MONOTONE RATIONAL SERIES AND MAX-PLUS ALGEBRAIC MODELS OF REAL-TIME SYSTEMS

Albert Benveniste* Stéphane Gaubert† Claude Jard‡

*IRISA/INRIA, Campus de Beaulieu, 35042 Rennes cedex, France. email: Albert.Benveniste@irisa.fr
†INRIA, Domaine de Voluceau, 78153 Le Chesnay cedex, France. email: Stephane.Gaubert@inria.fr
‡IRISA/CNRS, Campus de Beaulieu, 35042 Rennes cedex, France. email: Claude.Jard@irisa.fr

Abstract

In the modelling of timed discrete event systems, one traditionally uses dater functions, which give completion times, as a function of numbers of events. Dater functions are non-decreasing. We extend this modelling to the case of multiform logical and physical times, which are needed to model concurrent behaviors. We represent event sequences and time instants by words. A dater is a map, which associates to a word a word, or a set of words, and which is non-decreasing for the subword order. The formal series associated with these generalized dater functions live in a finitely presented semiring, which is equipped with some remarkable relations, due to the monotone character of daters. The implementation of this semiring relies on a theory of rational and recognizable series whose coefficients form a non-decreasing sequence in an idempotent semiring, that we sketch. Finally, we apply this formalism to the modelling and analysis of an elementary example of real time system.

Keywords
Max-plus algebra, real-time systems, rational series, performance evaluation.

1 Introduction

In the classical max-plus algebraic modelling of timed event graphs, the three following approaches are common (see [7] and [1, Ch. 5]).

1. *Time domain*. With this modelling, the behavior of a transition is represented by a *counter variable*, which is a non-decreasing map[1] $x : \mathbb{N} \to \mathbb{N}$. The quantity $x(t)$ gives the number of the last firing of the transition occurring before, or exactly at, time t. The counter variables associated with the different transitions of the timed event graph satisfy linear recurrent equations over the min-plus semiring, which can be analysed using min-plus spectral theory (see e.g. [1, §3.7]). Equivalently, the generating series of counter variables are rational series in a single indeterminate, usually denoted δ, with coefficients in the min-plus semiring. Then, performance evaluation and verification issues can be dealt with using the efficient machinery of commutative rational expressions, see [8, Ch. 11] and its min-plus and max-plus versions [11, Ch. VI,§1], [17, 12, 13].

2. *Event domain*. Dually, we can represent the behavior of a transition by a *dater variable*[1], which is a non-decreasing map $x : \mathbb{N} \to \mathbb{N}$, which to an integer k, associates the time of the firing numbered k of the transition. The analysis of dater variables uses linear systems over the max-plus semiring, and rational series in a single indeterminate, usually denoted γ, with coefficients in the max-plus semiring.

3. *Information domain*. A synthesis of these two approaches was made in [7] and [1, Ch. 5], by introducing the notion of *information*. The behavior of a transition is represented by a subset L of $\mathbb{N} \times \mathbb{N}$. An element $(n, t) \in L$ is interpreted as the *information* or *constraint*: "the firing number n occurs at the earliest at time t". Then, the physical behavior of the transition is the earliest that is compatible with the set of informations L. Algebraically, the information (n, t) can be denoted by the monomial $\gamma^n \delta^t$, where γ and δ are two commuting indeterminates. Since the union of informations is represented by the *sum* of formal series, the following relations hold[2]:

$$
\begin{array}{llcll}
\text{(1a)} & \gamma^n \oplus \gamma^p & = & \gamma^{\min(n,p)} & \forall n, p \in \mathbb{N} \ , \\
\text{(1b)} & \delta^s \oplus \delta^t & = & \delta^{\max(s,t)} & \forall s, t \in \mathbb{N} \ .
\end{array}
$$

The $\mathcal{M}_{\text{in}}^{\text{ax}}[[\gamma, \delta]]$ semiring[3] is precisely the quotient of the semiring of series with Boolean coefficients in two commuting indeterminates γ and δ, by the congruence generated by the (eponymous) relations (1). The analysis of timed event graphs reduces to the computation of rational elements in this special semiring.

The isomorphism theorem stated in [1, Lemma 5.16], and the discussion of [1, § 5.5], show that, despite some subtle

This work was partially supported by the European Community through the Esprit LTR-SYRF project (EP 22703) and the TMR network ALAPEDES ("The Algebraic Approach to Performance Evaluation of Discrete Event Systems", RB-FMRX-CT-96-0074).

Proc. of the Fourth Workshop on Discrete Event Systems

[1] Modulo certain modelling precautions detailed in [1, Ch. 5], we may define more generally a counter or dater function as a non-decreasing map $\mathbb{Z} \cup \{\pm\infty\} \to \mathbb{Z} \cup \{\pm\infty\}$.

[2] Indeed, a little thinking should convince the reader that for all t, the two informations: (1): "event n occurs at the earliest at time t and event p occurs at the earliest at time t"; and (2): "event $\min(n, p)$ occurs at the earliest at time t" are identical. This justifies rule (1a). A dual argument can be invoked to justify rule (1b).

[3] In [1, Ch. 5], negative exponents of γ and δ are allowed. But the main theorem (Theorem 5.39) of [1, Ch. 5], which states the equivalence of rational, realizable, and periodic series, requires a causality assumption, which, in loose terms, is equivalent to the non-negativity of the exponents. Hence, there is no real loss of generality in allowing only non-negative exponents, as we will do throughout the paper.

differences, the time domain, event domain, and information domain approaches to the modeling of timed event graphs are essentially equivalent. The main argument in favor of the $\mathcal{M}_{\mathrm{in}}^{\mathrm{ax}}[[\gamma,\delta]]$ formalism is perhaps of an aesthetic nature: in this approach, the physical time (dating events) and the logical time (counting events) play completely symmetric roles.

These three approaches agree on the fact that events and time are measured by integers. However, in many applications, and in particular, in order to represent concurrency phenomena, it is desirable to associate time instants with event sequences, which are *words*. For instance, a word may represent a particular execution of a program, and two distinct executions with different inputs or environments, abstracted by different letters, may have distinct completion times. Hence, the *event domain* approach was extended in [14] to the case of generalized dater functions, which are maps $\Sigma^* \to \mathbb{R}\cup\{-\infty\}$, where Σ^* is the set of words over a finite alphabet Σ. The effective manipulation of these daters relies on the theory of automata with multiplicities [10, Ch. VI] over the max-plus semiring, or equivalently, of non-commutative rational (or recognizable) series [5]. The modeling of discrete event systems via max-plus automata was pursued in [16, 15], where it was shown that max-plus automata can represent "heaps of pieces", i.e. systems with concurrent access to a set of resources. The same observation was done independently in [19, 6].

Since to some extent, the event domain approach has been extended to the case of a multiform logical time, modeled by words, we ask: *does an analogue of the time domain or information domain approaches exist, for concurrent systems ?* We ask, in particular, whether there exists an analogue of the notion of *counter* function, which, in this new context, should determine the *set* of possible events, as a function of the physical time. The importance of counter representations in the verification and analysis of real time systems should be clear: a number of verification issues reduce to checking whether something (good, or bad) can happen within a time period.

We show that there is a conceptually very simple answer to these questions: a timed behavior can be described by a *rational transduction* [4], i.e. by a relation between two monoids, that can be represented by a rational expression. The monotonicity of the behavior can be expressed in terms of the *subword* or *division* order (see [18, Ch. 6]). This answer can probably be regarded as satisfactory, theoretically, and æsthetically. This is less the case from the algorithmic point of view: rational transductions are complex objects, by comparison with the simple rational series of $\mathcal{M}_{\mathrm{in}}^{\mathrm{ax}}[[\gamma,\delta]]$, and much additional work is needed to establish the dictionary from practically relevant verification issues to algebraically tractable problems. We postpone these technical questions to another paper, and we focus here on the basic constructions and properties.

The second author is glad to acknowledge some joint unpublished work with Alessandro Giua, in 1995, on the modeling of discrete event systems via rational transductions: the notion of event-time transduction in §3 has been inspired by this joint work.

2 Monotone Series with Coefficients in an Idempotent Semiring

2.1 Preliminary Definitions

If Σ is a set (alphabet), the free monoid on Σ is, by definition, the set Σ^* of finite words with letters in Σ, equipped with concatenation. A word $w \in \Sigma^*$ can be written as a sequence $w = a_1 \dots a_p$, with $a_1, \dots, a_p \in \Sigma$ and $p \in \mathbb{N}$. We denote by $|w| = p$ the *length* of w. The unit of Σ^* is the empty word (sequence of length 0). It is denoted e. A monoid is free (and finitely generated) iff it is of the form Σ^*, for some (finite) alphabet Σ. If M_1, M_2 are monoids, the *Cartesian product* $M_1 \times M_2$ is the set of couples $(m_1, m_2) \in M_1 \times M_2$, equipped with the componentwise product: $(m_1, m_2)(m_1', m_2') = (m_1 m_1', m_2 m_2')$.

In the sequel, unless otherwise stated, M will denote a finite Cartesian product of free, finitely generated, monoids. We will sometimes write explicitly $M = \Sigma_1^* \times \cdots \times \Sigma_k^*$, where $\Sigma_1, \dots, \Sigma_k$ are finite alphabets. If $w = (w_1, \dots, w_k) \in \Sigma_1^* \times \cdots \times \Sigma_k^*$, we define the *length* of w, $|w| = |w_1| + \cdots + |w_k|$. The *set of generators* of M, denoted by G_M, is by definition, the set of elements with length 1. The monoid M is equipped with the *division* or *subword* order: we say that w is a subword of w', and we write $w \leq w'$ iff there exists a factorization $w' = a_1 b_1 a_2 \dots a_p b_p a_{p+1}$, with $a_1, b_1, \dots, b_p, a_{p+1} \in M$, and $w = b_1 b_2 \dots b_p$. Dually, we say that w' is a *supword* of w. The subword order is compatible with the monoid structure of M: $u \leq v \implies xuy \leq xvy$, for all $x, y, u, v \in M$. Clearly, if $w \leq w'$, then $|w| \leq |w'|$.

Example 1. If $k = 1$, then $M = \Sigma_1^*$, and the subword order on M is the usual subword or division order on Σ_1^* (see [18, Ch. 6]). A subword $w \leq w'$ is obtained by deleting letters of w', e.g. if $\Sigma_1 = \{a, b, c\}$, $w = ab$ is a subword of $w' = accaacb$.

Example 2. If $\Sigma_1 = \{a_1\}, \dots, \Sigma_k = \{a_k\}$ are one letter alphabets, then $M = \Sigma_1^* \times \cdots \times \Sigma_k^*$ can be identified to the additive monoid $\mathbb{N}^k$ (the isomorphism sends $e \times \dots e \times a_i \times e \dots \times e$ to the i-th vector of the canonical basis of $\mathbb{N}^k$). The subword order on M coincides with the ordinary partial order on vectors of $\mathbb{N}^k$.

Let $\mathcal{D}$ denote a semiring[4] whose addition is idempotent: $a \oplus a = a$. We will equip $\mathcal{D}$ with the *canonical* order relation: $a \leq b \iff a \oplus b = b$, which is such that $a \oplus b$ is the least upper bound of $\{a, b\}$. An idempotent semiring $\mathcal{D}$ is *complete* if any non-empty set $X \subset \mathcal{D}$ admits a least upper bound, denoted $\sup X$ or $\bigoplus_{x \in X} x$, and if the right and left distributivity properties hold for arbitrary sums. If $\mathcal{D}, \mathcal{D}'$ are complete idempotent semirings,

[4] A semiring is a set $\mathcal{S}$ equipped with two laws $\oplus$ and $\otimes$, such that $(\mathcal{S}, \oplus)$ is a commutative monoid (the zero element is noted ε), $(\mathcal{S}, \otimes)$ is a monoid (the unit element is noted e), the sum is left and right distributive over the product, and the zero element is absorbing. As usual, we will omit $\otimes$ for the product, writing ab instead of $a \otimes b$, for $a, b \in \mathcal{S}$.

a *morphism* $f : \mathcal{D} \to \mathcal{D}'$ is a morphism of semirings, that preserves infinite sums (i.e. $f(\sup X) = \sup f(X)$, for all $X \subset \mathcal{D}$).

A series with coefficients in $\mathcal{D}$ and indeterminates in M is simply a map $s : M \to \mathcal{D}$. The set of these series is denoted by $\mathcal{D}\langle\langle M\rangle\rangle$. We will represent a series s by a formal sum $s = \bigoplus_{w\in M} s_w w$. If $s_w = \varepsilon$, we need not write the monomial $s_w w$ in the sum. In particular, we will identify an element $w \in M$ with the series ew. We will identify a subset $X \subset M$ to its indicator series: $\bigoplus_{w\in X} w$. If s_w is zero for all but finitely many values of $w \in M$, we say that s is a *polynomial*. The set $\mathcal{D}\langle\langle M\rangle\rangle$, equipped with the componentwise sum $(s \oplus s')_w = s_w \oplus s'_w$ and Cauchy product $(ss')_w = \bigoplus_{uv=w} s_u s'_v$ is clearly a complete idempotent semiring. We will also use the *shuffle product* $\circ$, which can be defined inductively by the following rule, which holds for all $u, v, w \in M$ and $a, b \in G_M$,

$$w \circ e = e \circ w = w \ , \tag{2a}$$

$$au \circ bv = a(u \circ bv) \oplus b(au \circ v) \ , \tag{2b}$$

and then for all $s, s' \in \mathcal{D}\langle\langle M\rangle\rangle$,

$$s \circ s' \stackrel{\text{def}}{=} \bigoplus_{w,w'\in M} s_w s'_{w'} w \circ w' \ . \tag{2c}$$

Contrary to the Cauchy product, the shuffle product is always commutative. Due to the idempotency of addition, if M is commutative, the shuffle and Cauchy products coincide.

Example 3. If $M = \Sigma_1^*$, the shuffle of two words $w = a_1 \dots a_p$ and $w' = b_1 \dots b_q$ is simply the sum of words of the form $c_1 \dots c_{p+q}$, such that $w = c_{i_1} \dots c_{i_p}$ for some $1 \le i_1 < \dots < i_p \le p+q$, and $w' = c_{j_1} \dots c_{j_q}$, where $1 \le j_1 < \dots < j_q \le p+q$ denote the complementary subsequence of $i_1, \dots, i_p$ in $1, \dots, p+q$. E.g., if $\Sigma_1 = \{a, b, c\}$, $a \circ bc = abc \oplus bac \oplus bca$.

Remark 4. If M is not a Cartesian product of free monoids, it may be meaningless to define the shuffle product by (2). Indeed, in this case, two elements w and w' will have different factorizations as a product of generators and the result of a recursive application of (2) may depend of these factorizations. E.g., consider the monoid M with generators a, b, c and relation $ac = ca$. Formally, M can be defined as the quotient of the free monoid $\{a, b, c\}^*$ by the least congruence (equivalence relation compatible with the monoid structure) $\sim$ such that $ac \sim ca$. This is an example of *partially commutative monoid*, or *trace monoid* [9]. In this monoid, $ac = ca$, but $abc \ne cba$. An application of (2) yields:

$$ac \circ b = a(c \circ b) \oplus b(ac \circ e) = acb \oplus abc \oplus bac \ ,$$
$$ca \circ b = c(a \circ b) \oplus b(ca \circ e) = cab \oplus cba \oplus bca \ .$$

Since $cab = acb$, $bca = bac$, but $cba \ne abc$, these two series differ.

2.2 Non-decreasing Series

We say that a series $s \in \mathcal{D}\langle\langle M\rangle\rangle$ is *non-decreasing* if, for all $u, v \in M$, $u \le v \implies s_u \le s_v$ (in the left hand side, $\le$ denotes the subword order of M, in the right hand side, $\le$ denotes the canonical order of $\mathcal{D}$). We denote by $\mathcal{D}^{\uparrow}\langle\langle M\rangle\rangle$ the set of non-decreasing series.

We will call *non-decreasing* envelope of a series $s \in \mathcal{D}\langle\langle M\rangle\rangle$ the series $s^{\uparrow}$, which is defined by:

$$s^{\uparrow}_u = \bigoplus_{v\le u} s_v \ .$$

The proof of the following four results is detailed in [2].

PROPOSITION 5. *The series $s^{\uparrow}$ is the minimal non-decreasing series that is above s. Moreover, $s^{\uparrow} = s \circ M$.*

PROPOSITION 6. *The set of non-decreasing series $\mathcal{D}^{\uparrow}\langle\langle M\rangle\rangle$, equipped with sum and Cauchy product, is a complete idempotent semiring.*

PROPOSITION 7. *The map $s \mapsto s^{\uparrow}$ is a surjective morphism of complete idempotent semirings $\mathcal{D}\langle\langle M\rangle\rangle \to \mathcal{D}^{\uparrow}\langle\langle M\rangle\rangle$.*

Let $\equiv_{\min}$ denote the congruence[5] generated by the relations: $\forall a \in G_M,\ \ e \oplus a = e$.

THEOREM 8. *The complete idempotent semirings $\mathcal{D}^{\uparrow}\langle\langle M\rangle\rangle$ and $\mathcal{D}\langle\langle M\rangle\rangle / \equiv_{\min}$ are isomorphic.*

From the algorithmic point of view, we are interested in series which are produced by a suitable finite device. The two following notions are instrumental. We say that a series $s \in \mathcal{D}\langle\langle M\rangle\rangle$ is *recognizable* if there exists an integer n, a morphism of multiplicative monoids $\mu : M \to \mathcal{D}^{n\times n}$, a row vector $\alpha \in \mathcal{D}^{1\times n}$ and a column vector $\beta \in \mathcal{D}^{n\times 1}$ such that $s_w = \alpha\mu(w)\beta$. The triple (α, μ, β) is called a *linear representation* of s. We denote by $\mathcal{D}_{\text{rec}}\langle\langle M\rangle\rangle$ the set of recognizable series. A *rational expression* is obtained by a finite number of applications of the grammar rule[6]:

$$X \mapsto X \oplus X,\ XX,\ X^*,\ \lambda a \ , \tag{4}$$

where X is a variable, and $\lambda \in \mathcal{D}$, $a \in G_M$ are arbitrary elements. A series is *rational* if it is given by a rational expression. We denote by $\mathcal{D}_{\text{rat}}\langle\langle M\rangle\rangle$ the set of rational series. E.g., if $\mathcal{D} = (\mathbb{N} \cup \{-\infty\}, \max, +)$ and $M = \{a, b\}^*$, the series $2a(a \oplus 5b)^* \oplus 3a = 3a \oplus 2a^2 \oplus 7ab \oplus \dots$ is rational.

Classically, the Kleene-Schützenberger theorem [5] states that when M is finitely generated, $\mathcal{D}_{\text{rec}}\langle\langle M\rangle\rangle \subset \mathcal{D}_{\text{rat}}\langle\langle M\rangle\rangle$, and that the equality holds when M is free. We next state a non-decreasing analogue of this theorem, whose (simple) proof is detailed in [2]. A *supword-closed* rational expression is defined as in (4), replacing λa by $\lambda a \circ M$.

[5] A congruence of complete idempotent semiring $\mathcal{S}$ is an equivalence relation $\sim$ such that, for all (possibly infinite) families $\{x_i\}_{i\in I}, \{y_i\}_{i\in I} \subset \mathcal{S}$; if $x_i \sim y_i$ holds for all $i \in I$, then $\bigoplus_{i\in I} x_i \sim \bigoplus_{i\in I} y_i$; and for all $x, y, z \in \mathcal{S}$, $x \sim y$ implies $xz \sim yz$ and $zx \sim zy$. The congruence generated by a family of relations $u_j = v_j$, $j \in J$, is the intersection of the congruences that contain the couples (u_j, v_j), for all $j \in J$ (we identify a relation to its graph, which is a subset of $\mathcal{S}^2$).

[6] Recall that in any complete idempotent semiring, the star operation $s^* = e \oplus s \oplus s^2 \oplus \cdots$ is well defined.

THEOREM 9. *Let M denote a Cartesian product of free monoids, and let $s \in \mathcal{D}\langle\langle M\rangle\rangle$, where $\mathcal{D}$ is a complete idempotent semiring. The following assertions are equivalent:*

1. s is rational and non-decreasing;

2. s can be written as a supword-closed rational expression;

3. s is non-decreasing, and there exists a rational series in the equivalence class of s modulo $\equiv_{\min}$.

Moreover, if M is free and finitely generated, the above conditions are equivalent to any of the following:

4. s is recognizable and non-decreasing;

5. s admits a linear representation (α, μ, β) such that for all $a \in G_M$, $\mu(a) \geq I$ (where I denotes the identity matrix);

6. s is non-decreasing, and there exists a recognizable series in the equivalence class of s modulo $\equiv_{\min}$.

2.3 Non-increasing Series

Dually, we say that a series $s \in \mathcal{D}\langle\langle M\rangle\rangle$ is *non-increasing* if for all $u, v \in M$, $u \leq v \implies s_u \geq s_v$. The role of the shuffle product by M will be played, in the case of non-increasing series, by the *Magnus transformation*[7] $\mathfrak{m}$ which is defined by $\mathfrak{m}(a) = e \oplus a$, for all generators $a \in G_M$. It is extended to an element $w = a_1 \dots a_p \in M$, with $a_1, \dots, a_p \in G_M$, by setting

$$\mathfrak{m}(w) = \mathfrak{m}(a_1) \dots \mathfrak{m}(a_p) = \bigoplus_{u \leq w} u \quad . \tag{5a}$$

If s is a series, we define

$$\mathfrak{m}(s) = \bigoplus_{w \in M} s_w \mathfrak{m}(w) \quad . \tag{5b}$$

We next state the analogue of the results of section 2.2 (the proofs are similar).

PROPOSITION 10. *Let $s \in \mathcal{D}\langle\langle M\rangle\rangle$. The series*

$$s^{\downarrow}_w = \bigoplus_{w' \geq w} s_{w'}$$

is the minimal non-increasing series that is above s. Moreover $s^{\downarrow} = \mathfrak{m}(s)$.

PROPOSITION 11. *The map $s \mapsto s^{\downarrow}$ is a surjective morphism of complete idempotent semirings $\mathcal{D}\langle\langle M\rangle\rangle \to \mathcal{D}^{\downarrow}\langle\langle M\rangle\rangle$.*

Let $\equiv_{\max}$ denote the congruence generated by the relations: $e \oplus a = a, \ \forall a \in G_M$.

THEOREM 12. *The complete idempotent semirings $\mathcal{D}^{\downarrow}\langle\langle M\rangle\rangle$ and $\mathcal{D}\langle\langle M\rangle\rangle / \equiv_{\max}$ are isomorphic.*

A *subword closed* rational expression is defined as in (4), replacing λa by $\mathfrak{m}(\lambda a) = \lambda(e \oplus a)$.

[7]This transformation is borrowed to the theory of group presentations. In [18, Ch. 6], it is used to count subwords.

THEOREM 13. *Let M denote a Cartesian product of free monoids, and let $s \in \mathcal{D}\langle\langle M\rangle\rangle$, where $\mathcal{D}$ is a complete idempotent semiring. The following assertions are equivalent:*

1. s is rational and non-increasing;

2. s can be written as a subword-closed rational expression;

3. s is non-increasing, and there exists a rational series in the equivalence class of s modulo $\equiv_{\max}$.

Note that there is an essential lack of symmetry here: there is no natural non-increasing analogue of the second half of Theorem 9, relative to recognizable series.

3 Event-Time Transductions

3.1 The $\mathcal{M}_{\text{in}}^{\text{ax}}\langle\langle \mathcal{E} \times \mathcal{T}\rangle\rangle$ Semiring

Let $\mathcal{E}$, $\mathcal{T}$ denote two finite Cartesian products of free, finitely generated monoids. An element $w \in \mathcal{E}$ will be interpreted as a *sequence of events*, an element $t \in \mathcal{T}$ will be interpreted as a completion time. We call *time behavior* or *event-time transduction* an arbitrary subset

$$x \subset \mathcal{E} \times \mathcal{T} \quad .$$

We will adopt the following semantics: $(w, t) \in x$ represents the information: "the event sequence w is completed at the earliest at time t".

In particular, if γ and δ denote two indeterminates, and if $\mathcal{E} = \{\gamma\}^* \simeq \mathbb{N}$, $\mathcal{T} = \{\delta\}^* \simeq \mathbb{N}$, we obtain exactly the semantics of [1]. For instance, if x represents the behavior of a transition of a timed event graph $(\gamma^n, \delta^t) \in x$, with $n, t \in \mathbb{N}$, simply mean: the firing numbered n of the transition occurs at the earliest at time t.

We say that a time behavior $x \subset \mathcal{E} \times \mathcal{T}$ is *monotone* if

$$(w, t) \in x \text{ and } w \leq w' \implies (w', t) \in x \ , \tag{6a}$$

$$(w, t) \in x \text{ and } t \geq t' \implies (w, t') \in x \ . \tag{6b}$$

These two assumptions are consistent with the above semantics. Indeed, if the sequence w is completed at the earliest at time t, a fortiori, we can say that it is completed at the earliest at any time $t' \leq t$, which justifies the second rule. A dual argument justifies the first rule.

Let $\mathbb{B} = \{\varepsilon, e\}$ denote the Boolean semiring. The $\mathcal{M}_{\text{in}}^{\text{ax}}\langle\langle \mathcal{E} \times \mathcal{T}\rangle\rangle$ semiring is the quotient of the semiring of formal series $\mathbb{B}\langle\langle \mathcal{E} \times \mathcal{T}\rangle\rangle$ by the congruence $\equiv_{\max}^{\min}$ generated by the relations:

$$\forall a \in G_{\mathcal{E}}, \quad e \oplus a \quad = \quad e \tag{7a}$$

$$\forall t \in G_{\mathcal{T}}, \quad e \oplus t \quad = \quad t \ . \tag{7b}$$

In $\mathbb{B}\langle\langle \mathcal{E} \times \mathcal{T}\rangle\rangle$, we define the Magnus transformation, so that it only affects the $\mathcal{T}$-coordinate:

$$\forall t \in G_{\mathcal{T}}, \mathfrak{m}(t) = e \oplus t, \forall a \in \mathcal{E}, \mathfrak{m}(a) = a \ ,$$

and we extend $\mathfrak{m}$ to series by (5b). The following result establishes a bijective correspondence between monotone behaviors and elements of $\mathcal{M}_{\text{in}}^{\text{ax}}\langle\langle \mathcal{E} \times \mathcal{T}\rangle\rangle$.

THEOREM 14. *A series $s \in \mathbb{B}\langle\langle \mathcal{E} \times \mathcal{T} \rangle\rangle$ has a unique monotone representative modulo $\equiv^{\min}_{\max}$, namely*

$$\mathfrak{m}(s) \circ \mathcal{E} = \mathfrak{m}(s \circ \mathcal{E}) \ , \tag{8}$$

where $\mathcal{E}$ is identified to $\mathcal{E} \times e$.

We are particularly interested in the following questions. 1). What is the set x^d_w of *execution time constraints* that an event sequence w carries ? Formally,

$$x^d_w = \{t \in \mathcal{T} \mid (w, t) \in x\} \ .$$

2). What is the set of events x^c_t that are constrained at time t? Formally,

$$x^c_t = \{w \in \mathcal{E} \mid (w, t) \in x\} \ .$$

We will identify as usual such subsets to their indicator series. Then, using the canonical isomorphisms $\mathbb{B}\langle\langle \mathcal{E} \times \mathcal{T} \rangle\rangle \simeq (\mathbb{B}\langle\langle \mathcal{E} \rangle\rangle)\langle\langle \mathcal{T} \rangle\rangle \simeq (\mathbb{B}\langle\langle \mathcal{T} \rangle\rangle)\langle\langle \mathcal{E} \rangle\rangle$, it is easily seen that the functions x^c and x^d are simply obtained by ordering the series x in $\mathcal{T}$ and $\mathcal{E}$, respectively:

$$x = \bigoplus_{w \in \mathcal{E}} x^d_w w \in (\mathbb{B}\langle\langle \mathcal{T} \rangle\rangle)\langle\langle \mathcal{E} \rangle\rangle \ ,$$

$$x = \bigoplus_{t \in \mathcal{T}} x^c_t t \in (\mathbb{B}\langle\langle \mathcal{E} \rangle\rangle)\langle\langle \mathcal{T} \rangle\rangle \ .$$

The following proposition is an elementary consequence of the definition of monotone time behaviors.

PROPOSITION 15. *Let $x \in \mathbb{B}\langle\langle \mathcal{E} \times \mathcal{T} \rangle\rangle$ denote a monotone time behavior.*

1. *For all $t \in \mathcal{T}$, $x^c_t \in \mathbb{B}^{\uparrow}\langle\langle \mathcal{E} \rangle\rangle$.*
2. *For all $w \in \mathcal{E}$, $x^d_w \in \mathbb{B}^{\downarrow}\langle\langle \mathcal{E} \rangle\rangle$.*
3. *The map $x^c : \mathcal{T} \to \mathbb{B}^{\uparrow}\langle\langle \mathcal{E} \rangle\rangle$ is non-increasing.*
4. *The map $x^d : \mathcal{E} \to \mathbb{B}^{\downarrow}\langle\langle \mathcal{E} \rangle\rangle$ is non-decreasing.*

This allows us to identify a monotone time behavior x (or equivalently, the element of $\mathcal{M}^{\mathrm{ax}}_{\mathrm{in}}\langle\langle \mathcal{E} \times \mathcal{T} \rangle\rangle$ that x represents) to the counter $x^c \in (\mathbb{B}^{\uparrow}\langle\langle \mathcal{E} \rangle\rangle)^{\downarrow}\langle\langle \mathcal{T} \rangle\rangle$ or to the dater $x^d \in (\mathbb{B}^{\downarrow}\langle\langle \mathcal{T} \rangle\rangle)^{\uparrow}\langle\langle \mathcal{E} \rangle\rangle$. Hence, the theory of non-decreasing and non-increasing rational/recognizable series, sketched in sections 2.2,2.3 above, can be applied to elements of $\mathcal{M}^{\mathrm{ax}}_{\mathrm{in}}\langle\langle \mathcal{E} \times \mathcal{T} \rangle\rangle$.

From Theorem 14, it is clear that $\mathfrak{m}(s) \circ \mathcal{E} = \mathfrak{m}(s \circ \mathcal{E})$ is the *maximal* element in the equivalence class of s modulo $\equiv^{\min}_{\max}$. However, to implement the semiring $\mathcal{M}^{\mathrm{ax}}_{\mathrm{in}}\langle\langle \mathcal{E} \times \mathcal{T} \rangle\rangle$, i.e. to code its elements as economically as possible, we rather need a *minimal* representative. The next section gives a partial answer to this problem.

3.2 Minimal Counter Representation of Monotone Behaviors

We first recall a classical (and beautiful) order-theoretical result, which can be found in [18, Ch. 6].

THEOREM 16 (HIGMAN). *Let $\mathcal{E}$ denote a free, finitely generated, monoid. If $\mathcal{X}$ is a subset of $\mathcal{E}$ composed of pairwise incomparable elements for the subword order, then, $\mathcal{X}$ is finite.*

COROLLARY 17 (MINIMAL REPRESENTATION). *For all $x \in \mathbb{B}\langle\langle \mathcal{E} \times \mathcal{T} \rangle\rangle$, there is a minimal function $y : \mathcal{T} \to \mathbb{B}\langle\langle \mathcal{E} \rangle\rangle$, such that*

$$x \equiv_{\min} \bigoplus_{t \in \mathcal{T}} y_t t \ , \tag{9}$$

and for all $t \in \mathcal{T}$, y_t is a polynomial.

This corollary, which needs no rationality or recognizability hypotheses, states in particular that the constraints on the events at time t can always be represented by a *finite* set. There is no simple analogue of this result for minimal *dater* representations, because there are infinite increasing sequences for the subword order.

4 Application to Partial Order Automata Models of Real Time Systems

Consider the partial order automaton depicted in Fig 1. With the action labels a, b, c are associated directed

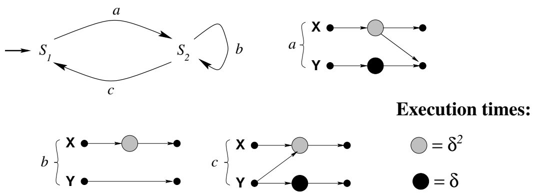


Figure 1: A (timed) partial order automaton.

graphs (partial orders), as shown on Fig. 1. With a word $w = u_1 \dots u_p \in \{a, b, c\}^p$ accepted by the automaton, we associate a partial order, obtained by composing the partial orders associated with $u_1, \dots, u_p$, as illustrated in Fig. 2 for $w = abc$. Black and gray events cost one (δ) or two (δ^2) units of time respectively. The *execution time* $\tau(w)$ of the run w is the maximal number of occurrences of a δ symbol, in a chain in this graph. E.g., we can check by mere inspection of Fig. 2 that $\tau(abc) = 6$. As detailed in [3], such partial order automata are a simple model of real time computations: the Boolean automaton describes the possible action sequences of the program, and each action sets up a dependence relation between data, which is abstracted by a partial order.

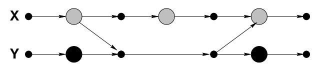
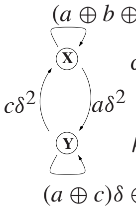


Figure 2: Partial order associated with abc.

The execution time map τ is recognized by the automaton with multiplicities over the max-plus semiring, which is shown in Fig. 3, together with its linear representation α, μ, β, which is such that $\tau(w) = \alpha\mu(w)\beta$, for all $w \in \{a, b, c\}^*$ (e.g., $\tau(abc) = \alpha\mu(a)\mu(b)\mu(c)\beta = 6$). We have to take into account, however, the fact that only



$$\alpha = [0, 0],\ \beta = \alpha^T,\ \mu(a) = \begin{bmatrix} 2 & 2 \\ -\infty & 1 \end{bmatrix}$$

$$\mu(b) = \begin{bmatrix} 2 & -\infty \\ -\infty & 0 \end{bmatrix},\ \mu(c) = \begin{bmatrix} 2 & -\infty \\ 2 & 1 \end{bmatrix}$$

Figure 3: Max-plus automaton.

the sequences accepted by the automaton of Fig. 1 are of interest. The natural way to do this is to compute the *tensor product* of this automaton by the max-plus automaton of Fig. 3 (see [14, §VI] for details), which is depicted in Fig. 4. The timed behavior of the system can be represented by the series $x \in \mathcal{M}_{\text{in}}^{\text{ax}}\langle\langle\{a, b, c\}^* \times \{\delta\}^*\rangle\rangle$ which is obtained by summing the weights of the paths of this graph. Formally, the series x is given by $\alpha'(\mu')^*\beta'$, with $\alpha'_i = e$, for the initial nodes $i = 1, 2$, $\alpha'_i = \varepsilon$, for $i = 3, 4$, $\beta'_i = e$, for the final nodes $i = 1, 2, 3, 4$, and

$$\mu' = \begin{bmatrix} \varepsilon & \varepsilon & a\delta^2 & a\delta^2 \\ \varepsilon & \varepsilon & \varepsilon & a\delta \\ c\delta^2 & \varepsilon & b\delta^2 & \varepsilon \\ c\delta^2 & c\delta & \varepsilon & b \end{bmatrix} .$$

Expanding $(\mu')^*$, or enumerating the paths in Fig. 4, we obtain, thanks to the simplification rules (7),

(10)
$$x = \alpha'(\mu')^*\beta' = e \oplus a\delta^2 \oplus ab\delta^2 \oplus ac\delta^3 \oplus \cdots \quad ,$$

which means: that event a is completed at the earliest at time 2, that the event sequences ab and ac are completed at the earliest at times 2 and 3, respectively. We cannot infer directly from (10) that a is completed *exactly* at time 2, or that ab and ac are completed *exactly* at times 2 and 3, respectively, unless we know that there are no other occurrences of a, ab, ac (or of their subwords) as coefficients of $\delta^4, \delta^5, \ldots$ in (10). This is the case for a which does occur at time 2, but this is not the case for ab and ac. Indeed, pursuing the computation in (10), we get

$$\begin{aligned} x &= e \oplus a\delta^2 \oplus ab\delta^2 \oplus ac\delta^3 \oplus (ac \oplus ab)\delta^4 \oplus \cdots \\ &= e \oplus a\delta^2 \oplus (ab \oplus ac)\delta^4 \oplus \cdots \quad , \end{aligned}$$

which shows that ab and ac occur at the earliest at time 4. In general, when $w\delta^t$ appears in a *minimal* expression of x (which cannot be further simplified using (7)), t can be interpreted as the effective execution time of w.

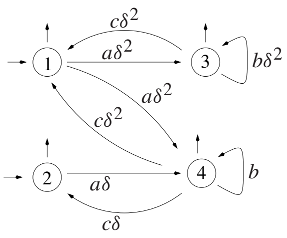


Figure 4: Tensor product automaton

This elementary example only illustrates the case when $\mathcal{E} = \Sigma^*$ and $\mathcal{T} = \{\delta\}^*$. The case of commutative monoids $\mathcal{T}$ and $\mathcal{E}$ was used in [11, Ch. IX] to compute symbolically the throughput of a flexible workshop as a function of the numbers of pallets (represented by different commuting indeterminates). More generally, worst case evaluation problems for partial order automata can be reduced to the (much easier) commutative case. E.g., considering again the example of Fig. 1, we may wish to compute the physical time elapsed, as a function of the numbers of events a. This can be done by replacing letters b, c by the unit e, and by making only states 3, 4 (which have an incoming a) final in Fig. 4. We obtain the series $\overline{x} = a\delta^2 \oplus a^2\delta^*$, which means that an observer ignoring events b and c will see runs in which a occurs at time 2, and event a^2 occurs at an arbitrarily large time. Such worst case evaluations can be done at a reasonable algorithmic price [11, Ch. IX,App. B].

References

[1] F. Baccelli, G. Cohen, G.J. Olsder, and J.P. Quadrat. *Synchronization and Linearity*. Wiley, 1992.

[2] A. Benveniste, S. Gaubert, and C. Jard. Extended version of the present paper. 1998.

[3] A. Benveniste, C. Jard, and S. Gaubert. Algebraic techniques for timed systems. In *Proceedings of CONCUR'98*, Nice, France, September 1998. to appear.

[4] J. Berstel. *Transductions and Context-Free languages*. Teubner, Stuttgart, 1979.

[5] J. Berstel and C. Reutenauer. *Rational Series and their Languages*. Springer, 1988.

[6] M. Brilman and J.M. Vincent. Synchronisation by resources sharing : a performance analysis. Technical report, MAI-IMAG, Grenoble, France, 1995.

[7] G. Cohen, P. Moller, J.P. Quadrat, and M. Viot. Algebraic tools for the performance evaluation of discrete event systems. *IEEE Proceedings: Special issue on Discrete Event Systems*, 77(1), Jan. 1989.

[8] J.H. Conway. *Regular algebra and finite machines*. Chapman and Hall, 1971.

[9] V. Diekert and G. Rosenberg, editors. *The book of traces*. World Scientific Publ., 1995.

[10] S. Eilenberg. *Automata, Languages and Machines*, volume A. Acad. Press, 1974.

[11] S. Gaubert. *Théorie des systèmes linéaires dans les dioïdes*. Thèse, École des Mines de Paris, July 1992.

[12] S. Gaubert. On rational series in one variable over certain dioids. Rapport de recherche 2162, INRIA, Jan. 1994.

[13] S. Gaubert. Rational series over dioids and discrete event systems. In *Proc. of the 11th Conf. on Anal. and Opt. of Systems: Discrete Event Systems*, number 199 in Lect. Notes. in Control and Inf. Sci, Sophia Antipolis, June 1994. Springer.

[14] S. Gaubert. Performance evaluation of (max,+) automata. *IEEE Trans. on Automatic Control*, 40(12), Dec 1995.

[15] S. Gaubert and J. Mairesse. Modelling and analysis of timed petri nets using heaps of pieces. To appear in IEEE-TAC. Also technical report LITP 97/14. Abridged version in the proceedings of the ECC'97, Bruxells, 1997.

[16] S. Gaubert and J. Mairesse. Task resource systems and (max,+) automata. In J. Gunawardena, editor, *Idempotency*, Publications of the Newton Institute. Cambridge University Press, 1998. (accepted in final form in Aug. 1995).

[17] D. Krob and A. Bonnier-Rigny. A complete system of identities for one letter rational expressions with multiplicities in the tropical semiring. *J. Pure Appl. Algebra*, 134:27–50, 1994.

[18] M. Lothaire. *Combinatorics on Words*. Encyclopedia of Mathematics and its applications. Addison-Wesley, 1983. Reprinted 1997, Cambridge University Press.

[19] J.M. Vincent. Some ergodic results on stochastic iterative discrete events systems. *DEDS: Theory and Applications*, 7(2):209–233, 1997.