Stability of interval polynomials

- **A single polynomial**
  - A polynomial
  $$F(z) = p_0 + p_1 z + p_2 z^2 + \cdots + p_n z^n \tag{1}$$
  is called stable if all its roots are in the LHP.
  - The Routh-Hurwitz test checks using only $O(n^2)$ operations if a polynomial is stable.

- **A family of polynomials**
  - Let we are given an infinite set of **interval polynomials** of the form (1)

$$IP = \{F(z) \text{ of the form } (1)\} \quad \text{where} \quad \overbrace{\underline{p_i} \le p_i \le \overline{p_i}}^{\textbf{intervals}}$$

- **A Question:** Is there any way to check if all the polynomials in IP are stable?

The classical Kharitonov's theorem

- Let we are given an interval polynomial

$$F(z) = p_0 + p_1 z + p_2 z^2 + \cdots + p_n x^n \quad \text{where} \quad \underline{p_i} \leq p_i \leq \overline{p_i} \tag{2}$$

- Kharitonov (1978): The infinite set of polynomials of the form (5) is stable if only the following four "boundary" polynomials are stable:

$$F_{min,min}(z) = F_{e,min}(z) + F_{o,min}(z), \qquad F_{min,max}(z) = F_{e,min}(z) + F_{o,max}(z)$$

$$F_{max,min}(z) = F_{e,max}(z) + F_{o,min}(z), \qquad F_{max,max}(z) = F_{e,max}(z) + F_{o,max}(z)$$

where

$$F_{e,min}(z) = \underline{p}_0 + \overline{p}_2 z^2 + \underline{p}_4 z^4 + \overline{p}_6 z^6 + \ldots,$$
$$F_{e,max}(z) = \overline{p}_0 + \underline{p}_2 z^2 + \overline{p}_4 z^4 + \underline{p}_6 z^6 + \ldots,$$
$$F_{o,min}(z) = \underline{p}_1 z + \overline{p}_3 z^3 + \underline{p}_5 z^5 + \overline{p}_7 z^7 + \ldots,$$
$$F_{o,max}(z) = \overline{p}_1 z + \underline{p}_3 z^3 + \overline{p}_5 z^5 + \underline{p}_7 z^7 + \ldots,$$

A connection to structured matrices?

Example I. Stability of Quasi-polynomials

- Control engineering: **retarded feedback time delay system**

$$\frac{dy}{dt} = Ay(t) + \sum_{r=1}^{p} By(\overbrace{t - \tau_r}^{\textbf{delays}}) \tag{3}$$

- After Laplace transformation one gets

$$F(s) = \det(sI - A - \sum_{r=1}^{p} B_r e^{-\tau_r s}) = \underbrace{f_0(s) + e^{-sT_1} f_1(s) + \cdots + e^{-sT_m} f_m(s)}_{\textbf{a quasi-polynomial}} \tag{4}$$

where $f_k(s)$ are polynomials.

- Stability of (3) $\Leftrightarrow$ all the roots of $F(s)$ in (4) are in the left half plane.

Recall the classical Kharitonov's theorem

- Let we are given an interval polynomial

$$F(z) = p_0 + p_1 z + p_2 z^2 + \cdots + p_n x^n \quad \text{where} \quad \underline{p_i} \leq p_i \leq \overline{p_i} \tag{5}$$

- Kharitonov (1978): The infinite set of polynomials of the form (5) is stable if only the following four "boundary" polynomials are stable:

$$F_{min,min}(z) = F_{e,min}(z) + F_{o,min}(z), \qquad F_{min,max}(z) = F_{e,min}(z) + F_{o,max}(z)$$

$$F_{max,min}(z) = F_{e,max}(z) + F_{o,min}(z), \qquad F_{max,max}(z) = F_{e,max}(z) + F_{o,max}(z)$$

where

$$F_{e,min}(z) = \underline{p}_0 + \overline{p}_2 z^2 + \underline{p}_4 z^4 + \overline{p}_6 z^6 + \ldots,$$

$$F_{e,max}(z) = \overline{p}_0 + \underline{p}_2 z^2 + \overline{p}_4 z^4 + \underline{p}_6 z^6 + \ldots,$$

$$F_{o,min}(z) = \underline{p}_1 z + \overline{p}_3 z^3 + \underline{p}_5 z^5 + \overline{p}_7 z^7 + \ldots,$$

$$F_{o,max}(z) = \overline{p}_1 z + \underline{p}_3 z^3 + \overline{p}_5 z^5 + \underline{p}_7 z^7 + \ldots,$$

The Kharitonov theorem revisited

- The meaning of max and min.

$$F_{e,min}(z) = \underline{p}_0 + \overline{p}_2 z^2 + \underline{p}_4 z^4 + \overline{p}_6 z^6 + \dots,$$
$$F_{e,min}(iz) = \underline{p}_0 - \overline{p}_2 z^2 + \underline{p}_4 z^4 - \overline{p}_6 z^6 \pm \dots,$$

- Kharitonov (1978): If only four polynomialls

$$F_{min,min}(z) = F_{e,min}(z) + F_{o,min}(z), \quad F_{min,max}(z) = F_{e,min}(z) + F_{o,max}(z)$$

$$F_{max,min}(z) = F_{e,max}(z) + F_{o,min}(z), \quad F_{max,max}(z) = F_{e,max}(z) + F_{o,max}(z)$$

are stable then all the polynomials

$$F(z) = \underbrace{F_e(z)}_{even} + \underbrace{F_o(z)}_{odd}$$

are stable provided that (for $z = \overline{z}$)

$$\frac{F_{o,min}(iz)}{iz} \leq \frac{F_o(iz)}{iz} \leq \frac{F_{o,max}(iz)}{iz}.$$

$$F_{e,min}(iz) \leq F_e(iz) \leq F_{e,max}(iz)$$

(Classical) Kharitonov via Hermite-Biehler. I

- THM (Hermite-Biehler). Let

$$F(z) = \underbrace{F_e(z)}_{even} + \underbrace{F_o(z)}_{odd}$$

Then the polynomial $F(z)$ is stable if and only if the following two conditions hold true.

1. The **roots** of the polynomials $F_e(iz)$ and $F_o(iz)$ are all real and they interlace.
2. There is at least one point $z_0 \in \mathbb{R}$ such that

$$F_e(iz_0)F_o'(iz_0) - F_e'(iz_0)F_o(iz_0) > 0.$$

Kharitonov via Hermite-Biehler. II

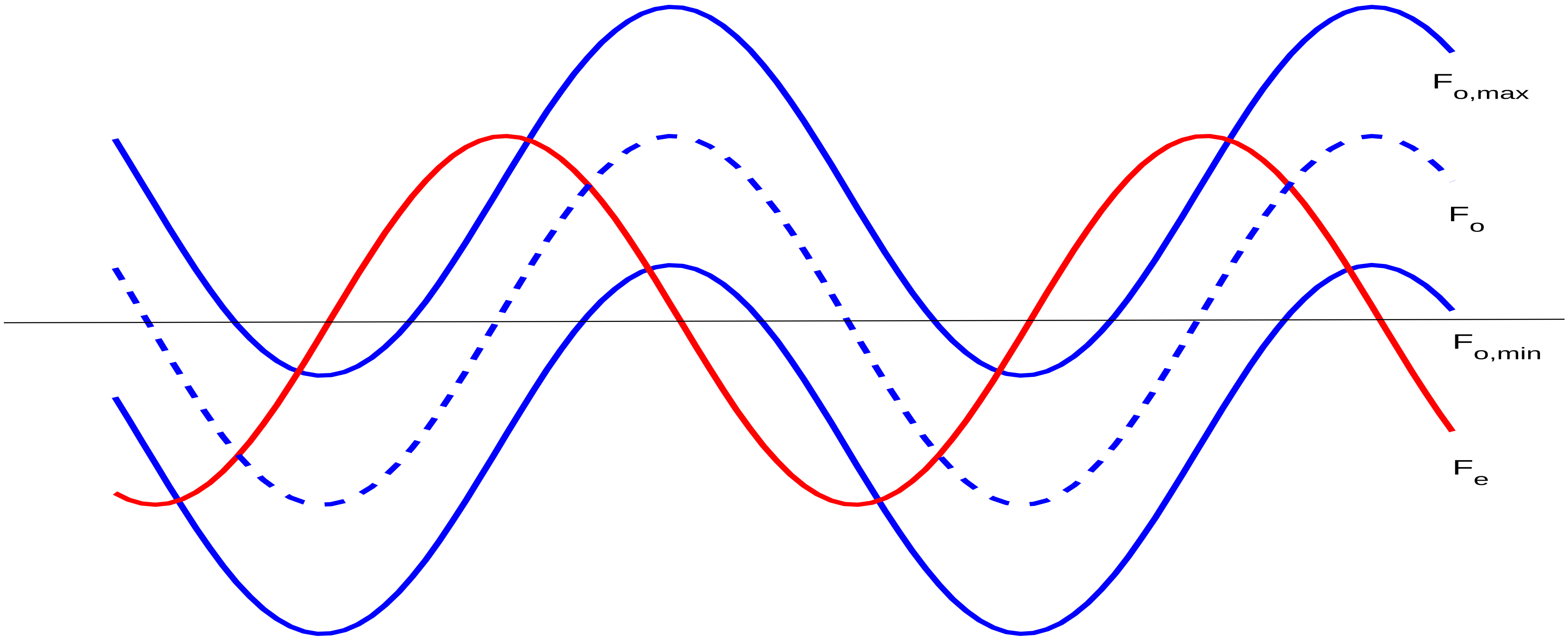

Illustration for the Proof of the classical Kharitonov theorem for polynomials via the Hermite-Biehler.