System Representation and Characterization

Course Feedback Control and Real-time Systems

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A model is a description of a (physical, biological, economical, etc...) phenomenon, in a given language (for example, mathematical language).

A model is defined by a collection of variables and describes their evolution over time:

- **Predict** the values of the variables
- **Explain** complex phenomena from simpler or more general phenomena/principles
Modelling steps

- **Formalisation**: define the input and output variables, and the equations describing their relations. The equations may contain parameters.
- **Identification**: determine the parameter values in a given context.
- **Validation**: verify if the model is coherent with the observations.

**Simulation**: solving the equations to find the relations between the input and output variables. The resolution can be analytical, numerical, *etc*.

**Other usages of models**: design of controllers, formal verification of properties, code generation.
A signal is an application from time to a domain \( X : T \rightarrow D_X \)

- \( T \) can be either continuous in \( R \), or logical/discrete \( N, Z \)
- \( D_X \) specifies the signal type, \( R, N, Bool \)

A system is a signal transformer: \( S : (T \rightarrow D_X) \rightarrow (T \rightarrow D_Y) \)

Example 1: A modem transforms a binary signal into a continuous electrical signal (system in open loop, i.e. the output is determined directly from the input)

Example 2: A thermostat is a system in closed loop (with a feedback loop from the output to the input)
A system: establishes a cause-effect link between the input signals (excitations) and the output signals (responses).

Among the inputs, we distinguish:

- the **controls**
- the **disturbances**
A system is a transformer of signals
\[ S : (T \to D_X) \to (T \to D_Y) \]

To define a system

- **Identify the inputs and outputs**: a plant (inputs: raw material, outputs: products), a computer (input/output: information coming from the input/output interfaces)
- **Choose the types of the input/output signals**: \( D_X \) and \( D_Y \)
- **Choose the domain of time** \( T \): \( Z, N \) (discrete time), or \( R, R_+ \) (continuous time), or collection of moments at which some events occur
Define directly the function $S$ is **difficult**!!

We need to use associated analysis tools

- For continuous-time systems: differential and integral calculus
- For discrete-time systems: algebra
• Block diagrams: graphical description of connections between the components. Each component is associated with a function of signal transformation
• Connection $\Rightarrow$ composition of functions
• Hierarchical, easy to understand
Composition using block diagrams (2)

The global system $S_3 : U \to Y$ t.q.
$\forall u \in U : S_3(u) = S_2(S_3(u), S_1(u))$

The connection between $y$ et $w$ is called ‘feedback’. We need to solve the equation $z = S_2(z, S_1(z))$
Linear systems vs non-linear systems

A system is **linear** iff it satisfies the following properties:

- **Properties of additivity**: If the input is \( x_1(t) \), the output is \( y_1(t) \). If the input is \( x_2(t) \), the output is \( y_2(t) \). Thus, if the input is \( x(t) = x_1(t) + x_2(t) \), the output is \( y(t) = y_1(t) + y_2(t) \).

- **Properties of homogeneity**: If the input is \( x_1(t) \), the output is \( y_1(t) \). Thus, for \( \forall \alpha \neq 0 \), if the input is \( x(t) = \alpha x_1(t) \), the output is \( y(t) = \alpha y_1(t) \).
Properties, characteristics of a system (2)

**Stationary systems (time-invariant)**

More formally, we say that the system commutes with a delay:

\[ S(x(t - \delta)) = (Sx)(t - \delta) \]

The linear stationary systems form a class important historically and practically.
Causal system

Principle of causality: the effects should not precede the causes.

If the input $x(t)$ is nul for $t < 0$, then the output $y(t)$ is also nul for $t < 0$. 

**Instantenous system vs dynamical system**

*Instantenous system* (without memory or static): at a given instant, the output depends only on the input at that instant.

For example, \( y(t) = a(t)x(t) \) defines a static system.

*Dynamical system*: non-static, with memory.
Dynamical system In continuous system, memory is formalised by an integrator. The input/output relation is described by differential equations involving $y(t)$ and their derivatives

$$y'(t) = \lim_{h \to 0} \frac{(y(t) - y(t - h))}{h}$$

- $y'(t)$ : information about the growth
- $y(t)$ : present instant
- $y(t - h)$ : past instant
Consider a continuous dynamical system described by the following first-order linear differential equation:

\[ a_1 y'(t) + a_0 y(t) = x(t) \]

\[ y'(t) = -\frac{a_0}{a_1} y(t) + \frac{1}{a_1} x(t) \]

This schema is not optimised in the sense that one single integrator would suffice.
For a continuous system:

- Differential equations
- Functional representation
- State space representation
- Representation in a transformed space, for example via the Laplace transform
The functional schema is deduced from differential equations and allows a more direct way to numerical simulation

- It is a program in a graphical language connecting the functional blocks
- A compiler translates this schema into a computer program for the numerical resolution of differential equations
- To describe a linear continuous system, we need the following functional blocks: Gain, Sum/Subtraction, integrators (memory blocks).
- This representation is not unique
Consider a continuous dynamical system described by:

\[ a_1 y'(t) + a_0 y(t) = x(t) \]

\[ y'(t) = -\frac{a_0}{a_1} y(t) + \frac{1}{a_1} x(t) \]

Integration uses initial conditions:

\[ y(t) = \int_0^t y'(t) dt + y_0 \]
How to obtain a functional representation (1)

Obtain systematically a functional representation associated with a differential equation ($x(t)$ is input and $y(t)$ is output)

\[
\frac{d^n y(t)}{dt^n} = -a_{n-1} \frac{d^{n-1} y(t^{n-1})}{dt} - \ldots - a_1 \frac{dy(t)}{dt} - a_0 y(t) \\
+ b_m \frac{d^m x(t)}{dt^m} + \ldots + b_1 \frac{dx(t)}{dt} + b_0 x(t)
\]

Example 1:

\[
\frac{d^2 y(t)}{dt^2} = -a_1 \frac{dy(t)}{dt} - a_0 y(t) + b_0 x(t)
\]
How to obtain a functional representation (2)

Example 2:
\( f_a(t) \) as input and \( x(t) \) as output.

The system is represented by the differential equation:

\[
mb \frac{d^3 x(t)}{dt^3} + m k_1 \frac{d^2 x(t)}{dt^2} + b(k_1 + k_2) \frac{dx(t)}{dt} + k_1 k_2 x(t) = b \frac{df_a(t)}{dt} + k_1 f_a(t)
\]
How to obtain a functional representation (3)

\[ \frac{d^2 y(t)}{dt^2} = -a_1 \frac{dy(t)}{dt} - a_0 y(t) + b_0 x(t) \]

We write \( u_2(t) = y(t) \) and \( u_1'(t) = -a_0 y(t) + b_0 x(t) \). Hence, \( u''_2 = -a_1 u'_2 + u'_1 \).

After the first integration \( u'_2 = -a_1 u_2 + u_1 \).

Now in integrating \( u'_2(t) \) we obtain \( y(t) \)

\[
\begin{align*}
  u'_1(t) &= -a_0 y(t) + b_0 x(t) \\
  u'_2(t) &= -a_1 u_2(t) + u_1(t) \\
  y(t) &= u_2(t)
\end{align*}
\]
How to obtain a functional representation (4)

\[
\frac{d^2 y(t)}{dt^2} = -a_1 \frac{dy(t)}{dt} - a_0 y(t) + b_1 \frac{dx(t)}{dt} + b_0 x(t)
\]

We write \( u_1'(t) = -a_0 y(t) + b_0 x(t) \) et \( u_2(t) = y(t) \). Hence,

\[ u_2'' = -a_1 u_2' + b_1 x' + u_1'. \]
Notion of state

- To specify the function $S$, we often need a collection $\mathcal{X}$ of internal states.

- **More formally**, the state is a vector containing a minimal number of variables such that:
  The initial output value $y(t_0)$ is known $\Rightarrow$ for all $t > t_0$, $y(t)$ can be determined uniquely if the input $x(t)$ is known for the interval $[t_0, t]$.
Example of capacitor $i(t) = C \frac{dv(t)}{dt}$

$$v(t) = \frac{1}{C} \int_{-\infty}^{t} i(\tau) d\tau = \frac{1}{C} \int_{-\infty}^{t_0} i(\tau) d\tau + \frac{1}{C} \int_{t_0}^{t} i(\tau) d\tau$$

$$= v(t_0) + \frac{1}{C} \int_{t_0}^{t} i(\tau) d\tau$$

• Specifying $v(t_0)$ is more “economical” than specifying all the evolution $i(t)$ from $t = -\infty$ to $t = t_0$
• The state at the instant $t_0$ of the system must form the memory of the system
• The state can be a representation more compact than the complete history of the system.
• The state at the instant $t_0$ of the system must form the minimal memory of the past, necessary to determine the future

• The state represented by the internal variables provides a complete description of the evolution of the system

• This formalism allows transforming all the linear differential equations of order $n$ into a system of differential equations of order 1.

• **The choice of state representation is not unique**
Consider the precedent example

\[
\frac{d^2 y(t)}{dt^2} = -a_1 \frac{dy(t)}{dt} - a_0 y(t) + b_1 \frac{dx(t)}{dt} + b_0 x(t)
\]

We have set

\[
\begin{align*}
    u_1'(t) &= -a_0 y(t) + b_0 x(t) = -a_0 u_2(t) + b_0 x(t) \\
    u_2'(t) &= -a_1 u_2(t) + b_1 x(t) + u_1(t) \\
    y(t) &= u_2(t)
\end{align*}
\]

In a matrix form

\[
\begin{pmatrix}
    u_1' \\
    u_2'
\end{pmatrix} = 
\begin{pmatrix}
    0 & -a_0 \\
    1 & -a_1
\end{pmatrix} 
\begin{pmatrix}
    u_1 \\
    u_2
\end{pmatrix} + 
\begin{pmatrix}
    b_0 \\
    b_1
\end{pmatrix} x
\]

and

\[
y = \begin{pmatrix}
    0 & 1
\end{pmatrix} 
\begin{pmatrix}
    u_1' \\
    u_2'
\end{pmatrix}
\]
State space representation

\[ u' = Au + Bx \]
\[ y = Cu + Dx \]

- The matrix \( A \): **state matrix**, of dimension \( n \times n \)
- The matrix \( B \): **input matrix**, of dimension \( n \times p \)
- The matrix \( C \): **output matrix**, of dimension \( q \times n \)
- The matrix \( D \): **coupling matrix**, of dimension \( q \times p \)
From a structural viewpoint

System of first order (second ordre, …):

<table>
<thead>
<tr>
<th>differential</th>
<th>recurrent, automaton, object program</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X(0)$</td>
<td>$X(0)$</td>
</tr>
<tr>
<td>$X'$  =  $F(X, U)$</td>
<td>$X_{n+1}$  =  $F(X_n, U_{n+1})$</td>
</tr>
<tr>
<td>$Y$   =  $G(X, U)$</td>
<td>$Y_n$   =  $G(X_n, U_n)$</td>
</tr>
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Order of the system: **dimension of $X$**

Remark: not intrinsic

Finite-state system: Automaton, finit-state machine