Computing Schedules for Multithreaded Real-Time Programs using Geometry

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Abstract

We describe a novel technique for computing efficient schedules for multi-threaded real-time programs. The technique makes use of abstractions which are constructed by embedding the model of the program in a geometric space and then constructing a decomposition of this space. This embedding uses the model of **PV diagrams**. We introduce a timed version for PV programs and diagrams, which allows us to define the worst-case response time of the schedules, and then to use the geometric abstractions for computing efficient schedules.

1 Introduction

With the decreasing cost of embedded systems, constructing "more intelligent" embedded systems becomes possible. This possibility has been exploited, and now product designers ask for more functionalities from an embedded system. This increases the portion of the software in the system, to the point that some applications require programming with threads—that is, what was before designed as separate hardware components of a chip can now be conceived of as separate threads. These threads can then be executed on one or several (in the case of multi-processing) hardware components. But parallel programming in the real-time context is rather new, and much work is to be done in order to be able to analyse the real-time behaviour of such programs. Indeed, simple extensions of existing analysis tools for sequential programs will not suffice: parallelism with threads involves purely parallel-specific phenomena like deadlocks.

In this paper we examine the behaviour of a class of multi-threaded programs, from the point of view of the **worst-case response time** (WCRT). In order to address this complex issue, we employ a geometric approach, which enables us not only to better "see" what happens, but also to exploit the geometric nature of the model in order to deal with the problem of state explosion arising in the analysis of concurrent programs. We will also address the issue of scheduling parallel programs on a limited number of processors, using the same approach.

Our ideas of exploiting the geometry of parallel programs are inspired by the work on **PV diagrams**. This model for geometrically describing interactions of concurrent processes was introduced by Dijkstra [9]. It has been used, since the beginning of the 90's, for the analysis of concurrent programs [13, 11] (see [15] for a good survey). In particular, in this paper, we follow the spirit of [11], where the geometry of the diagrams is used to construct an efficient analysis algorithm. We use the notion of **timed PV diagrams**, which can be used in the context of real-time concurrent programming. We focus on a particular problem: finding a schedule which is safe (no deadlocks), and quick—that is, though we are not looking for the quickest possible schedule, we want to find one which is as quick as possible within a reasonable computational time. The motivations for this are:

- the program under consideration might be part of a global system and subject to a deadline. However, if no precise timing analysis result is available, then to determine the worst-case response time of the program one must assume the worst case, i.e., all threads share a single protected resource (with a single access). This amounts to sequentialization of the threads, and the WCRT is the sum of the WCRT of each thread considered individually. This measure can easily be greater than the deadline, while the real worst case concurrent behaviour of the program is probably better. The best case is when no resource is shared, then the maximum among the WCRTs of each thread gives the WCRT of the program. Here we are interested in analysing the WCRT of **real** executions of the program.
- Using our method, finding a deadlock-free schedule comes "for free" when looking for a quick schedule.
- Finally, in a more general consideration, if finding a quick schedule is computationally feasible, we believe that it is a good investment to design systems using such efficient schedules. For example, by reducing the computation time one can reduce energy consumption.

The paper is structured as follows. In Section 2 we recall basic definitions and concepts related to PV programs and diagrams. Here, PV diagrams are described in the discrete world—Z^N. In the next section we describe our timed version of PV programs and diagrams. Then we introduce the notion of the worst case response time for a given schedule and discuss its computation. In Section 4 we explain an abstraction of efficient schedules, and show how this abstraction serves to find efficient schedules (w.r.t. execution time). Section 5 describes how to construct this abstraction using the geometry of timed PV diagrams and presents a spatial decomposition method, which is suitable for the exploration of the abstraction. In Section 7 we describe some related work on timed PV diagrams and on scheduling of concurrent programs. In Section 8 we conclude and present future work.

2 PV Programs and Diagrams

In this section we briefly present PV programs and PV diagrams. We adapt the vocabulary to our application domain: we use "threads" instead of "processes", and we call a set of threads running together a "program" or a "PV program". We first explain the model with the classical example of the "Swiss flag".

PV Programs. "P" and "V" are actions on semaphores. "P" is for "proberen", "to test" in Dutch, and "V" is for "verhogen" ("to increment"), as applied on Dijkstra semaphores. In multithreaded programs vocabulary, P is for "lock", and V for "unlock" or "release". In PV programs, only lock and unlock actions are considered. The Swiss flag program is:

```
A = Pa.Pb.Vb.Va
B = Pb.Pa.Va.Vb
```

where `a` and `b` are 1-semaphores. In this program threads `A` and `B` run concurrently, for example they might be executed on two processors—one for each thread.

PV Diagrams. PV programs have a geometric representation. The PV diagram of the Swiss flag program is shown in Figure 1.

The meaning of the diagram is that a schedule for the program is represented by a sequence of arrows from the bottom left corner of the diagram, point $(\bot_A, \bot_B)$, to the top right corner, point $(\top_A, \top_B)$. Indeed, any possible schedule is a particular order of events (P or V) of threads `A` and `B`. A schedule is shown in the diagram, drawn in solid arrows.

In this diagram the black circles indicate the "forbidden points", that is those that are not possible in a schedule. For example, point $(2, 1)$ is forbidden because its associated combination of actions, $(P_{\mathtt{b}}, P_{\mathtt{b}})$, means that both threads lock resource `b` at the same time, which is not

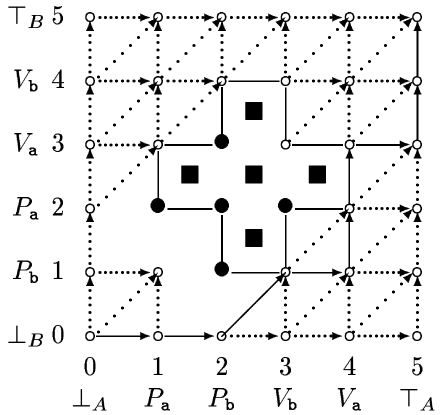


Figure 1: The Swiss flag diagram; a schedule

possible since b is a 1-semaphore. Consequently, we do not draw the arrows that have black points as source or target. We draw in dotted line all the arrows that a schedule could follow. The small black squares mark the squares of the diagram which are "forbidden squares", which are the "expansion" of each forbidden point to the adjacent upper-right little square. The "Swiss flag" name of the example comes from the cross form of the union of these forbidden squares.

The advantage of such diagrams is that they allow to visualize special behaviours of a program. In this example, we can see two special cases: point $(1, 1)$, which is a **deadlock**; and point $(3, 3)$, which is an **unreachable point**.

2.1 PV Diagrams: Formal Definitions

We now formalize the above explanation and provide the basis for our subsequent development of a timed version of PV programs and diagrams. We use partial orders to model threads. When B is a partial order, we use the term "arc" or "arrow" to refer to an element $(b, b') \in |B| \times |B|$ if $b \sqsubseteq b'$, and we denote it by $\langle b, b' \rangle$.

2.1.1 Orders

Resources. Shared resources are represented by a set $\Re$ of **resource names**. Each resource is protected by a **semaphore**, which is represented with a function $limit : \Re \to \mathbb{N}_+$. We suppose that each resource has a finite limit, since this is the case which interests us. An **action** (by a thread) is the locking or unlocking of a resource. If $r \in \Re$, the action of locking r is denoted by P_r, and the action of unlocking r is denoted by V_r.

Threads. We consider a set of N **threads**, which we index with integers, for convenience: $E_1, \ldots, E_N$. Each thread E_i is a partial order of **events**. A thread event e has one associated action. We denote by $act(e)$ the action associated with thread event e, for example $act(e) = P_r$. The set of events of thread i is denoted by $|E_i|$, and the order relation on it by $\sqsubseteq_{E_i}$ (also written simply $\sqsubseteq$ when no confusion is possible). This order is total (no branching considered in the present study.) Each thread E_i contains at least two events: its *start event*, $\bot_{E_i}$, which is the bottom element of the order, and its *end event*, $\top_{E_i}$, which is the top element of the order. The threads we consider are *well-behaved*, in the sense that for each resource $r \in \Re$, the thread has form: $B^*(P_r B^* V_r)^* B^*$, where B is the set of actions $P_{r'}$ or $V_{r'}$ with $r' \neq r$.

We say that thread i is **accessing** resource r at event e if and only if P_r has occurred before or at e, and the corresponding release V_r occurs (strictly) after e. Formally, this is the case if there exist an $e' \sqsubseteq e$ with $act(e') = P_r$, and an e'' with $e \sqsubset e''$ and $act(e'') = V_r$ such that $e' \sqsubseteq e \sqsubset e''$, and for all e''' with $e' \sqsubseteq e''' \sqsubset e''$, $act(e''') \neq P_r$, $act(e''') \neq V_r$.

The running together of N threads is formalized by the product of N partial orders, $\mathcal{E} = \prod_{i=1,\ldots,N} E_i$. We denote by $\bot$ the bottom $(\bot_{E_1}, \ldots, \bot_{E_N})$ of this partial order, and by $\top$ its maximum $(\top_{E_1}, \ldots, \top_{E_N})$. We denote by $\preccurlyeq$ the order of $\mathcal{E}$. We will use letters $\epsilon, \epsilon', \ldots$ to denote elements of $\mathcal{E}$. Given $\epsilon = (\epsilon_1, \ldots, \epsilon_i, \ldots, \epsilon_N)$, ϵ_i is the event that belongs to thread E_i.

Forbidden Elements. For each element ϵ of $\mathcal{E}$, and each resource $a \in \Re$, we compute the number of threads which access resource a at this element. A point is forbidden if there is at least one resource to which the number of concurrent accesses is greater than its initial semaphore value. Formally, the element ϵ is forbidden if and only if $\bigvee_{r \in \Re}(\sum_{i=1,\ldots,N} accessing_i(r, \epsilon) > limit(r))$, where $accessing_i(r, \epsilon) = 1$ if thread i is accessing resource r at ϵ_i, $accessing_i(r, \epsilon) = 0$ otherwise.

We denote by F the set of all forbidden elements of $\mathcal{E}$, and we denote by $\mathcal{A}$ (for "allowed") the restriction of order $\mathcal{E}$ to non-forbidden elements (elements of $\mathcal{E} \upharpoonright F$).

Strings and Schedules. We use in the remainder of this paper the following notation: if $e \in B$ and $e \neq \bot_B$, where B is a total order, then $\mathrm{pred}_B(e)$ denotes the direct predecessor of e in B. That is, $\mathrm{pred}_B(e) \sqsubset e$, and $\forall e' \in B \ : \ \mathrm{pred}_B(e) \sqsubseteq e' \sqsubset e \implies e' = \mathrm{pred}_B(e)$. When the order B considered is clear in the context, we will simply write $\mathrm{pred}(e)$.

Among arrows in relation $\preccurlyeq$, we distinguish the "small steps". An arrow $\langle \epsilon', \epsilon \rangle \in \preccurlyeq$ is a small step if : $\forall\ i = 1, \ldots, N \ : \ \mathrm{pred}(\epsilon_i) \sqsubseteq {\epsilon'}_i \sqsubseteq \epsilon_i$. For example, in the diagram of Figure 1, the dotted arrows are small steps from $\preccurlyeq$.

Definition 1 *A* ***string*** *s is subset of $\mathcal{A}$, which forms a path from an element $\epsilon \in \mathcal{A}$ to an element $\epsilon' \in \mathcal{A}$ with $\epsilon \preccurlyeq \epsilon'$, such that for each element ϵ'' in $s \setminus \{e\}$, arrow $\langle pred_s(\epsilon''), \epsilon'' \rangle$ is a small step. A string which forms a path from $\bot$ to $\top$ is called a* ***schedule*** *(for the program).*

An element of a string is called **state**. From now on, the letter ρ will denote a schedule.

2.1.2 Geometric Realization

Now we define the mapping of a program and its schedules to a diagram and trajectories, which we call the geometrization mapping. The idea is to map the set of schedules to trajectories inside an N-dimensional cube, going from the bottom left corner (for $\bot$) to the top right corner (for $\top$) of the cube. Since we want to stay in the discrete world, we describe geometric realization in $\mathbb{Z}^N$. We use notation "$\overline{}$" for the mapping; hence, $\overline{\rho}$ is the image of schedule ρ by this mapping. We map threads E_i onto a subset of $\mathbb{N}$ as follows. Each event e of thread E_i is associated with an ordinate $c(e)$. The ordinates are defined inductively as follows:

- $c(\bot_{E_i}) = 0$.
- $c(e) = c(\mathrm{pred}(e)) + 1$ if $e \neq \bot_{E_i}$.

The order of E_i is mapped onto the order $\leq$ between the integers $c(e)$. We denote by $\overline{E_i}$ the resulting partial order $(\{c(e) \mid e \in |E_i|\}, \leq)$. This mapping is clearly an isomorphism of partial orders.

Mapping the Product of the Threads. Since $\overline{E_i}$ is isomorphic to E_i, the product of partial orders $\prod_{i=1,\ldots,N} \overline{E_i}$ is isomorphic to $\mathcal{E} = \prod_{i=1,\ldots,N} E_i$. We denote by $\overline{\mathcal{E}}$ this product: it is indeed the geometrization of $\mathcal{E}$. If looked onto an N-dimensional discrete Euclidian space, elements of $\overline{\mathcal{E}}$ are points of an N-dimensional grid. More precisely, the mapping sends every $\epsilon = (e_1, \ldots, e_N) \in \mathcal{E}$ to the point $\overline{\epsilon} = (c(e_1), \ldots, c(e_N))$. So for example, $\overline{\bot}$ is $(0, \ldots, 0)$, and $\overline{\top}$ is $(c(\top_{E_1}), \ldots, c(\top_{E_N}))$.

Mapping Forbidden Elements and Strings. The set of forbidden elements F is mapped onto $\overline{F}$; $\overline{F}$ has an intuitive form geometrically: if every point of $\overline{F}$ lends a colouring of the adjacent top right "little box", then we see a union of N-dimensional boxes, which we call "forbidden boxes" or forbidden regions.

As a sub-order of $\mathcal{A}$, any string s is mapped onto $\overline{s}$, which is the set $\{\overline{\epsilon} \mid \epsilon \in s\}$ of points of $\overline{\mathcal{A}}$, together with the order it inherits from $\overline{\mathcal{A}}$. Geometrically schedules are trajectories that avoid touching the front boundary of the forbidden boxes.

3 Timed PV Programs and Diagrams

In this section we present our timed version of PV programs and diagrams. This version differs from existing versions of timed PV programs and diagrams [14, 10]. These latter works are briefly presented in Section 7, where we also explain why we introduce a new version of timed PV programs and diagrams.

3.1 Timed PV Programs

Our version of timed PV programs is an enrichment of untimed PV programs with a task duration between any two consecutive events of each thread. This is motivated by considerations of practical real-time programming, where one may measure the duration of the execution of the program code between two events. Such measures are usually done to foresee the worst case, so this duration is a *worst-case execution time* (WCET). After defining our timed version of PV programs, our goal is to define the duration of a given schedule. And then we aim at finding a quick schedule, in the sense of the schedule that makes the execution of all threads finish as soon as possible.

Adding Duration of Tasks. In our definition of timed programs, we associate with each event e in a thread E_i the duration (the WCET) of the task, i.e., the part of the program code which is performed between the direct predecessor of e and e. We denote by E the union $\bigcup_{i=1,\ldots,N} |E_i|$. The task durations are given in form of a function $d : E \to \mathbb{N}$. We define $d(\perp_{E_i}) = 0$ for each thread E_i.

Example: the Timed Swiss Flag Program. A timed version of the Swiss flag program is as follows:

```
A = 1.Pa.1.Pb.2.Vb.5.Va.2
B = 1.Pb.3.Pa.1.Va.0.Vb.1
```

Timed Schedules. A schedule in our timed version is, as in the untimed case, an order of events of the threads.

3.2 Geometric Realization

We now define the mapping of a timed PV program and its schedules into a diagram and trajectories. In principle, we could use the geometric realization for the untimed case, since the involved orders are the same. However, it is more convenient to have a diagram where one can visualize durations. To this end, we only have to change the ordinate function c as follows. Each event e of thread E_i is associated with ordinate $c(e)$. Ordinates are chosen so as to visually reflect task durations in the Euclidian dimension (in one dimension). A special case is tasks with zero duration, for which we choose a fixed length $\alpha > 0$ to represent the order geometrically. The ordinates are defined inductively as follows:

- $c(\perp_{E_i}) = 0$.

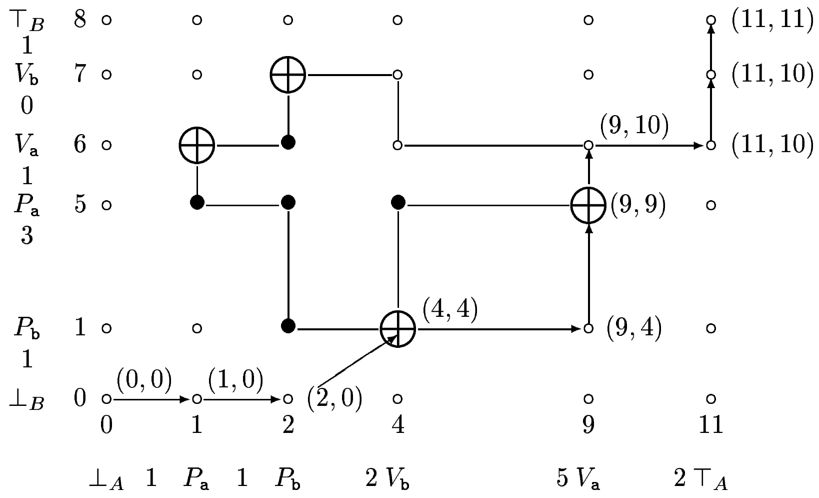


Figure 2: A timed schedule

- $c(e) = c(\text{pred}(e)) + d(e)$ if $e \neq \bot_{E_i}$ and $d(e) \neq 0$.
- $c(e) = c(\text{pred}(e)) + \alpha$ if $e \neq \bot_{E_i}$ and $d(e) = 0$.

The order of E_i is mapped onto the order $\leq$ between integers $c(e)$. We denote by $\overline{E_i}$ the resulting partial order $(\{c(e) \mid e \in |E_i|\}, \leq)$. The timed diagram for the timed Swiss Flag program is shown in Figure 2 (with $\alpha = 1$).

3.3 3D Example: the Timed Dining Philosophers

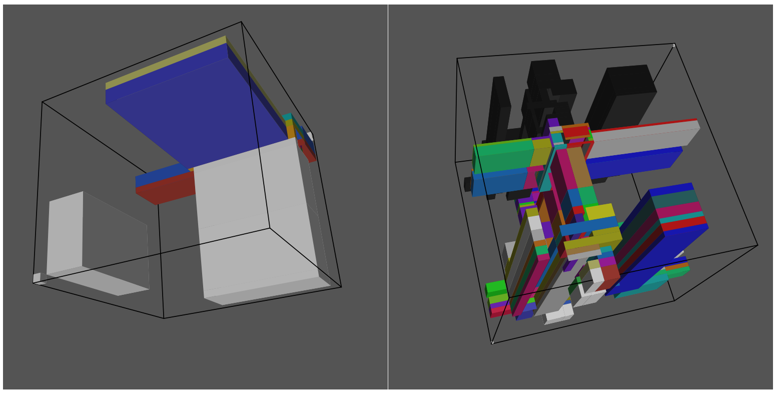
We also give a timed version of the 3 philosophers problem. The philosophers, as usually, have to get their left and right forks for eating. In the program forks are named `a`, `b`, and `c`: the left fork of philosopher `A` is `a`, and its right fork is `b`; and so on. The forks are 1-semaphores. We add a 2-semaphore for controlling an access to a small thinking room which can contain no more than 2 philosophers at a time. Each philosopher thinks in the thinking room, then walks to the eating room (which can contain the three philosophers), and eats. Non-zero task durations are given for thinking, walking, and eating. The program is the following:

```
A = 0. P room  .5. V room .4. P a .0. P b .15. V a .0. V b .0
B = 0. P room .14. V room .6. P b .0. P c  .5. V b .0. V c .0
C = 0. P room  .9. V room .9. P c .0. P a  .2. V c .0. V a .0
```

Then the trajectory for a schedule has to be taken in the cube shown in Figure 3 (a). We add little white cubes to indicate the $\bot$ and $\top$ corners. The forbidden regions for the forks are the three intersecting bars. The forbidden region for the thinking room is the cube at the bottom left of the overall cube. We show also, in Figure 3 (b), the geometry of a more complex version which has concurrent access to an anti-stress, and a small ashtray, etc.

3.4 Duration of Strings

Now we explain how the **duration of a string** (and hence of a schedule) is defined. We have added durations between events, which are WCETs. The duration we consider for a string corresponds to the case where all the tasks take their WCET as effective duration; thus the duration of a string is its worst-case response time.



(a) (b)

Figure 3: Forbidden regions of the three philosophers problem: (a) simple version; (b) enriched version

Waits. The computation of the duration can be understood in terms of a **logic of waits**. More concretely, we assume that a thread *could* begin its tasks as soon as the necessary resources are available. However, the real "permission" depends on the schedule under consideration. For example, a thread A might be ready to begin a task after event e but is forced to wait until another thread B performs an event e', if the schedule indicates that event e cannot happen before event e'.

New Events. For convenience, we introduce the notion of *new events* along a schedule. New events are the events that happen at an element in a string. Given a string s and an element $\epsilon \in s$, the set of new events, denoted by $new_s(\epsilon)$, that occur at ϵ along the string s is defined as: if $\epsilon \neq \bot_s$, $new_s(\epsilon) = \{\epsilon_i\ (i = 1, \ldots, N) \mid (\mathrm{pred}_s(\epsilon))_i \neq \epsilon_i\}$. If $\epsilon = \bot_s$, then $new_s(\epsilon)$ is defined as $\{\epsilon_i\ (i = 1, \ldots, N)\}$.

Algorithm to Compute the Duration of a String. Consider a string s (which can be a schedule). The duration of string s, which we denote by $d(s)$, is computed with the following algorithm. The algorithm iterates over the states of the string, beginning at $\bot_s$ and ending at $\top_s$. Its goal is to find "what time is at least at $\top_s$" when time is 0 at $\bot_s$. To this end, the algorithm uses clocks: N **local clocks**—one for each thread—, and one **global clock**. The global clock is not indispensable, but eases the explanation. We call the variable for the global clock `H`, and `h` the array (of size N) of the local clocks, with indices from `1` to `N`: `h[i]` is the local clock for thread i. The algorithm is as follows.

- First all clocks, global and local, are initialized to 0.
- Then we iterate over the sequence of states of s, beginning from the element just above its bottom element. For each element ϵ of the sequence, do, in the following order:

1. Update the global clock according to all threads i that have a new event at ϵ:
   `H := max(H, h[i] +` $d(\epsilon_i)$`)` for all i such that $\epsilon_i \in new_s(\epsilon)$.
2. Update the local clocks of all threads i that have a new event at ϵ:
   `h[i] := H` for all i such that $\epsilon_i \in new_s(\epsilon)$.

When element $\top_s$ has been processed, the algorithm returns the final value of `H` which is the duration of the string, $d(s)$.

We explain the algorithm: when arriving at a state ϵ one observes which events occur at this state. Let i the index of a thread that has a new event at ϵ.

(1) The last time an event of thread i happens is stored as value h[i]. Now, since that point, time has elapsed by at least $d(\epsilon_i)$ time units, since we now observe event ϵ_i. Therefore, the global time at state ϵ must be at least h[i] + $d(\epsilon_i)$). So we update the global clock accordingly. The "max" function is needed because it is possible that value h[i] + $d(\epsilon_i)$) is in fact not greater than the last H recorded. An example of this case is given below.

(2) After the global clock has been updated in step (1), the local clocks of the threads that have new events have to be synchronized. Indeed, we know that current time is now *at least* H, so the local clocks are updated accordingly.

Example. The algorithm is illustrated with the schedule shown in Figure 2. The vector-like annotations that accompagny the trajectory indicate the values of the local clocks during the execution. We have not indicated the global clock, since its value at one state is always the maximum of the values of the local clocks. We execute the algorithm on the sequence of states of the schedule, and we explain below what happens at some particular states. We identify states by their coordinates in the diagram.

- State $(1, 0)$: at this state, a new event of thread A happens. Since only A has a new event, the global clock is updated to $\max(0, 0 + 1) = 1$, and thread A updates its local clock to 1. Hence the vector of local clocks is $(1, 0)$ at this state.
- State $(4, 1)$: at this state a new event happens to each thread A and B. The global clock H becomes 4, and both local clocks are updated. The schedule implies that action P_b of thread B does not happen before thread A performs V_b. Since thread A runs for 4 time units before executing V_b, B cannot execute its action P_b before that time point. The fact that the local clock of B is updated to 4 shows that the soonest B can access b (with this schedule) is at $t = 4$. So B has a lapse of 4 time units for executing its task of duration 1. For example, if it executes this task immediately—beginning at date 0—, at global time 1 it has finished, it is forced to **wait** for 3 time units until A releases resource b.
- State $(11, 7)$: at this state, a new event of thread B happens. But the duration of the task before this event is zero, so there is no change to be made.

The final value of the global clock is 11. This defines the WCRT for the considered schedule.

4 Abstraction of Efficient Schedules

The Scheduling Problem and Approach. We are interested in finding a quick schedule. Let us first assume that we are looking for the quickest possible one (in the sense of a schedule has the minimal WCRT). We observe that the approach of computing the duration for each possible schedule and then picking the schedule with the minimal duration is not feasible in general. Indeed, the combinatorial explosion comes not only from the number of possible states, but also from the total number of possible schedules from bottom to top. If we also count the forbidden schedules (which pass through forbidden regions), to simplify computations, we get the following numbers: for the timed Swiss flag example, $6 \times 6 = 36$ states and 1683 possible schedules; for the timed philosophers example, $8 \times 8 \times 8 = 512$ states and 75494983297 possible schedules; for the enriched version of the timed philosophers, $16 \times 18 \times 26 = 7488$ states and more than 5×10^{30} possible schedules.[1]

Given this complexity problem, we propose to exploit the geometry of the diagrams to construct abstractions that can make the computation of one or all shortest paths feasible. In this section we define these abstractions, and we will describe in the next section a method to compute them.

[1] —5589092438965486974774900743393, to be precise.

Eager Strings. We focus on a class of strings which is interesting w.r.t looking for efficient schedules: **eager strings** are the strings that make no unnecessary wait—that is, a wait in the string is necessarily induced by waiting for a locked resource to be unlocked.

Notice the difference between being eager and being the quickest schedule: while the quickest schedule is necessarily eager, the converse is not true. For example, in the example in Figure 1, a string from $\perp$ to $\top$ that goes *above* the cross could be eager, but will not be optimal. Indeed, since thread A has to wait for the resources a and b to be unlocked by thread B, the quickest string that goes above the cross will have duration $5 + 1 + 9 = 15$ time units.

We give also an example of a non-necessary wait in a schedule (which eager strings do not have). In the time Swiss flag example a schedule with an unnecessary wait would go, for example, through points $(4, 0)$ and $(9, 0)$ before going to $(9, 1)$: this corresponds to B waiting for A to release resource a before accessing resource b, while resource b is already available. As a result, the local clocks in this case would be $(9, 9)$ at point $(9, 1)$ and $(9, 12)$ at point $(9, 5)$, reflecting the time spent on waiting.

Studying eager strings, we are interested in what we call the **critical exchange points**: the points where a resource is exchanged, and which border a forbidden region. Those are the only points where a wait can be justified (or necessary). In the Swiss flag diagram critical exchange points are indicated with the circled addition symbols.

In conclusion, an eager string waits only at critical exchange elements, and between any two such elements makes no wait (since it would be unnecessary). Thus an eager string is characterized by the critical points it passes through. We need to add $\perp$ and $\top$ in the set of critical exchange points, since it is possible that a quickest schedule does not touch the forbidden regions. This characterization of eager strings by critical exchange points is the basis for our abstraction method for looking for efficient schedules. In the following we will prove that looking only at critical exchange points is sufficient to construct an abstraction of all the quickest schedules. To do so, we need first to introduce an abstraction of wait-free strings.

Bows: Abstractions of Wait-Free Strings. In order to define abstractions for eager strings, we first define abstractions for their wait-free parts. For this we introduce the notion **bow**. Intuitively, a bow is an arc $\langle e, e' \rangle$ from $\mathcal{A}$ such that the longest side of the cube (in the geometric realization) whose bottom left and top right corners correspond to e and e' is equal to the duration of the quickest strings between e and e'.

We first introduce the abstraction which we will use for the duration of wait-free strings.

Definition 2 *The **distance** between two elements $\epsilon, \epsilon' \in \mathcal{E}$ with $\epsilon \preccurlyeq \epsilon'$ is defined as: $\|\langle \epsilon, \epsilon' \rangle\| = \max_{i=1,\dots,N}(s(\epsilon_i) - s({\epsilon'}_i))$, where for any thread E_i and event $e \in E_i$: $s(e) = \sum_{\perp_{E_i} \sqsubset e' \sqsubseteq e} d(e)$.*

Note that $s(e) \neq c(e)$ in general: $c(e)$ is the ordinate of e for the geometrization, while $s(e)$ is the "true ordinate" of e in term of the sum of the WCETs of the tasks. The case $s(e) \neq c(e)$ when there is at least one $e' \sqsubseteq e$ that has $d(e') = 0$: then $s(e) < c(e)$.

We want to use arcs of $\mathcal{A}$ as abstractions of strings, so we introduce the following operation.

Definition 3 *Given any arc $\langle \epsilon, \epsilon' \rangle$ from $\mathcal{A}$, the **stringing** of $\langle \epsilon, \epsilon' \rangle$, which we denote by $\langle \epsilon, \epsilon' \rangle \searrow$, is the set of all the strings from ϵ to ϵ' that have the smallest duration.*

This set is not empty, since $\epsilon \preccurlyeq \epsilon'$ implies that there is a sequence of small steps from ϵ to ϵ' in $\mathcal{A}$. We call the **tightened length** of an arc $\langle \epsilon, \epsilon' \rangle$ from $\mathcal{A}$, the duration of any element of $\langle \epsilon, \epsilon' \rangle \searrow$. For simplicity of discussion we extend notation d, the duration of a string, to sets of strings that have the same duration. Then the tightened length of $\langle \epsilon', \epsilon \rangle$ is written $d(\langle \epsilon', \epsilon \rangle \searrow)$.

Now in abstracting wait-free strings, we want to be **conservative** with respect to looking for the quickest schedule. So we look at arcs whose distance is not smaller than the duration of the strings they could abstract.

Definition 4 *A **bow** is an arc $\langle \epsilon, \epsilon' \rangle$ from $\mathcal{A}$, such that $\epsilon \prec \epsilon'$ and $\|\langle \epsilon, \epsilon' \rangle\| \geq d(\langle \epsilon, \epsilon' \rangle \searrow)$.*

The **height** of a bow $\langle \epsilon, \epsilon' \rangle$ is the distance $\|\langle \epsilon, \epsilon' \rangle\|$. In fact, $d(\langle \epsilon, \epsilon' \rangle \searrow) \leq \|\langle \epsilon, \epsilon' \rangle\| \implies d(\langle \epsilon, \epsilon' \rangle \searrow) = \|\langle \epsilon', \epsilon \rangle\|$. This is summarized as:

Lemma 1 *For any bow $\langle \epsilon, \epsilon' \rangle \in \mathcal{A}$, $d(\langle \epsilon, \epsilon' \rangle \searrow) = \|\langle \epsilon, \epsilon' \rangle\|$.*

Proof: see Appendix A.

Example. The notion of a **bow** is best explained on an example. Consider again the Swiss flag diagram in Figure 2. Arc $\langle (9,0), (11,6) \rangle$ is a bow, while arc $\langle (0,1), (9,8) \rangle$ is not. Indeed, the latter arc has length $\|\langle (0,1), (9,8) \rangle\| = 9$, while its tightened length is 11 (the quickest string from $(0,1)$ to $(9,8)$ exchanges resource b at point $(2,7)$, and thread A has to wait for it for at least 2 time units).

Critical Potential Exchange Points. We define critical potential exchange points—the only points where an eager string can wait. A **potential exchange point** is an element ϵ of $\mathcal{A}$ where a resource can be exchanged. That is, there exist at least one resource $r \in \Re$, and two indices i, j, such that $\epsilon_i = V_r$ and $\epsilon_j = P_r$. We use the term "potential" because in order to be a real exchange point, it must be the element of a schedule ρ which has $\epsilon_i \in new_\rho(\epsilon)$ and $\epsilon_j \in new_\rho(\epsilon)$.

Definition 5 *A potential exchange point for a resource r with $accessing(r, \epsilon) = limit(r)$ is called a* ***critical potential exchange point****.*

The Abstraction Graph. We are now ready to define our **abstraction of all the eager strings** (and hence also of all the quickest schedules). It is the graph constructed from the critical potential exchange points, having bows as arrows. We call it the **abstraction graph**.

We denote by C the union of all critical potential exchange points for the PV program with $\{\perp, \top\}$. The abstraction graph is then defined as a relation $G \subseteq C \times C$, characterized by: $\epsilon \, G \, \epsilon'$ if and only if and $\langle \epsilon, \epsilon' \rangle$ is a bow. We label each arc (ϵ, ϵ') of G with a weight which is $\|\langle \epsilon, \epsilon' \rangle\|$.

Definition 6 *A* ***path*** *p in graph G from an element $\epsilon \in C$ to $\epsilon' \in C$ is any sequence of critical exchange points, $p : \{0, \ldots, K\} \to C$, with $p(0) = \epsilon$, $p(K) = \epsilon'$, and $\forall i \in \{1, \ldots, K\}$ $\langle p(i-1), p(i) \rangle$ is a bow.*

The **length of a path** p in G, denoted by $l(p)$, is defined as $l(p) = \sum_{i=1,\ldots,K} \|\langle p(i-1), p(i) \rangle\|$. For an arc $\langle \epsilon, \epsilon' \rangle \in G$, we denote by $\widehat{\langle \epsilon, \epsilon' \rangle}$ the set of the shortest paths from ϵ to ϵ'. And by abuse of notation, we denote by $l(\widehat{\langle \epsilon', \epsilon \rangle})$ the length of any of the paths in $\widehat{\langle \perp, \top \rangle}$.

Example. We look again at the Swiss flag example in Figure 2. The critical potential exchange points (except for $\perp$ and $\top$) are indicated by circled addition symbols. The arrows (of G) between them are from $(0,0)$ to $(4,1)$, from $(4,1)$ to $(9,5)$, from $(9,5)$ to $(11,8)$; from $(0,0)$ to $(1,6)$, from $(1,6)$ to $(2,7)$, from $(2,7)$ to $(11,8)$; and from $(4,1)$ to $(11,8)$ and from $(1,6)$ to $(11,8)$. Here we see that a bow is not completely tied to the geometry: the last two bows, if represented as line segments between the points in the space, *do cross* the forbidden region.

Property of the Abstraction Graph. In the example of Figure 2, we see that the shortest path has length $4 + 5 + 2 = 11$. The following theorem states an important property of the graph G:

Theorem 1 *The duration of a quickest schedule is the length of a shortest path in G.*
More formally: $d(\langle \perp, \top \rangle \searrow) = l(\widehat{\langle \perp, \top \rangle})$.

Proof: see Appendix A.

An interesting computational implication of Theorem 1 is that the size of the graph G is reasonable since the number of critical potential exchange points is much smaller than the number of elements in $\mathcal{E}$; hence the shorstest paths in G can be efficiently computed. We will discuss this in more detail in the following section.

5 Finding Efficient Schedules using Geometric Realization

The construction of graph G has two parts: 1) find the critical potential exchange points; 2) find the bows between these points. Then the shortest path in graph G is computed. Notice that this approach automatically finds a **deadlock-free path**. Indeed, if a path in G leads to a deadlock point, no bow goes from it; and a shortest path from $\perp$ to $\top$ is, above all, a path from $\perp$ to $\top$, and hence contains no deadlock.

We use geometry for the construction. Notice however that our method does not depend on the coordinates $c(e)$, in the sense that the function c of the untimed case would give the same results. This is because we use the *structure* of the geometry of $\overline{\mathcal{E}}$ (the forbidden boxes), not the distances in the embedding. We use a function c which uses $d(e)$ only for visual intuition (the "max" measure is still close to the Euclidian distance).

Notice that is it possible, after we have found a satisfying path p in G, to actually *construct* an eager string abstracted by this path. The construction operates bow by bow. For one bow $\langle\epsilon, \epsilon'\rangle$ the quickest string abstracted by it is one that just makes no unnecessary wait, so a possible procedure is to start from ϵ and to pick the adjacent small step to an ϵ'' which increases the least the duration (there may be several), among those that have not ${\epsilon''}_i \sqsupset {\epsilon'}_i$ in one of the dimensions i.

5.1 Computing the Critical Potential Exchange Points

The critical potential exchange points are given by some points on the boundary the forbidden regions: in dimension 2, these are the bottom-right and top-left points of the forbidden regions; in dimension 3, all points on some edges of the boundary; etc. The (quite straitforward) formal characterization of this geometric aspect of critical potential exchange points is not given in this paper to save space.

Computing the Forbidden Regions. In this section we describe briefly the algorithm we use to compute the forbidden regions from the timed PV program. Clearly checking for each element whether it is forbidden is not a reasonable approach. We use instead the *access intervals* of the threads. A thread E_i creates an access interval when it accesses resource r (P_r) at an event e, and releases it some time after (V_r), at event $e' \sqsupset e$: this access interval is stored as the triplet of integers $(i, c(e), c(e'))$. Moreover the algorithm proceeds resource by resource: for each resource $r \in \Re$, we compute the forbidden regions created by access to r by more than $limit(r)$ threads concurrently. This set R_r is computed as follows.

1. For each thread E_i, construct the set $accesses(r, i)$ of *access intervals* by i to resssource r.
2. Then for efficiency we proceed as follows. First we determine the *abstract occurrences* of forbidden concurrent accesses. This is when there are more than $limit(r)$ accesses concurrently to resource r. So, from the set of all sets $accesses(r, i)$ which are not empty (this set contains $m \leq N$ elements), we compute its subsets of cardinal $limit(r) + 1$: those are the abstract occurrences.
3. Then we compute the *concrete occurrences* of forbidden concurrent accesses from each abstract one, by combining the access intervals. When the cardinal of a concrete occurrence is less than N, it means that one (or more) thread(s) k are not concerned by this forbidden concurrent access: then dimension k is added as access interval $(k, c(\perp_{E_k}), \top_{E_k})$,

because geometrically the forbidden access holds for all ordinates of k. This each concrete occurrence defines the coordinates of an N-dimensional box.

In step (2) of the procedure, computing the parts with a cardinal greater than $\mathit{limit}(r)+1$ is not necessary because those occurrences are included (geometrically) in the regions computed for the $(\mathit{limit}(r)+1)$-occurrences.

Example. We consider the three philosophers program of page 6. We compute the forbidden regions for resource `c`. Suppose threads `A`, `B` and `C` have respective indices 1, 2 and 3. (1) We get: $\mathit{accesses}(\mathtt{c}, 1) = \emptyset$, $\mathit{accesses}(\mathtt{c}, 2) = \{(2, 22, 28)\}$, $\mathit{accesses}(\mathtt{c}, 3) = \{(3, 19, 22)\}$. (2) The non-empty sets among those are $\{(2, 22, 28)\}$ and $\{(3, 19, 22)\}$. Since $\mathit{limit}(\mathtt{c}) = 1$, the abstract occurrences must have cardinal 2. There is only one such abstract occurrence here: $\{\{(2, 22, 28)\}, \{(3, 19, 22)\}\}$. (3) This abstract occurrence of a forbidden access results in a single concrete occurrence $\{(1, 0, 28), (2, 22, 28), (3, 19, 22)\}$ which defines a 3-dimensional box whose bottom and top vertices are $(0, 22, 19)$ and $(28, 22, 19)$ respectively.

5.2 Finding the Arrows of the Abstraction Graph

From the forbidden boxes we can compute the critical potential exchange points, which are the nodes of the abstraction graph G. But it remains to compute the bows between the critical potential exchange points. A simple method to determine whether an arc $\langle \epsilon, \epsilon' \rangle$ is a bow is to determine the tightened length of the arc by enumerating all the strings from ϵ to ϵ' and then check the condition of Definition 4. However, this method is clearly very expensive and, to remedy this, we will exploit some properties of the geometrization.

We use a method which uses some arcs which are necessarily bows: we use a decomposition of forbidden-point-free regions. Using this approach we may not find a quickest schedule but we can find a good schedule. This decomposition approach and the strategies for looking for the quickest schedule are discussed in the following.

Finding Efficient Schedules using Decomposition. We denote $\mathcal{B} = [0, c(\top_1)] \times \ldots \times [0, c(\top_N)] \subset \mathbb{R}^N$. In $\mathbb{R}^N$, $\overline{\mathcal{E}}$, that is the image by the geometrization mapping of the product $\mathcal{E}$ of all the threads, forms a (non-uniform) N-dimensional grid over the box B. A potential exchange point ϵ corresponds to a *grid point*, denoted by $g(\epsilon)$; therefore, a bow corresponds to a line segment connecting two grid points, and a path in the graph G corresponds to a sequence of such line segments. It is important to note that while the graph G is used to model the schedules with the shortest duration, it does not capture resource conflicts. Consequently, to construct the graph G we need to consider the bows which do not cause a resource conflict. In this geometric setting, the forbidden regions is a union of boxes whose vertices are grid points. This union is indeed an *orthogonal polyhedron* [5], denoted by P_F. Let $P_A = \mathcal{B} \setminus P_F$ denote the *allowed polyhedron*. We now make the following observation: if a box contains no forbidden points, then any two points on its boundary form a bow if there are grid points. Indeed, intuitively, the line segment between them does not intersect polyedron P_F. This motivates considering a decomposition of the polyedron P_A.

Definition 7 (Decomposition) *We define a* decomposition *of an orthogonal polyhedron P as a set $\mathcal{D}_P = \{B_1, \ldots, B_k\}$ where each B_i ($i \in \{1, \ldots, k\}$) is a full-dimensional box such that the following conditions are satisfied:*

1. *For all $i \in \{1, \ldots, k\}$ the vertices of B_i are grid points.*
2. $P = \bigcup_{i \in \{1, \ldots, k\}} B_i$
3. *For all $i, j \in \{1, \ldots, k\}, i \neq j$ the boxes B_i and B_j are non-overlapping, that is their interiors do not intersect with each other.*

Note that the vertices of the boxes in a decomposition are not necessarily critical exchange points. If all the vertices of a box are grid points then it is called *grid box*. Additionally, if a grid box does not contain any other grid boxes, then it is called *elementary box.* We will use in the sequel two types of decompositions that we call elementary and compact. Given a decomposition $\mathcal{D}_P = \{B_1, \ldots, B_k\}$, $\mathcal{D}_P$ is called *elementary* if all B_i are elementary boxes; $\mathcal{D}_P$ is called *compact* if there exists no pair of B_i and B_j with $i \neq j$ such that $B_i \cup B_j$ is a grid box. Intuitively, in a elementary decomposition none of its boxes can be split into smaller grid boxes, and in a compact decomposition no pair of its boxes forms a grid box. Note that there exists a unique elementary decomposition of a given orthogonal polyhedron, however there may be many different compact decompositions.

We now show how to use decompositions to construct the abstraction graph G. Let $\mathcal{D}_{P_A}$ be a decomposition of the allowed polyhedron P_A. We first recall the observation we use to reduce the complexity of the search for bows: a line segment connecting two vertices of a box $B_i \in \mathcal{D}_{P_A}$ which are critical exchange points corresponds to a bow (since it is a direct path which does not cross the forbidden polyhedron P_F). It is however clear that even when $\mathcal{D}_{P_A}$ is the elementary decomposition, the set of all such edges does not allow to cover all possible bows since two vertices of two different boxes might also form a bow. However, if our goal is to find one path with the shortest duration that respects resource constraints, it is not necessary to construct the whole graph G but we need to include all the bows that form such a path. It can be proved that there exists a decomposition such that the vertices of its boxes are enough to discover a shortest path. We call such a decomposition an *effective decomposition*, and it is of great interest to find such a decomposition, which is our ongoing work. Other possible heuristics to approach such decomposition is discussed in the next paragraph.

We finish this section by briefly describing our current method for computing a compact decomposition of orthogonal polyhedra. The essential idea of the method is as follows. From a given starting box we try to merge it with other elementary boxes, along one or more axes, so as to maximize the volume of the resulting box. To do so, we make use of the efficient algorithms for Boolean operations and membership testing developed based on a compact and canonical representation of such polyhedra (see [5]). In some cases, the criterion of maximizing the volume of merged boxes may not be the best one with respect to including the shortest path in the graph. Alternative criteria are merging as many as possible boxes along a fixed axis. Intuitively, a shortest path tends to approach the diagonal between the bottom left and top right corners of the box B while avoiding the forbidden regions; hence, we can combine different merging criteria depending on the relative position to the forbidden regions.

5.3 Experimental Results

We demonstrate in this section the effectiveness of our method. We have written a prototype which implements the exposed method. For computing the forbidden regions we use a program written in the language Maude [6] and executed with the Maude system. The execution time for computing the forbidden regions is negligible. The program for the decomposition (construction of allowed boxes from the forbidden boxes), the construction of the abstraction graph from the allowed boxes, and the search of the shorstest path in this graph is written in C++. The construction of the allowed boxes from the forbidden ones is rather quick, and most of the time in the execution of this program is spent in the construction of the graph from the allowed boxes—due to the number of vertices we use, as we explain below. We present in the table below some experiments with this program.

We first test with the philosophers problem, in 3 dimensions and more. That is, we use N forks—one per philosopher—and one thinking room which can take only $N-1$ philosophers. Then we take the same program, but with a thinking room which can contain only half the philosophers ("phil. s.th.-r" is for "philosophers with small thinking room"). Program "enr. phil." is the enriched version of the philosophers problem whose geometry is shown in Figure 4 (b). Program "more enr. phil." is when we add still more actions to this enriched version. Program "enr. phil. 4D" is when we add a fourth philosopher to the enriched version. Program "3 phil. 2 procs" is the program of Section 6, whose geometry is shown in Figure 4. In the table, "na" stands for "not available"—the computation was not finishing in less than 10 minutes. We have used a PC with a Xeon processor of 2.40 GHz frequency, 1 Go of memory and 2 Go of swap.

program	dim	#states	#forbid	#allowed	#nodes	#edges	t (sec.)
3 phil.	3	512	4	35	151	773	0.58
4 phil.	4	4096	5	107	743	7369	17.38
5 phil.	5	32768	6	323	3632	67932	571.12
6 phil.	6	262144	7	971	na	na	na
3 phil. s.th.-r.	3	512	6	59	227	1271	1.50
4 phil. s.th.-r.	4	4096	8	199	1147	13141	60.24
5 phil. s.th.-r.	5	32768	15	1092	na	na	na
6 phil. s.th.-r.	6	262144	21	3600	na	na	na
enr. phil.	3	7488	26	390	1468	7942	51.01
more enr. phil.	3	29568	137	1165	4616	30184	461.18
enr. phil. 4D	4	119808	44	5447	na	na	na
3 phil. 2 procs	3	1728	12	78	352	2358	2.56

One can observe that the number of allowed boxes is very reasonable compared with the number of states. The number of nodes reflects the fact in our current prototype, we add in the graph some of the vertices of the allowed boxes which are not critical exchange points, to compensate for the fact that we do not currently include inter-allowed-box bows: thus we can find paths whose length approximate (conservatively) the weight of such inter-box bows. The advantage of this approach is that any decomposition can serve to find a relatively good schedule. Its inconvenient is that the number of considered vertices for a box is of order 2^N. Thus the number of threads considered is the main obstacle in our current implementation.

We find good schedules: in the case of the enriched 3 philosophers program, the durations of the threads are 24, 25 and 20 respectively, and the found schedule has duration 39, which is good. In the case of the enriched version (3D), the threads have respective durations 83, 94, and 95, and the found schedule has duration 160, which is also good in view of the many forbidden regions which bar the direct way.

Our future experiments will use the following heuristics: using, for each box in the decomposition, only its bottom and top elements. Intuitively, quick schedules follow diagonals, so this heuristics could be useful. It addresses the main obstacle of our method—the number of vertices considered per allowed box (we descend from 2^N points per box to only 2). On the other hand, how close one then gets to the quickest schedule depends on the decomposition, as discussed in the previous section.

6 Limited Number of Available Processors

The Problem. We have defined the WCRT of a schedule *assuming that the threads run concurrently.* But in concrete terms, this implies that N processors are available. It might be possible that less than N are needed, for example when thread migration is allowed and $N-1$ processors are enough for this schedule because the schedule has some particular waiting patterns. Therefore the true question is: what does the WCRT of the schedule become when there are only $M < N$ processors available?

The problem of defining the mapping of the N threads (or processes) onto M processors, that we call the thread distribution mapping, has already been treated in [7]. But this is in the untimed context, and aims at building a scheduler that avoids deadlock states. We are looking not only for safe schedules using a limited number of processors, but also *efficient* schedules.

We distinguish two approaches: 1) first compute an efficient schedule with the method shown in the previous section; and then compute a good mapping of this particular schedule onto

$M < N$ processors. The advantage of this approach is that it separates "abstract scheduling" and mapping. The inconvenient is that there may be some schedules that were not considered efficient in the abstract world, but that could do very well on $M < N$ processors. 2) Integrating the mapping problematics into the model, and computing an efficient schedule that takes this constraint into account. The advantage of this approach is that it is more precise. But it can also lead to state explosion, as we discuss in the following.

In this section we examine the second solution, because it gives some geometric intuition on the mapping, and in addition, for many practical cases the complexity of the computation is reasonable.

A Solution. The idea is to model the resource limitation in terms of available processors, as a M-semaphore. This modelling assumes that the threads have no preference on which processor to run on. This is reasonable in the case of a homogeneous architecture—all the processors are the same. It also ignores issues to communication optimisation, so it implicitly assumes a shared memory architecture. The advantage of using a semaphore is that it makes a drastic combinatorial simplification: when 2 threads A and B, among a pool of 3 concurrent threads A, B, C, are running on 2 processors p_1 and p_2, we do not have to say whether A is running onto p_1 and B onto p_2 or vice-versa. Knowing that A and B are running, and not C, is what interests us from the point of view of scheduling. The effective distribution of the threads onto the processors can then be done statically, or at run time, but in any case, **after** we have already determined the schedule.

We use a *manual* locking and releasing of a processor in a PV program. This corresponds to **manual proposition of preemption**: the programmer decides when a thread gives a chance to other threads of taking the processor. If the schedule which is eventually chosen does not use this preemption opportunity, then of course in the implementation of this schedule the thread does not need to preempt itself.

Example. As an example we use the simple version of the three philosophers problem. Here the programmer decides that a philosopher keeps the processor for thinking and walking to the eating room, and before entering the thinking room makes a proposition of preemption so as to give the opportunity for other threads to get the processor. We denote by `p` the semaphore for the processors. The program of philosopher `A` is modified as follows (the modification is similar for philosophers `B` and `C`):

```
A =  0. P p      .0. P room .5. V room .4.             V p
    .0. P p .0.      P a .0. P b .15. V a .0. V b .0.   V p .0
```

The geometry of the new program is shown in Figure 4. We see that a trajectory must go through the "canyons" between the `p`-forbidden boxes, as well as avoiding the parts of the previous forbidden regions that still emerge from these new boxes. Notice that the `room`-forbidden box is now included in the bottom left `p`-forbidden box. Indeed, the `room` semaphore served to forbid acces to the room by more than two philosophers, which is no longer necessary.

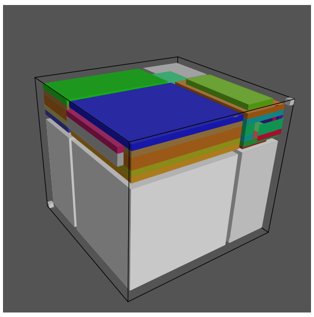


Figure 4: Forbidden regions of the three philosophers problem with two processors

Limitations of the Approach. Remark that since each philosopher accesses 2 times a processor (through a lock of semaphore p), we indeed get $2^3 = 8$ boxes that form the corresponding forbidden regions. Computationally, it means that a thread should not propose preemption too often. On the other hand, finding the optimal schedule "for all possible preemptions" would imply, on the contrary, proposing a preemption between each event of the original program (which can be done automatically). But this would induce an exponential number of forbidden regions (p-forbidden regions).

On the other hand, this geometric approach can give new ideas for optimizations of the control of programs that run on a limited number of processors. For example, in the previous example, the geometry indicates that, in the given preemption is implemented, then the implementation can dispense with the `room` semaphore.

7 Related Works

Timed PV Diagrams. Some other versions of timed PV diagrams have been proposed. We have not used them, for the reasons we explain below.

- The work [10], which presents a timed version of PV programs and diagrams, attempts to model multiple clocks, as in timed automata [4]. In the present paper we do not use the timed automaton model. Moreover, in the approach of [10] time is modeled as an additional dimension—one per clock. Thus, with one clock and three threads, a 4-dimensional space is studied. In this paper we consider each thread (or process) dimension as a "local time dimension", and then define the synchronization of local time dimensions.
- The work [14] exploits the dimension of each process as a time dimension. In this aspect, this work is close to ours. However there are important differences. First, the definitions in [14] are given in a continuous setting, and therefore topological spaces are considered, such that the duration of a schedule is described with an integral. In our work we stay in a discrete domain, an the definition of the duration of a schedule is given by an algorithm on a discrete structure. On the other hand, the fact that the definitions in [14] are tied to geometry implies, in particular, that zero-delays between two consecutive actions in a process (for example two successive locks, which often happens in programs that share resources) are not possible since the two actions would be the same in the geometry. In our approach, while we exploit the geometry to construct abstractions, the notion of duration itself is not geometric. Consequently, zero-delays are possible. This is of particular interest if one considers that the practical delay, on most architectures, between two consecutive locks, is too small to be modelled as a non-zero value. We conjecture that our version of timed PV diagrams is a discretized version of the continuous version of [14] (in the case of no zero-delays in the program).

Timed Automata. A large class of real-time systems can be adequately modelled with **timed automata** [4], and in this framework the problem of scheduling has been addressed [3, 1, 2, 16, 17], often closely related to the context of **controller synthesis**. A timed PV program has a direct representation using timed automata. First, each thread is modelled as an automaton, where each node represents an event, and each transition from node e to node e' is labeled with constraint "$t \geq d(e')$" plus a reset of the clock. The global automaton is the product of all the thread-automata. Semaphores can be represented via variables. Such a product of automata is very close to that of [16], where the aim is also to schedule multi-threaded programs. In this work a *scheduler* is constructed to guarantee that a schedule does not go into deadlock states or deadline-breaking directions. We look for a complete schedule which is not only safe but also efficient; however our model is not as rich as the timed automata model: we have not yet included deadlines, branching, and looping.

Scheduling and the Polytope Model. Another geometry-based method for scheduling concurrent programs is the **polytope model** (see, e.g., [8]), which is used in the context of automatic parallelization. However the semantics of the points in the geometric space is not the same as in PV diagrams: each point inside a polytope represents a task which has to be executed, while in PV diagram each point is a possible state and only a very small number of these states have to be represented in the implementation. Also the polytope model does not consider resource sharing, and has no task durations.

8 Conclusion and Future Work

In this paper, we defined a timed version of PV programs and diagrams which can be used to model a large class of multithreaded programs sharing resources. We also introduced the notion of the worst-case response time of a schedule of such programs. This framework was then used to find efficient schedules for multithreaded programs. In particular, to tackle the complexity problem, we define an abstraction of the quickest schedules and we show how to exploit the geometry of PV diagrams to construct this abstraction and compute efficient schedules as well as a quickest one. This work demonstrates an interesting interplay between geometric approaches and real-time programming. An experimental implementation allowed us to validate the method and provided encouraging results.

Our future work will explore the following directions.

- When developing a real-time system one is often interested in the worst-case response time of the whole program, if it is part of a larger system, *for any schedule*. As a definition, this WCRT could be given as the duration of the eager schedule that has the longest duration. We conjecture that we could use abstraction graph G for computing the longest eager schedule by computing the longest path in a subgraph of G. Defining this subgraph is a topic of our future research.
- We are able to find schedules, but it remains to see how they can be implemented. An obvious solution is controlling the computed schedule so as to enforce *exactly* the order of events it describes. But an interesting question is: among those control points, which can we "forget" while guaranteeing that the real execution will not diverge from the planned schedule as far as critical exchanges of resources are concerned? Indeed, in practice tasks can take less time than the WCET: control is needed for ensuring that such behaviour does not make the trajectory follow a direction which does not correspond to the schedule.
- We are currently investigating the problem of adding *deadlines* in our model. This extension is not straightforward since the "symmetry" with the definition of a lower bound to the duration spent by a thread between two consecutive events (the WCET of the task) is not trivial. We also intend to examine the possibility of lifting to the timed case the existing studies on the geometry of loops [12] or branching (`if-then-else` constructs) in PV programs.
- Another approach to treat deadlines is to integrate our geometric abstractions into existing tools that use timed automata, such as [16]. These tools suffer from the problem of state explosion. Since our model is close to a product of automata, integrating our geometric approach into these tools could allow to handle larger systems.

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A Proof of the Theorem

In this section we prove theorem 1. For this we first prove lemma 1, and then we introduce some useful notions.

Proof of lemma 1. We want to prove that for any bow $\langle \epsilon, \epsilon' \rangle \in \mathcal{A}$, $d(\langle \epsilon, \epsilon' \rangle \searrow) = \| \langle \epsilon, \epsilon' \rangle \|$.

Pick a string s in $\langle \epsilon, \epsilon' \rangle \searrow$. This string must execute, for each thread j, all of the tasks whose durations are the $d(e)$, $\epsilon_j \sqsubset e \sqsubseteq \epsilon'_j$ (see the definition of the duration of string). Thus the duration of s is greater than or equal to $\max_{i=1,\ldots,n}(s(\epsilon'_i) - s(\epsilon_i))$. But the latter is (by definition) $\| \langle \epsilon, \epsilon' \rangle \|$. Thus $d(\langle \epsilon, \epsilon' \rangle \searrow) = d(s) \geq \| \langle \epsilon, \epsilon' \rangle \|$. We can conclude that $d(\langle \epsilon, \epsilon' \rangle \searrow) = \| \langle \epsilon, \epsilon' \rangle \|$. □

Abstraction. Abstractions of eager strings are paths. This is formalized here.

Definition 8 *The* ***pathing of a string*** s, *which we denote by* $s\nearrow$, *is the path which is constituted of all the critical potential exchange points contained in* s. *This operation is authorized only if both* $\bot_s$ *and* $\top_s$ *are critical potential exchange points, and* s *is eager.*

The construction is correct:

Fact 1 *If* s *is an eager string, then* $s\nearrow$ *is a path in* G.

Proof: Let $s\nearrow: [0, \ldots, K] \to C$. We want to prove that for each $i = 1, \ldots, K$, $\langle s\nearrow(i-1), s\nearrow(i) \rangle \in G$. That is, we want to prove: for any $i \in [1, \ldots, K]$: $\| \langle s\nearrow(i-1), s\nearrow(i) \rangle \| = d(\langle s\nearrow(i-1), s\nearrow(i) \rangle \searrow)$.

Take $i \in [1, \ldots, K]$. Between $s\nearrow(i-1)$ and $s\nearrow(i)$, there is no critical potential exchange point (otherwise it would have been included in the pathing). But critical potential exchange points are the only elements which can induce a necessary wait, and the string, which is eager, has waits only at critical exchange points. Thus between $s\nearrow(i-1)$ and $s\nearrow(i)$ the string has no unnecessary wait so its duration from elements $s\nearrow(i-1)$ to $s\nearrow(i)$ is exactly the maximum of the tasks to be executed, $\| \langle s\nearrow(i-1), s\nearrow(i) \rangle \|$. □

Concretization. The "reverse" operation of abstraction of strings, is concretization of paths of G into strings.

Definition 9 *The* ***stringing of a path*** $p : [0, \ldots, K] \to C$ *from* G, *which we denote by* $p\searrow$, *is the set of all the string from* $p(0)$ *to* $p(K)$ *which have the smallest duration and contain all points* $p(i)$, *for* $i = 1, \ldots, K-1$.

This set is not empty, since a string from $p(0)$ to $p(K)$ can be constructed from the strings from the sets $\langle p(i-1), p(1) \rangle \searrow$ $(i = 1, \ldots, K)$, which are not empty since bows are arcs of $\mathcal{A}$.

For a path p from G, we denote by $d(p\searrow)$ its **tightened length**.

Interaction of Abstraction and Concretization.

Lemma 2 *Let* $\langle \epsilon', \epsilon \rangle \in \mathcal{A}$, *with* $\epsilon', \epsilon \in C$. *Then there exists a string* $s \in \langle \epsilon', \epsilon \rangle \searrow$ *which is such that* $l(s\nearrow) = d(s)$.

In the following, a string s is said to be **optimal** if $d(s) = d(\langle \bot_s, \top_s \rangle \searrow)$.

Proof: Pick a s in $\langle \epsilon', \epsilon \rangle \searrow$ (any s). This is possible, since $\langle \epsilon', \epsilon \rangle \in \mathcal{A}$ and so the set $s \in \langle \epsilon', \epsilon \rangle \searrow$ is not empty. This string is optimal, so it is eager, so pathing is valid for it: $s\nearrow$ is a path in G. Let K be the number of elements in $s\nearrow$. That is: $s\nearrow: [0, \ldots, K] \to C$. The proof is by **induction on** K.

- **Case** $K = 1$. In this case, we have only one bow, $\langle s\nearrow(0), s\nearrow(1) \rangle$. Since s is optimal, $s \in \langle s\nearrow(0), s\nearrow(1) \rangle \searrow$. So by lemma 1, $l(s\nearrow) = \| \langle s\nearrow(0), s\nearrow(1) \rangle \| = d(\langle s\nearrow(0), s\nearrow(1) \rangle \searrow) = d(s)$. So the proposition is true for $K = 1$, with this string s.

- **Case** $K > 1$. We want to prove that the property is true for K assuming it is true for $K-1$.

  Element $s\nearrow(K-1)$ is a critical potential exchange point. We look at what happens from $s\nearrow(K-1)$ to $s\nearrow(K)$. For one (or more) dimension k, $\|\langle s\nearrow(K-1), s\nearrow(K)\rangle\| = s(s\nearrow(K)_k) - s(s\nearrow(K-1)_k)$, that is, the maximal sum of task durations between $s\nearrow(K-1)$ and $s\nearrow(K)$ is for dimension k. Then there are two possible cases:

  1. Thread k has a new event at $s\nearrow(K-1)$.
     Let $H \in \mathbb{N}$ be the value of the global clock at $s\nearrow(K-1)$. Then the global clock at $s\nearrow(K)$ is $H + \|\langle s\nearrow(K-1), s\nearrow(K)\rangle\|$. Thus $d(s) = H + \|\langle s\nearrow(K-1), s\nearrow(K)\rangle\|$ $= d_s(s\nearrow(K-1)) + \|\langle p(K-1), p(K)\rangle\|$. Then, using the recurrence hypothesis on the subpath from $s\nearrow(0)$ to $s\nearrow(K-1)$, one gets the desired property, with this string s.

  2. Thread k has no new event at $s\nearrow(K-1)$.
     That is, $act((s\nearrow(K-1))_k)$ happens in s before $s\nearrow(K-1)$.

     We construct a string s' from s, as follows: we substitute element $s\nearrow(K-1)$ with element $s\nearrow(K-1)[(s\nearrow(K-1))_k \longleftarrow (\mathrm{succ}_s((s\nearrow(K-1)))_k$, where $\mathrm{succ}_s(\epsilon)$ denotes the successor of ϵ in the total order s. That is, string s' goes from $\mathrm{pred}_s(s\nearrow(K-1))$ directly to an element where action $\mathrm{succ}((s\nearrow(K-1))_k$ occurs.

     (The proof that the new point is not k-forbidden is done by contradiction. Suppose that it is the case, then thread k would have to go around a k-forbidden region (and wait) between $s\nearrow(K-1)$ and $s\nearrow(K)$, which is not possible since it is the "leader" thread for this bow, i.e., k is the dimension that determines the distance between $s\nearrow(K-1)$ and $s\nearrow(K)$.)

     The substitution does not change the duration of the string. Indeed, only the dimension k is affected, and thread k had no new event at $s\nearrow(K-1)$. Now there are two cases:

     (a) the substitution replaces a point of C with a point of $|\mathcal{A}| \setminus C$. Then $K' = K-1$, and we use the recurrence hypothesis to show that string s' satisfies the desired property.

     (b) the substitution replaces a point of C with another point of C. But at this new point thread k has a new event. So the situation is as in case (1) but string s' replaces string s.

Conclusion: the proposition is true for $K = 1$ and $K > 1$, so it holds for all $K \geq 1$. □

Optimal Paths. A path p in G from ϵ to ϵ' is said to be **optimal** if $p \in \widehat{\langle \epsilon, \epsilon' \rangle}$.

Lemma 3 *Let $\langle \epsilon, \epsilon' \rangle \in \mathcal{A}$, with $\epsilon, \epsilon' \in C$. If string $s \in \langle \epsilon, \epsilon' \rangle\searrow$ is such that $l(s\nearrow) = d(s)$, then $s\nearrow$ is an optimal path.*

Proof: By contradiction. Existence of such a string s with $l(s\nearrow) = d(s)$ is given by lemma 2. Now suppose $s\nearrow$ is not an optimal path. Then there exists an optimal path q from ϵ to ϵ' with $l(q) < l(s\nearrow)$. Then we get: $d(q\searrow) \leq l(q) < l(s\nearrow) = d(s)$, which means that there are strings from ϵ to ϵ' whose duration is smaller that of s, which is not possible by the optimality of s. So $s\nearrow$ must be an optimal path. □

Proof of theorem 1. We can now prove $l(\widehat{\langle \bot, \top \rangle}) = d(\langle \bot, \top \rangle\searrow)$.

Take ρ in $\langle \bot, \top \rangle\searrow$, such that $l(\rho\nearrow) = d(\rho)$ (which is possible by lemma 2). By lemma 3, $\rho\nearrow$ is optimal. Thus $d(\langle \bot, \top \rangle\searrow) = d(\rho) = l(\rho\nearrow) = l(\widehat{\langle \bot, \top \rangle})$. □