Automatic circuit equation formulation for non-smooth electrical circuits

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Circuit branch

A branch is described by a pair of branch variables: the tension $U_a$ and the current $I_a$. Moreover, the Branch Constitutive Equation can be expressed in a general implicit form:

$$F(U_a, I_a, \frac{dU_a}{dt}, \frac{dl_a}{dt}, ...) = 0 \quad (1)$$

Some examples with a resistor, a capacitor and a current source:

$$U_a = RI_a \quad I_a = C\frac{dU_a}{dt} \quad I_a = \alpha U_b$$

We will see the different forms of this relation.
The Kirchhoff Current Law (KCL)

At any node in an electrical circuit where charge density is not changing in time, the sum of currents flowing towards that node is equal to the sum of currents flowing away from that node.

\[
\begin{align*}
-I_1 + I_2 + I_3 &= 0 \quad (KCL1) \\
-I_3 + I_4 &= 0 \quad (KCL2) \\
I_1 - I_2 - I_4 &= 0 \quad (KCL0)
\end{align*}
\]

or the matrix formulation:

\[ AI = 0 \]

where A is known as the incidence matrix and I is the vector of branch currents.
The Kirchhoff Voltage Law (KVL)

*The directed sum of the electrical differences around a closed circuit must be zero.*

With this example

\[ U_1 + U_2 = 0 \]
\[ -U_2 + U_3 + U_4 = 0 \]
\[ U_1 + U_3 + U_4 = 0 \]

or the matrix formulation:

\[ BU = 0 \]

where B is known as the loop matrix and U is the vector of branch tensions.
Relation between A and B

\[ BA^t = 0 \]
Relation between A and B

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Proof:

\[ B = \begin{pmatrix} b_1 \\ \vdots \\ b_{n_l} \end{pmatrix}, \quad A = \begin{pmatrix} KCL_1 \\ \vdots \\ KCL_{n_n} \end{pmatrix} \]

Let \( p \in \{1, n_l\} \) and \( q \in \{1, n_n\} \). We proof that \( b_p.KCL^t_q = 0 \).

- The node \( q \) is not on the loop \( p \), then \( b_p \) and \( KCL_q \) have no common non null coordinate.
- The node \( q \) is on the loop \( p \):

\[ b_p = (\ldots 1 \ldots - 1 \ldots), \quad KCL_q = (\ldots 1 \ldots 1 \ldots) \]

The other coordinates are not simultaneous not null. Therefore:

\[ b_p.KCL^t_q = 0 \]
An other KVL formulation

It consists in writing:

\[ \forall p \in \{1, n_b\} \quad U_p = V_{j_p} - V_{k_p} \]

This system of equation implies \( BU = 0 \).

The matrix formulation is:

\[ U - A^t V = 0 \]

where \( V \) is the vector of nodes potential and \( U \) the vector of branches tensions.

⇒ In further consideration, \( BU = 0 \) is eliminated.

Example:

\[ U_1 = V_1 - V_0 \quad U_2 = V_0 - V_1 \]
\[ U_3 = V_2 - V_1 \quad U_4 = V_0 - V_2 \]
The Sparse Tableau Approach STA leads to the following system:

\[ A I = 0 \quad (KCL) \]
\[ U - A^t V = 0 \quad (KVL) \]

For all branches: \( F(U_a, I_a, \ldots) = 0 \quad (BCE) \)

Example:

\[ U_L \quad \text{L} \quad U_C \]
\[ I_L \quad \text{C} \quad I_C \]

The vector of unknowns: \((I_L, I_C, U_L, U_C, V_1)^t\)

\[ -I_L + I_C = 0 \quad (KCL1) \]

\[ U_L + V_1 = 0 \quad U_C - V_1 = 0 \quad (KVL) \]

\[ CU_C' - I_C = 0 \quad LI_C' - U_L = 0 \quad (BCE) \]
A Current-Defined Branch (CD)

The branch is current-defined if its current is a function of its own voltage, controlling variable or their time-derivatives:

\[ I_a = F_i(U_a, U_b, I_c, \frac{dU_a}{dt}, \frac{dU_b}{dt}, \frac{dl_c}{dt}) \] (2)

A Voltage-Defined Branch (VD)

The branch is voltage-defined if its voltage is a function of its own current, controlling variable or their derivatives:

\[ U_a = F_u(I_a, U_b, I_c, \frac{dl_a}{dt}, \frac{dU_b}{dt}, \frac{dl_c}{dt}) \] (3)

Examples:
A resistor is a voltage-defined branch because \( U_a = RI_a \).
A inductor is a voltage-defined branch because \( U_a = L \frac{dl_a}{dt} \).
A capacitor is a current-defined branch because \( I_a = C \frac{dU_a}{dt} \).

MNA Hypothesis:

The M.N.A. assumes that smooth branches are explicit functions of current or voltage. It means each branch is either Voltage Defined or Current Defined.
Modified Nodal Analysis

MNA unknowns

The vector of unknowns contains:

- All node potentials $V_i$
- Currents through all voltage defined branches

The following equations are written:

- The KCL is applied to every node.
- The Branch Constitutive Equation for all voltage defined branches.

Example:

![Circuit Diagram]

The vector of unknowns is: $(V_1, I_L)^t$

$$CV_1' - I_L = 0 \quad LI_L' + V_1 = 0$$
MNA and DAE

MNA leads to a DAE of the form:

\[ C(X)X' + J(X) - S(t) = 0 \]

With

- \( C(X) \) describing the dynamic elements.
- \( J(X) \) describing the static ones.
- \( S(t) \) describing the independent sources.

Index of the DAE: 0, 1, 2.

In SPICE, the discretization of this equation is linearized and solved using the Newton-Raphson iterations. But, the non-smooth components such as diodes or transistors, result in troubles in this iterative method. A novel approach is to model these components with piecewise linear functions and to formulate the problem as a Complementary Problem.
Ideal Diode

Branch Constitutive Equation of the diode:

\[ I_d = I_s (\exp(-U_D/C) - 1) \]

The complementary formulation:

\[ 0 \leq I_d \perp -U_D \geq 0 \]
Ideal Transistor

The complementary formulation:

\[ I_d = (c_1 \ldots c_{10}) \lambda \]

\[ I_d = -I_s \]

\[ Y = A \begin{pmatrix} V_d \\ V_g \\ V_s \end{pmatrix} + I\lambda + C \]

\[ 0 \leq Y \perp \lambda \geq 0 \]
Extended MNA

It consists in replacing the Constitutive Equation of the non-smooth branch with the complementary formulation.

**Unknowns**

Before go ahead, the unknowns vector $X$ is decomposed in:

- $x$ which contains only the dynamic unknowns (currents in inductor and tensions from capacitor branches)
- $Z_s$ which contains only the non dynamic unknowns (Voltage nodes, ...).

A good choice of unknowns yields the following system:

\[
\begin{aligned}
    x' &= A_{2x}x + A_{2zs}Z_s + R\lambda + A_{2s} \\
    0 &= B_{2x}x + B_{2zs}Z_s + B_{2\lambda}\lambda + B_{2s} \\
    Y &= D_{2x}x + D_{2zs}Z_s + D_{2\lambda}\lambda + D_{2s} \\
    0 &\leq Y \perp \lambda \geq 0
\end{aligned}
\]

\(\text{(4)}\) \hspace{1cm} \(\text{(5)}\) \hspace{1cm} \(\text{(6)}\) \hspace{1cm} \(\text{(7)}\)
Mixed Linear Complementary System

Given the matrices $A_x \in \mathbb{R}^{n \times n}$, $A_z \in \mathbb{R}^{n \times p}$, $A_v \in \mathbb{R}^{n \times m}$, $B_x \in \mathbb{R}^{p \times n}$, $B_z \in \mathbb{R}^{p \times p}$, $B_v \in \mathbb{R}^{p \times m}$, $C_x \in \mathbb{R}^{m \times n}$, $C_z \in \mathbb{R}^{m \times p}$, $C_v \in \mathbb{R}^{m \times m}$, and the vectors $a \in \mathbb{R}^n$, $b \in \mathbb{R}^p$, $c \in \mathbb{R}^m$, the explicit MLCS denoted by $EMLCS(A_x, A_z, A_v, B_x, B_z, B_v, C_x, C_z, C_v, a, b, c)$ consists in finding three vectors $x \in \mathbb{R}^n$, $z \in \mathbb{R}^p$ and $v \in \mathbb{R}^m$ such that

\[
\begin{aligned}
    x' &= A_x x + A_z z + A_v v + a \\
    0 &= B_x x + B_z z + B_v v + b \\
    0 &\leq v \perp C_x x + C_z z + C_v v + c \geq 0
\end{aligned}
\]  

(8)

The time discretization of a MLCS leads to:

Mixed Linear Complementary Problem

Given the matrices $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{m \times m}$, $C \in \mathbb{R}^{n \times m}$, $D \in \mathbb{R}^{m \times n}$, and the vectors $a \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, the MLCP denoted by $MLCP(A, B, C, D, a, b)$ consists in finding two vectors $u \in \mathbb{R}^n$ and $v \in \mathbb{R}^m$ such that

\[
\begin{aligned}
    Au + Cv + a &= 0 \\
    0 &\leq v \perp Du + Bv + b \geq 0
\end{aligned}
\]  

(9)

At each step, SICONOS solves a MLCP.
Cost evaluation

Let

- $N_b$ the number of branches.
- $N_n$ the number of nodes.
- $N = \max\{N_b, N_n\}$

The costs of the automatic circuit equation formulation:

- To parse the Netlist: It consists in reading and storing the Netlist. $O(N)$.
- A topological analysis to build the unknowns vector: The Minimum Spanning Tree algorithm complexity is $O(N\log(N))$.
- Stamping method: Each component writes its contribution in the table equation. $O(N)$.
- Matrix product: Multiplied two dense matrices costs $O(N^3)$. 
Diodes Bridge
Buck converter

![Buck converter circuit diagram]

Figure: Buck converter simulation

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Conclusions

Software aspects

The actual software version is able to parse and simulate a Netlist composed of:

- Non constant sources.
- Linear components: resistor, capacitor, inductance.
- Non smooth components: diode, MOS transistor, comparator.
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Theoretical aspects

- Formulation of nonsmooth circuits in MLCS
- New algorithms in Siconos/Numerics for MLCP
Perspectives

Software Aspects

- Forecast active set: use a local piecewise linear approximation to solve a smaller MLCP.
- Integration of the nonlinear SPICE component: MNCP solvers.
- Integration of piecewise smooth model of MOS transistor
- Specific electrical heuristics.
- Improve the component model.
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- Forecast active set: use a local piecewise linear approximation to solve a smaller MLCP.
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Theoretical Aspects

- Topological analysis for the index reduction of the DAE and the relative degree of the EMLCS
- Topological analysis to reduce the number of unknowns.
- Multi–grid algorithm: use a set of embedded piecewise linear approximation (from coarser to finer).
- Other Algorithms for MLCP and MNCP (Nonsmooth Newton methods and Projection methods for VI)
- Adaptive time–stepping scheme