilon)} H_0(FIFO_{2k}) < E_{(1,\varepsilon)} H_0(FIFO_{2k+1}) .$$

Besides, $E_{(1,\varepsilon)} H_0(FIFO_m) < E_{(1,\varepsilon)} H_0(LRU_m)$ if $m \geqslant 3$ and

$$\lim_{m \to +\infty} E_{(1,\varepsilon)} H_0(FIFO_m) = E_{(1,\varepsilon)} H_0(LRU_r) .$$

11 Mean hitting times on flowers

If the length m of the update rule is zero, then every tabu random walk is a simple random walk. Direct application of potential theory, see for example the book [3, p. 55], yields the equality

$$E_{(1,\varepsilon)} H_0 = \sum_{x=0}^{r-1} v(x) \deg(x) ,$$

where v is an harmonic function on $\{0,\dots,2\ell+1\}\setminus\{0,1\}$ such that $v(0)=0$ and $\sum_{x\in\mathcal{V}_1}(v(1)-v(x))=1$. Since for all integers x in $\{2,\dots,2\ell+1\}$, $v(x)=1$, we get

$$E_{(1,\varepsilon)}H_0(FIFO_0)=E_{(1,\varepsilon)}H_0(LRU_0)=E_{(1,\varepsilon)}H_0(RAND_0)=6\ell+1\ .$$

Consider the update rule $RAND_1$. By symmetry,

$$\begin{aligned}
E_{(1,\varepsilon)}H_0(RAND_1) =&1+\frac{2\ell}{2\ell+1}E_{(2,1)}H_0(RAND_1)\ ,\\
E_{(2,1)}H_0(RAND_1) =&1+\frac{1}{2}\left(E_{2,1}+E_{(3,2)}H_0(RAND_1)\right)\ ,\\
E_{(3,2)}H_0(RAND_1) =&1+E_{(1,2)}H_0(RAND_1)\ ,\\
E_{(1,2)}H_0(RAND_1) =&1+\frac{2(\ell-1)}{4\ell}\left(E_{(4,2)}H_0(RAND_1)+E_{(2,1)}H_0(RAND_1)\right)\\
&+\frac{1}{4\ell}\left(E_{(2,1)}H_0(RAND_1)+E_{(3,2)}H_0(RAND_1)\right)\\
E_{(4,2)}H_0(RAND_1) =&1+\frac{1}{4}\left(E_{(4,2)}H_0(RAND_1)+E_{(3,2)}H_0(RAND_1)\right)\\
&+\frac{1}{2}E_{(1,2)}H_0(RAND_1)\ .
\end{aligned}$$

Hence,

$$E_{(1,\varepsilon)}H_0(RAND_1)=\frac{40\ell^2+8\ell+3}{3(2\ell+1)}\ .$$

When $\ell=5$, we infer that $E_{(1,\varepsilon)}H_0(RAND_1)\approx 31.61$, while a previous expression yields $E_{(1,\varepsilon)}H_0(RAND_0)=31$. Thus, the mean hitting time $E_{(1,\varepsilon)}H_0(RAND_1)$ is strictly greater than $E_{(1,\varepsilon)}H_0(RAND_0)$.

Likewise,

$$E_{(1,\varepsilon)}=1+\frac{2\ell}{2\ell+1}\left(1+E_{(3,2\cdot1)}\right)\ ,$$

and the vectors

$$h=\begin{pmatrix}E_{(3,2\cdot1)}\\E_{(3,3\cdot2)}\\E_{(1,2\cdot1)}\\E_{(1,3\cdot2)}\\E_{(4,3\cdot2)}\\E_{(4,2\cdot1)}\\E_{(5,4\cdot2)}\\E_{(1,4\cdot2)}\\E_{(6,4\cdot2)}\\E_{(2,2\cdot1)}\end{pmatrix}\quad\text{and}\quad v=\begin{pmatrix}1\\1\\1\\1\\1\\1\\1\\1\\1\\1\end{pmatrix}$$

satisfy $h = Ah + v$, where A is the 10×10-matrix given by

$$A = \begin{pmatrix}
\frac{1}{6} & \frac{1}{6} & \frac{1}{3} & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2\ell} & 0 & 0 & 0 & 0 & \frac{2(\ell-1)}{2\ell} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{2(\ell-1)}{3(2\ell-1)} & \frac{4(\ell-1)}{3(2\ell-1)} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\
\frac{1}{3} & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & \frac{2}{3} & 0 & 0 \\
\frac{2}{3(2\ell-1)} & 0 & 0 & 0 & 0 & \frac{4\ell-6}{3(2\ell-1)} & \frac{2}{3(2\ell-1)} & 0 & \frac{2(\ell-2)}{3(2\ell-1)} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{2} & \frac{1}{6} & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

Thus,

$$h = \frac{1}{5(308\ell^2 - 137\ell + 12)} \begin{pmatrix}
11144\ell^3 - 11150\ell^2 + 4287\ell - 621 \\
2(5572\ell^3 - 6541\ell^2 + 2079\ell - 195) \\
11144\ell^3 - 13838\ell^2 + 6471\ell - 1032 \\
(7\ell - 2)(1592\ell^2 - 1634\ell + 225) \\
11144\ell^3 - 12158\ell^2 + 3207\ell - 144 \\
(22288\ell^3 - 19612\ell^2 + 5850\ell - 621)/2 \\
2(5572\ell^3 - 6541\ell^2 + 1809\ell - 90) \\
11144\ell^3 - 14622\ell^2 + 4033\ell - 135 \\
11144\ell^3 - 12158\ell^2 + 3045\ell - 81 \\
11144\ell^3 - 9610\ell^2 + 3602\ell - 561
\end{pmatrix}.$$

Consequently,

$$E_{(1,\varepsilon)}H_0(RAND_2) = \frac{22288\ell^4 - 16140\ell^3 + 7374\ell^2 - 1687\ell + 60}{5(2\ell+1)(308\ell^2 - 137\ell + 12)},$$

which implies $E_{(1,\varepsilon)}H_0(RAND_2) \approx 31.28 > 31$ on the flower F_5. We infer that $E_{(1,\varepsilon)}H_0(RAND_2)$ is strictly greater than $E_{(1,\varepsilon)}H_0(RAND_0)$ on F_5. Similarly, on the flower F_6, $E_{(1,\varepsilon)}H_0(RAND_2) \approx 38.4$ whereas a previous expression yields $E_{(1,\varepsilon)}H_0(RAND_1) \approx 38.23$. Thus $E_{(1,\varepsilon)}H_0(RAND_2)$ is strictly greater than $E_{(1,\varepsilon)}H_0(RAND_2)$ on F_6.

Now, for all positive integers ℓ, on the flower F_ℓ,

$$E_{(1,\varepsilon)}H_0(FIFO_1) = 1 + \frac{2\ell}{2\ell+1}\left(2 + E_{(1,2)}\right)$$

and

$$E_{(1,2)} = 1 + \frac{2\ell - 1}{2\ell}\left(2 + E_{(1,2)}\right).$$

Thus

$$E_{(1,\varepsilon)}H_0(FIFO_1) = \frac{12\ell^2 + 2\ell + 1}{2\ell + 1} \ .$$

Likewise,

$$E_{(1,\varepsilon)}H_0(FIFO_2) = 1 + \frac{2\ell}{2\ell + 1}\left(\frac{5}{2} + E_{(1,3\cdot 2)}\right)$$

and

$$E_{(1,3\cdot 2)}H_0(FIFO_2) = 1 + \frac{2\ell - 2}{2\ell - 1}\left(\frac{5}{2} + E_{(1,3\cdot 2)}H_0(FIFO_2)\right) \ .$$

Therefore,

$$E_{(1,\varepsilon)}H_0(FIFO_2) = \frac{14\ell^2 - 5\ell + 1}{2\ell + 1} \ .$$

Consequently, on the flower F_7, $E_{(1,\varepsilon)}H_0(FIFO_2) \approx 43.47 > 43$, thus $E_{(1,\varepsilon)}H_0(FIFO_2) > E_{(1,\varepsilon)}H_0(FIFO_0)$. Besides, on F_4, $E_{(1,\varepsilon)}H_0(FIFO_2) \approx 22.78 > 22.33 \approx E_{(1,\varepsilon)}H_0(FIFO_2)$.

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