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The development of the Verimag Verified Polyhedra Library

made of untrusted parts mixing Ocaml, C++, threads, ...

and just a little bit of Coq for certification

joint work, started in 2012, with

Sylvain Boulmé, Alexis Fouilhé, Alexandre Maréchal, David Monniaux, **Michaël Périn**, Hang Yu

Univ. Grenoble Alpes, CNRS, Grenoble INP, VERIMAG, France

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- OCAML implementation of standard **polyhedral operators** of a **relational abstract domain** ($\sqsubseteq$, $\sqcup$, $\sqcap$, elimination of variables, ...)

The Verimag Verified Polyhedra Library (Features)

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- OCAML implementation of standard **polyhedral operators** of a **relational abstract domain** ($\sqsubseteq$, $\sqcup$, $\sqcap$, elimination of variables, ...)
- certifying library by **a posteriori verification** of each computation
  1. OCAML operators generate witnesses
  2. witness are checked by a simple COQ checker

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- can be used as **a standalone** OCAML **library**, *e.g.* in the FRAMAC static analyzer [Buhler+VMCAI'2017]

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- can be used as **a standalone** OCAML **library**, *e.g.* in the FRAMAC static analyzer [Buhler+VMCAI'2017]
- used in a new COQ-**tactic for simplifying affine expressions (S.Boulmé, A.Maréchal)** (submitted)

The Verimag Verified Polyhedra Library (Contributions)

- polyhedra library in **constraint-only** representation
- **new algorithms**
  - precise polyhedral approximation of polynomial guards
  - minimization by raytracing
  - projection via Parametric Linear Programming
- **novel certification approach using factories**
  correct by construction (by A.Maréchal, S.Boulmé)
- **efficiency issues:** parallelization, floating-point computations, external libraries (GLPK,GMP,EIGEN,FLINT), reconstruction of the exact solution (on $\mathbb{Q}$)
- **available at** `github.com/VERIMAG-Polyhedra`
- state of the art **Parametric Linear Programming Solver**

Topics

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- Polyhedra: basics
- All you need to understand the field of Polyhedra
  Farkas combinations and Linear Programming
- Why certifying software verification tools?
- Certification by result verification
- ... with as little COQ as possible
- Why another polyhedra library?
- Why are polyhedra expensive?
- Revisiting the algorithmic
- Experimental results
- Will Polyhedra be usable?

Convex Polyhedra

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capture **affine relations between program variables** such as

inequalities: $x_1 - 2x_2 \geq 3x_3 \rightsquigarrow x_1 - 2x_2 - 3x_3 \geq 0$

boundaries: $2 \leq x_1 \leq 3 \rightsquigarrow x_1 - 2 \geq 0 \bigwedge -x_1 - 3 \geq 0$

equalities: $x_1 = x_2 + 2 \rightsquigarrow \left\{ \begin{array}{l} x_1 - x_2 - 2 \geq 0 \\ x_2 - x_1 + 2 \geq 0 \end{array} \right\}$

affine form = linear form + constant

Definition

*A **convex polyhedron** is a set of vectors* $(x_1, ..., x_n) \in \mathbb{Q}^n$ *satisfying*
a system of affine inequalities between variables $x_1, ..., x_n$

Remark (It is convex)

if two points are in the set, the segment also is.

A 3D polyhedron ...

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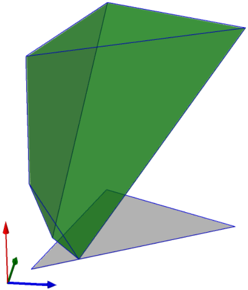
... as system of constraints

$$\left\{ \begin{array}{lllll} \boldsymbol{C_1} : & x_1 + 2x_2 - 2x_3 & \leq & 7, \\ \boldsymbol{C_2} : & x_1 - 2x_2 & \leq & -1, \\ \boldsymbol{C_3} : & -3x_1 + x_2 & \leq & 0, \\ \boldsymbol{C_4} : & x_3 & \leq & 10, \\ \boldsymbol{C_5} : & -x_1 - x_2 - x_3 & \leq & -5 \end{array} \right\}$$



Uses of Polyhedra

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- **Linear Programming**
  - optimization of a cost function under affine inequalities,
  - decide the existence of a solution fulfilling affine inequalities

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- **Linear Programming**
  - optimization of a cost function under affine inequalities,
  - decide the existence of a solution fulfilling affine inequalities
- **Loop Optimization**
  "The Polyhedron Model" [Feautrier and Lengauer, 2011]
  1. approximate by a polyhedron the cells of a n-dimensions array to be updated by a loop
  2. compute vectors that exactly describe that space of cells
  3. generate the optimized loop

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  3. generate the optimized loop
- **Static Analysis of Programs**
  POPL'78 [Cousot and Halbwachs, 1978]
  - capture affine relation between variables
  - discover implicit equalities
  - more precise than interval analysis but costlier

Why certifying results of Software Verification Tools?

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An old greek syllogism

Programs contain bugs.

Software Verification Tools are programs.

Thus, ___s contain ___s

Why certifying results of Software Verification Tools?

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An old greek syllogism

Programs contain bugs.
Software Verification Tools are programs.
Thus, <u>S</u><u>V</u><u>T</u>s contain <u>B</u><u>U</u><u>G</u>s

Why certifying results of Software Verification Tools?

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An old greek syllogism

Programs contain bugs.
Software Verification Tools are programs.
Thus, SVTs contain BUGs

… more than other programs

- mostly prototypes developed by several students
- complex underlying theory
- less users, less tested

Three ways to gain confidence in SVT

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1. **No tool is trustable ... but if they agree on the result**
   - Running each tool with (same inputs $\rightsquigarrow$ same answer) increases the confidence
   - **Quantifiable?** How **many tools** to reach P.Failure$=10^{-9}$?

Three ways to gain confidence in SVT

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   - Running each tool with (same inputs $\rightsquigarrow$ same answer) increases the confidence
   - **Quantifiable?** How **many tools** to reach P.Failure$=10^{-9}$?
2. **Only trust the proof-checker … which becomes critical**
   - extend SVT to **generate certificates**

     SAT, UNSAT, $\Longrightarrow_{Theory}$, $[\![\text{Program}]\!] \models$ Properties
   - $[\![\ldots]\!]$ is a formal semantics of the programming language

     (*e.g.* CompCert C semantics in COQ)
   - How long **can a bug stay silent** in the proof-checker?

     (the COQ-engine is not as simple as it used to be)

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     (*e.g.* COMPCERT C semantics in COQ)
   - How long **can a bug stay silent** in the proof-checker?

     (the COQ-engine is not as simple as it used to be)

3. **Correct by extraction using** COQ (*e.g.* COMPCERT)

$$\underbrace{\text{proof(algo} \models \text{spec)}}_{\text{in COQ}} \xrightarrow{\text{extraction}} \text{OCAML program}$$

Certification versus Result Verification

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Full COQ-certified development (COMPCERT, VERASCO S.A)

- time consuming (refactoring the code means adapting the proofs)
- requires proof skills
- the algorithms must be designed to be easy to prove
- **correctness of all results guaranteed** by a COQ-proof

Result verification (COMPCERT, B.S.A [Besson et al., 2010], VPL)

- external libraries (untrusted code)
- **correctness of each result is checked by** COQ
  1. external code generates witness of correctness
  2. verification by a **simple COQ-certified checker**

What must be proved for over-approximations?

- **join/union** operator $(P' := P_1 \sqcup P_2)$ is sound if $P_1 \sqsubseteq P'$ and $P_2 \sqsubseteq P'$
- **meet/intersection** operator $(P' := P_1 \sqcap P_2)$ is sound if $P' \sqsubseteq P_1$ and $P' \sqsubseteq P_2$
- **minimization** operator $(P' := \mathit{min}\ (P))$ is **sound** if $P \sqsubseteq P'$ and **precise** if $P' \sqsubseteq P$
- **elimination/projection** $(P' := \mathit{elim}\ \ \{x\}\ \ P)$ is sound if $P \sqsubseteq P'$ and x is unbounded in P'

Soundness boils down to inclusion [Besson et al., 2010]

$P_1 \sqsubseteq P_2$ can be proved by Farkas combinations

Each operator must provide **Farkas combinations** to prove the **required inclusions**

Farkas combinations I

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Example

$$\begin{aligned} P &= \left\{ \begin{matrix} \boldsymbol{C_1} : x_1 - x_2 - 1 \geq 0, \\ \boldsymbol{C_2} : x_1 + 2x_2 + 1 \geq 0 \end{matrix} \right\} \\ P' &= \{ \boldsymbol{C'} : 3x_1 - 1 \geq 0 \} \end{aligned}$$

The Farkas Combination $2 \times \boldsymbol{C_1} + 1 \times \boldsymbol{C_2}$ ... is $\boldsymbol{C'}$

$$\overbrace{2x_1 - 2x_2 - 2 \geq 0}^{2 \times \boldsymbol{C_1}} + \overbrace{x_1 + 2x_2 + 1 \geq 0}^{1 \times \boldsymbol{C_2}} = \underbrace{3x_1 - 1 \geq 0}_{\boldsymbol{C'}}$$

It shows that $\{\boldsymbol{C_1}, \boldsymbol{C_2}\} \sqsubseteq \{\boldsymbol{C'}\}$

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Farkas combination of constraints (linear version)

$\boldsymbol{x} \in P := \{\boldsymbol{C_1}, \ldots, \boldsymbol{C_k}\}$ *means*
$\boldsymbol{x}$ *satisfies* $\boldsymbol{C_1}(\boldsymbol{x}) \geq 0 \wedge \ldots \wedge \boldsymbol{C_k}(\boldsymbol{x}) \geq 0$ *then,*
for any ***non-negative scalars*** $\lambda_1, \ldots, \lambda_k \in \mathbb{Q}$

$$\underbrace{\lambda_1 \times \boldsymbol{C_1}(\boldsymbol{x})}_{\geq 0} + \ldots + \underbrace{\lambda_k \times \boldsymbol{C_k}(\boldsymbol{x})}_{\geq 0} \geq 0$$

Farkas combinations II

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for any ***non-negative scalars*** $\lambda_1, \ldots, \lambda_k \in \mathbb{Q}$

$$\underbrace{\lambda_1 \times \boldsymbol{C_1}(\boldsymbol{x})}_{\geq 0} + \ldots + \underbrace{\lambda_k \times \boldsymbol{C_k}(\boldsymbol{x})}_{\geq 0} \geq 0$$

Now, if $\boldsymbol{C'} = \lambda_1 \times \boldsymbol{C_1} + \ldots + \lambda_k \times \boldsymbol{C_k}$ *then*

$$\forall \boldsymbol{x},\ \boldsymbol{C_1}(\boldsymbol{x}) \geq 0 \wedge \ldots \wedge \boldsymbol{C_k}(\boldsymbol{x}) \geq 0 \Longrightarrow \boldsymbol{C'}(\boldsymbol{x}) \geq 0$$

which is the definition of the geometric inclusion

$$\{\boldsymbol{C_1}, \ldots, \boldsymbol{C_k}\} \sqsubseteq \{\boldsymbol{C'}\}$$

Farkas combinations III

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How to find the Farkas coefficient λ's

$$
\begin{array}{rcl}
P & = & \left\{ \begin{array}{lrrrl}
\boldsymbol{C_1}: & 1\,x_1 & -1\,x_2 & -1 & \geq 0, \\
\boldsymbol{C_2}: & 1\,x_1 & +2\,x_2 & +1 & \geq 0, \\
\boldsymbol{C_3}: & -1\,x_1 & +1\,x_2 & +1 & \geq 0
\end{array} \right\} \\
P' & = & \left\{ \begin{array}{lrrrl}
\boldsymbol{C'}: & 3\,x_1 & +0\,x_2 & -1 & \geq 0
\end{array} \right\}
\end{array}
$$

$$
\exists ? \lambda_i \geq 0 \quad \lambda_1 \times \boldsymbol{C_1} \; + \; \lambda_2 \times \boldsymbol{C_2} \; + \; \lambda_3 \times \boldsymbol{C_3} \; = \; \boldsymbol{C'}
$$

Farkas combinations III

How to find the Farkas coefficient λ's

$$
\begin{array}{rcl}
P & = & \left\{ \begin{array}{lrrrl}
\boldsymbol{C_1}: & \mathbf{1}\,x_1 & -1\,x_2 & -1 & \geq 0, \\
\boldsymbol{C_2}: & \mathbf{1}\,x_1 & +2\,x_2 & +1 & \geq 0, \\
\boldsymbol{C_3}: & \mathbf{-1}\,x_1 & +1\,x_2 & +1 & \geq 0
\end{array} \right\} \\
P' & = & \left\{ \begin{array}{lrrrl}
\boldsymbol{C'}: & \mathbf{3}\,x_1 & +0\,x_2 & -1 & \geq 0
\end{array} \right\}
\end{array}
$$

$$
\begin{array}{cccccccc}
\exists?\lambda_i \geq 0 & \lambda_1 \times \boldsymbol{C_1} & + & \lambda_2 \times \boldsymbol{C_2} & + & \lambda_3 \times \boldsymbol{C_3} & = & \boldsymbol{C'} \\
\hline\hline
(x_1) & \lambda_1 \times \mathbf{1} & + & \lambda_2 \times \mathbf{1} & + & \lambda_3 \times \mathbf{-1} & = & \mathbf{3}
\end{array}
$$

Farkas combinations III

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How to find the Farkas coefficient λ's

$$P = \left\{ \begin{array}{lrrrl} \boldsymbol{C_1}: & 1\,x_1 & -\mathbf{1}\,x_2 & -1 & \geq 0, \\ \boldsymbol{C_2}: & 1\,x_1 & +\mathbf{2}\,x_2 & +1 & \geq 0, \\ \boldsymbol{C_3}: & -1\,x_1 & +\mathbf{1}\,x_2 & +1 & \geq 0 \end{array} \right\}$$

$$P' = \left\{ \boldsymbol{C'} : \quad 3\,x_1 \quad +\mathbf{0}\,x_2 \quad -1 \quad \geq 0 \right\}$$

$$\begin{array}{cccccccc} \exists ? \lambda_i \geq 0 & \lambda_1 \times \boldsymbol{C_1} & + & \lambda_2 \times \boldsymbol{C_2} & + & \lambda_3 \times \boldsymbol{C_3} & = & \boldsymbol{C'} \\ \hline (x_1) & \lambda_1 \times 1 & + & \lambda_2 \times 1 & + & \lambda_3 \times -1 & = & 3 \\ (x_2) & \lambda_1 \times \textbf{-1} & + & \lambda_2 \times \mathbf{2} & + & \lambda_3 \times \mathbf{1} & = & \mathbf{0} \end{array}$$

Farkas combinations III

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$$P' = \left\{ \boldsymbol{C'} : \quad 3\,x_1 \quad +0\,x_2 \quad -\mathbf{1} \quad \geq 0 \right\}$$

$$\begin{array}{rlllll} \exists?\lambda_i \geq 0 & \lambda_1 \times \boldsymbol{C_1} & + & \lambda_2 \times \boldsymbol{C_2} & + & \lambda_3 \times \boldsymbol{C_3} & = & \boldsymbol{C'} \\ \hline\hline (x_1) & \lambda_1 \times 1 & + & \lambda_2 \times 1 & + & \lambda_3 \times -1 & = & 3 \\ (x_2) & \lambda_1 \times -1 & + & \lambda_2 \times 2 & + & \lambda_3 \times 1 & = & 0 \\ (cst) & \lambda_1 \times \textbf{-1} & + & \lambda_2 \times \mathbf{1} & + & \lambda_3 \times \mathbf{1} & = & \textbf{-1} \end{array}$$

Linear Algebra / Linear Programming

Gauss Resolution (Linear Algebra)

It gives solutions of the systems of equalities

$$\begin{array}{rl} (x_1) & \lambda_3 = -3 + \lambda_1 + \lambda_2 \\ (x_2) & \lambda_2 = 1 \\ (cst) & 0 \times \lambda_1 = 0 \quad \lambda_1 \textit{ is free, thus choose } \lambda_1 \geq 0 \end{array}$$

But some are not Farkas Combinations *i.e.* satisfying $\lambda_i \geq 0$

$$(\lambda_1, \lambda_2, \lambda_3) = \{ (0, 1, -2), (1, 1, -1), (2, 1, 0), (3, 1, 1), \ldots \}$$

$\exists ? \lambda_i \geq 0$ ... is not in the realm of Linear Algebra
but that of Linear Programming

Linear Algebra / Linear Programming

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The Simplex Algorithm (Linear Programming)

It's a way to choose pivots in Gauss elimination

$$\begin{array}{rl} (x_1) & \lambda_1 = 3 - \lambda_2 + \lambda_3 \\ (x_2) & \lambda_2 = 1 \\ (cst) & 0 \times \lambda_3 = 0 \quad \lambda_3 \textit{ is free, thus choose } \lambda_3 \geq 0 \end{array}$$

It focuses on Farkas Combinations *i.e.* satisfying $\lambda_i \geq 0$

$$(\lambda_1, \lambda_2, \lambda_3) = \{(2,1,0), (3,1,1), (4,1,2), (5,1,3), \dots\}$$

From efficient floating-point solver to exact solutions in $\mathbb{Q}$

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Efficient floating-point simplex algorithms (such as GLPK) do not provide exact solution (due to rounding)

$$(\lambda_1, \lambda_2, \lambda_3) = (1.99...,\ 1.0,\ 0.0...1)$$

But they are trustable on **variables that must be null** (*e.g.* $\lambda_3 = 0$) from which we can use **fast linear algorithm over the rationals (FLINT)** to solve the simplified system and obtain an exact solution $(\lambda_1, \lambda_2, \lambda_3) = (2, 1, 0)$

$\exists ? \lambda_i \geq 0$	$\lambda_1 \times \boldsymbol{C_1}$	$+$	$\lambda_2 \times \boldsymbol{C_2}$	$+$	$0 \times \boldsymbol{C_3}$	$=$	$\boldsymbol{C'}$
(x_1)	$\lambda_1 \times 1$	$+$	$\lambda_2 \times 1$			$=$	3
(x_2)	$\lambda_1 \times -1$	$+$	$\lambda_2 \times 2$			$=$	0
(cst)	$\lambda_1 \times -1$	$+$	$\lambda_2 \times 1$			$=$	-1

Polyhedra inclusion checker in COQ

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1. **Original work in 2010** [Besson et al., 2010]
   - results checking of Bytecode Static Analyzer
   - operations are performed by NewPolka [Jeannet and Miné, 2009]
   - witnesses are computed afterwards by solving Linear Programming problems
2. **VPL, started in 2012**
   - produces witnesses on-the-fly (no duplicate computation)
   - constraint-only representation

The inclusion checker (COQ code) (Definitions & Lemma)

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Definition Polyhedra := list (cstr $\mathbb{Q}$).

Definition *sat* $(x : \text{Vec})\ (p : \text{Polyhedra}) : \text{Prop} :=$
    $\text{List.Forall}\ p\ (\text{fun}\ (c : \text{csrt}\ \mathbb{Q}) \rightarrow c(x) \geq 0).$

Definition (infix $\sqsubseteq$) $(p_1\ p_2 : \text{Polyhedra}) : \text{Prop} :=$
    $\forall x : \text{Vec},\ \textit{sat}\ \ x\ p_1 \Rightarrow \textit{sat}\ \ x\ p_2.$

Lemma *Farkas* : $\forall\ (\Lambda : \text{list (list}\ \mathbb{Q}))\ (p_1\ p_2 : \text{Polyhedra})$
    $$(\forall \lambda \in \Lambda, \lambda \geq 0)\ \wedge\ (\underbrace{\textit{combine}}_{\simeq \textit{matrix-product}}\ \Lambda\ p_1) = p_2 \implies p_1 \sqsubseteq p_2$$

The inclusion checker (COQ code) (Program & Extraction)

Definition *inclusion_checker*
  (Λ : list (list $\mathbb{Q}$)) (p_1 p_2 : Polyhedra) : option ($p_1 \sqsubseteq p_2$) :=
 let *nn* := (*non_negative* Λ) **in**
 let *eq* := (*equal* (*combine* Λ p_1) p_2)
 in match (*nn*,*eq*) **with**
  | (**Some** proof_nn, **Some** proof_eq)
   $\rightarrow$ **Some** (*Farkas* Λ p_1 p_2 proof_nn proof_eq)
  | (_,_) $\rightarrow$ **None**
 end

COQ *inclusion_checker* $: (\Lambda, p_1, p_2) \rightarrow \begin{cases} \textbf{Some}\ (p_1 \sqsubseteq p_2) \\ \textbf{None} \end{cases}$

$\downarrow$ $\boxed{\textbf{extraction}}$

OCAML *inclusion_checker* $: (\Lambda, p_1, p_2) \rightarrow$ *bool*

Using the COQ checker in OCAML code

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Illustration on the convex-hull operator of the VPL

The convex-hull operator $P := P_1 \sqcup P_2$ is sound if

$$P_1 \sqsubseteq P \text{ and } P_2 \sqsubseteq P$$

which is proved using two Farkas inclusion witnesses Λ_1 and Λ_2 using

$$\textit{inclusion\_checker}\ (\Lambda_1, P_1, P) = \textbf{Some}(P_1 \sqsubseteq P)$$

$$\textit{inclusion\_checker}\ (\Lambda_2, P_2, P) = \textbf{Some}(P_2 \sqsubseteq P)$$

The convex-hull (OCAML code)

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```
type polyhedra = { ocaml: (rat cstr) list ; coq: (ℚ cstr) list }

let convex_hull  (p1:polyhedra) (p2:polyhedra) : polyhedra =
  let (f1, f2, pOcaml) =                                      } compute
       untrusted_convex_hull p1.ocaml  p2.ocaml
  in let Λ1    = rat_to_ℚ  f1                                  }
         Λ2    = rat_to_ℚ  f2                                  }
         pCoq = rat_to_ℚ  pOcaml                               }
  in  if (inclusion_checker Λ1  p1.coq  pCoq)                  } check
        && (inclusion_checker Λ2  p2.coq  pCoq)                }
     then { ocaml = pOcaml ; coq = pCoq }                      }
     else  error  "convex_hull"                                }
```

What is guaranteed by result verification?

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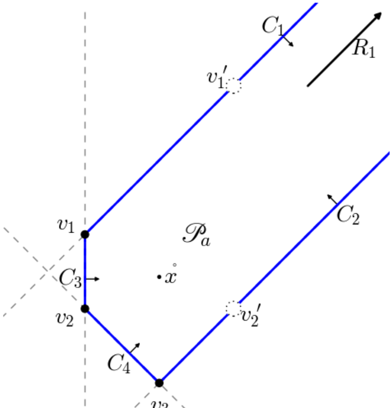
- The checker is extracted in OCAML but still uses COQ representations (trustable but inefficient)
  type nat = O | S of nat
  type positive = B1 of positive | B0 of positive | BH
  e.g. 5 ≃ S(S(S(S(S(O))))) ≃ B1(B0(B1(BH)))
- **12% overhead when the COQ checker is activated** (conversion into COQ representation then computations)
- **The COQ checker can be de/activated.**
- **The equality (p.ocaml = p.coq) cannot be guaranteed**
- **Guaranty:** the COQ side mimics the computations of the untrusted side and **the COQ side checks soundness**
- S.Boulmé noticed that *"the COQ type of polyhedra can even be an opaque abstract data type or a generic type 'α"* leading to new certification means using **factories**.

The Double Description of Polyhedra

used in most Polyhedra Libraries (NewPolka, PPL, Cudd, ...)

$$\left(\begin{array}{ccccc} \textbf{as constraints} & & & \textbf{as generators} & \\ \{\boldsymbol{C_1}, \boldsymbol{C_2}, \boldsymbol{C_3}, \boldsymbol{C_4}\} & , & \{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}\} & + & \{\boldsymbol{R_1}\} \\ & & \text{vertices} & + & \text{rays} \end{array} \right)$$



Why are polyhedra expensive ? I

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Polyhedra as generators: $\mathscr{G} \sqcap \{ C' \}$

The intersection with one constraint can **double the number of generators**.

Example (A sliced tube unbounded in one direction)

$$\mathscr{G} = \{ \boldsymbol{v_1}, \ldots, \boldsymbol{v_n} \} + \{ \boldsymbol{r_1} \}$$

$$\mathscr{G} \sqcap \{ \boldsymbol{C'} \} = \{ \boldsymbol{v_1}, \ldots, \boldsymbol{v_n} \} + \{ \boldsymbol{v'_1}, \ldots, \boldsymbol{v'_n} \}$$

Why are polyhedra expensive ? II

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Polyhedra as constraints: *elim* $(x_3, \mathscr{C})$ = projection on (x_1, x_2)

The elimination of one variable can **double the number of constraints**

Example (An orange segment)

$$\mathscr{C} = \{\boldsymbol{C'}, \boldsymbol{C''}, \ \ \boldsymbol{C_1}, \ldots, \boldsymbol{C_n}\}$$

$$\textit{elim}\ (x_3, \mathscr{C}) = \{\boldsymbol{C'_1}, \ldots, \boldsymbol{C'_n}, \boldsymbol{C''_1}, \ldots, \boldsymbol{C''_n}\}$$

$$\textit{elim}\ (\{x_3, x_2\}, \mathscr{C}) = \{b \le x_1, \ x_1 \le b'\}$$

Choosing the good representation $\mathscr{C}$? $\mathscr{G}$?

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- The polyhedra representation can double on basic operations ($\sqcap$, *elim*)
- Sequential eliminations of variables is exponential on constraints *elim* $[x_1 \ ; \ \ldots \ ; \ x_n]$ $\mathscr{C}$

  based on Fourier-Motzkin's elimination of **one** variable
- sequential intersections is exponential on generators

$$\mathscr{G} \sqcap [\boldsymbol{C_1} \ ; \ \ldots \ ; \ \boldsymbol{C_k}]$$

  based on Chernikova's intersection with **one** cutting plane
- $\mathscr{C}$ is needed for intersection and widening, used for inclusion and minimization
- DD can choose the best algorithm or an even better algorithm using $(\mathscr{C}, \mathscr{G})$

Why a polyhedra library in constraint-only?

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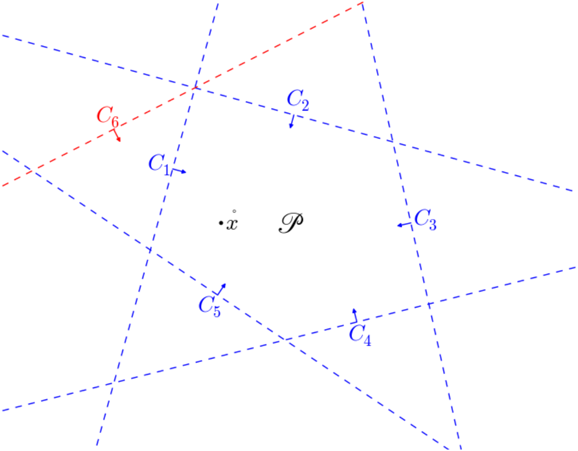
- **no polynomial algorithm to check equivalence** $(\mathscr{C}, \mathscr{G})$ (it probably does not exists)
- DD would have meant
  - **implementing** a naive version of **Chernikova**'s conversion algorithm **in COQ**,
  - proving it correct then
  - extracting to OCAML to get a correct but inefficient algorithm
- out of curiosity: no conversion, less memory space, **can it be as efficient as DD?**
- could we do better than **Fourier-Motzkin one-variable elimination** algorithm?

  [Howe and King, 2012]: Parametric Linear Programming can elimate several variables simultaneously
- could we improve **minimization**?

Minimization by Raytracing

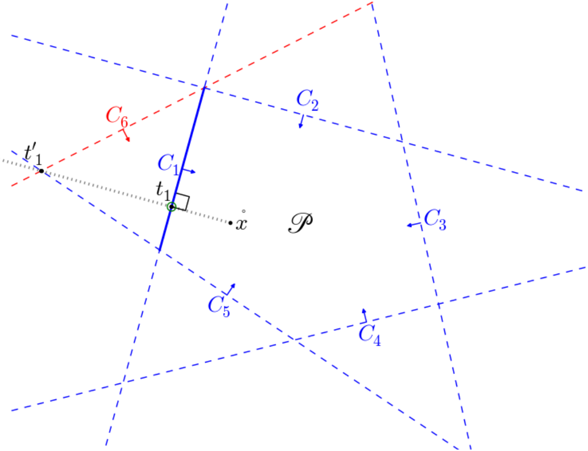
Launch a ray orthogonally to each constraint



the first constraint hit by the ray is irredundant

Minimization by Raytracing

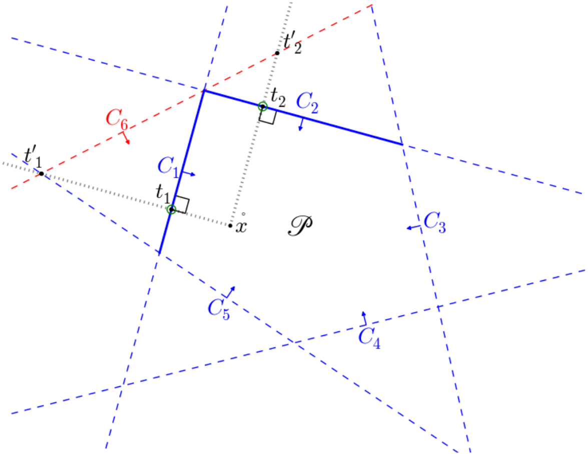
Launch a ray orthogonally to each constraint



the first constraint hit by the ray is irredundant

Minimization by Raytracing

Launch a ray orthogonally to each constraint



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Minimization by Raytracing

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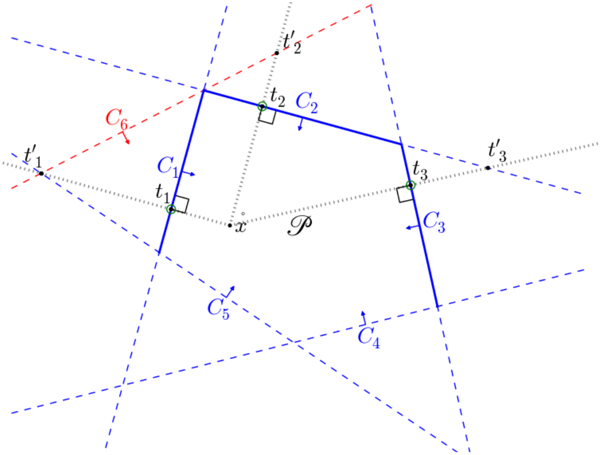

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Minimization by Raytracing

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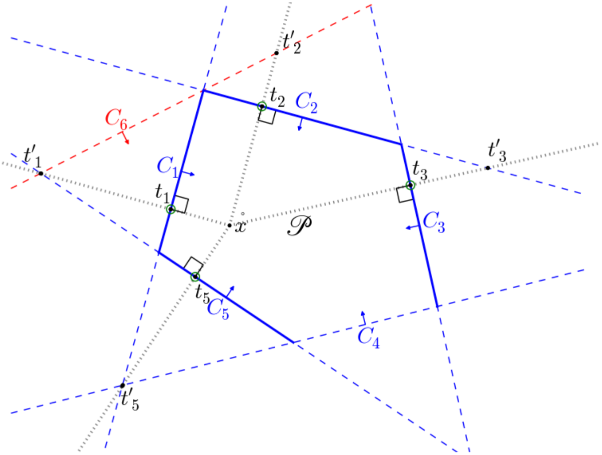

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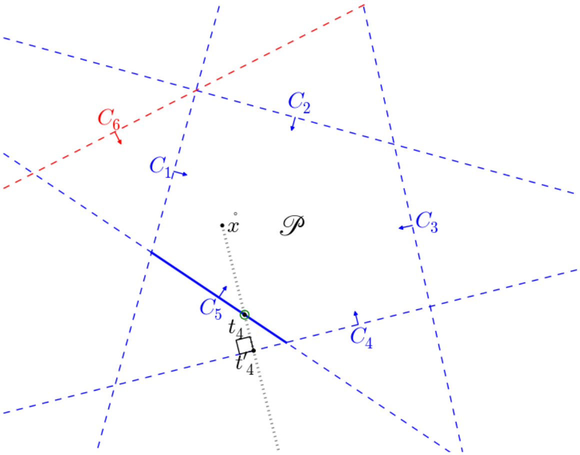
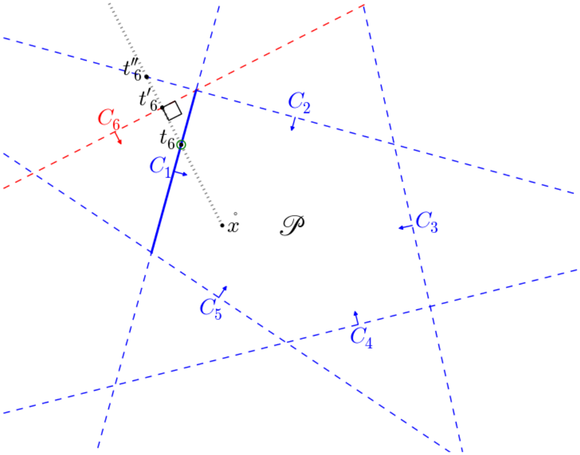


the first constraint hit by the ray is irredundant

Minimization by Raytracing

Launch a ray orthogonally to each constraint



the first constraint hit by the ray is irredundant

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When raytracing fails

We use the Standard Minimization Algorithm
= inclusion testing
= existence of a Farkas Combination

- $\{\boldsymbol{C_3}, \boldsymbol{C_5}\} \subseteq \{\boldsymbol{C_4}\}$? yes
- $\{\boldsymbol{C_1}, \boldsymbol{C_2}\} \subseteq \{\boldsymbol{C_6}\}$? no

Finally,
$\boldsymbol{C_1}, \boldsymbol{C_2}, \boldsymbol{C_3}, \boldsymbol{C_5}$ were determined without solving LP problems,
it only costs a matrix-matrix product:
matrix of constraints $\times$ matrix of rays

Experiments

We compared three algorithms:

1. The standard algorithm (SMA)
   Detecting redundancies by finding Farkas combinations.
   Requires one Linear Programming for each constraint.
   Each Linear Programming contains all the constraints.
2. Raytracing with rationals (RRT)
   using a rational simplex for finding Farkas combination
3. Raytracing with floating points (FRT)
   using a floating simplex (GLPK) then reconstruction

Experiments on random polyhedra

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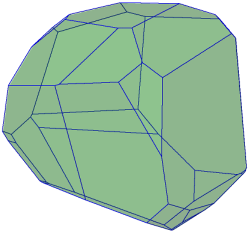
We generated random polyhedra to study the sensitivity of algorithms to

- the number of variables
- the number of constraints
- the number of generators
- the redundancy rate
- the density



Example (6 variables, density 50%)

$0\,x_1 + 21x_2 + 0\,x_3 + 26x_4 + 0\,x_5 - 13x_6 \leq 20$

Experimental results

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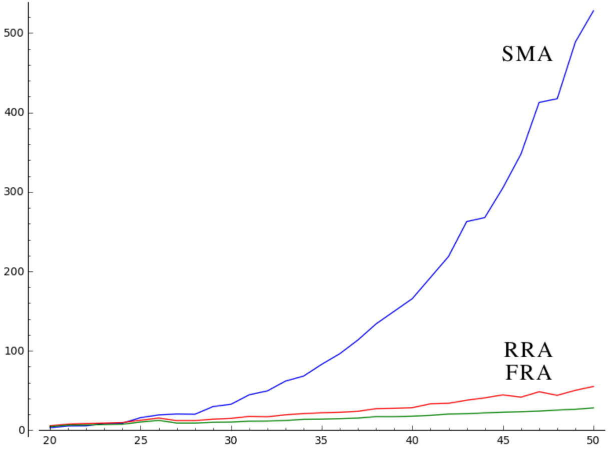

Time (ms) when varying then **number of constraints**, with 10 variables, a redundancy rate of 50%, and a density of 50%.

Comparison with Chernikova's conversion algorithm

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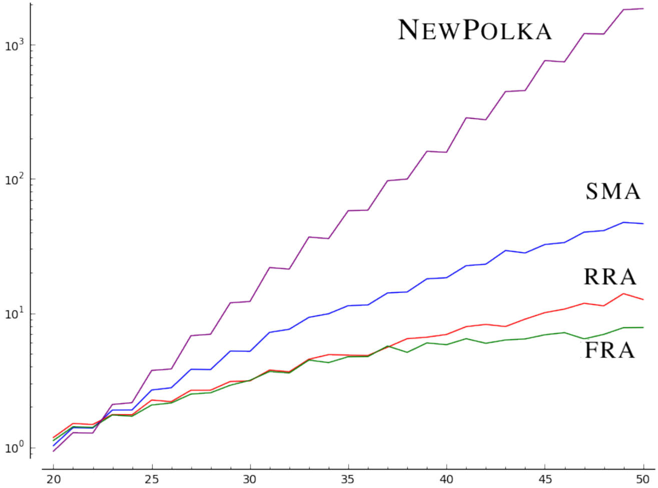

Time (ms) in log scale

Ongoing Work I (Alexandre MARÉCHAL)

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Experimentation is not easy

- **Comparing libraries: How to be fair?**
  - DD delay some operations (conversion, minimization)
  - start by a conversion to build $(\mathscr{C}, \mathscr{G})$,
  - n-dimenions hypercubes (2^n generators) kill them
- **Using which analyzer?**

  static analyzers have been designed for intervals, not ofr polyhedra (Frama-C, VERASCO)
  - too much (useless) variables involved
  - duplication of computations
  - $(P \sqcup P') = P'$ instead of $P \sqsubseteq P'$

Ongoing Work I (Alexandre MARÉCHAL)

An experimental setup is under development

Goal: **Profile each operator** on random polyhedra **to study sensitivity** to the number of variables, of constraints, of generators, of redundancies, and the density (number of nonzero coefficients)

- record the call to polyhedral operators during analysis of
- realistic programs with a polyhedra-aware static analyzer
- extract significant sequences of operations, *e.g.*

$$DD \ ; \ \textit{timer-start} \ ; (\sqsubseteq \ ; \sqcap ; \ := \ ; \sqcup)^* ; \ \textit{min} \ ; \ \textit{timer-stop}$$

- run the sequence on each library with random inputs

Ongoing Work II (Hang YU)

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Parallelization, floating-point computations then reconstruction

- minimization by raytracing
  independant determination of each constraint
- the Solver of Parametric Linear Problems
  (C, C++, GLPK, FLINT, EIGEN, COQ, OCAML)
- new algorithm for inclusion

Will someday Polyhedra be usable?

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May be with

- a polyhedra-aware static analyzer
- polyhedra used in a second phase where intervals failed
- dynamic packing of variables, removal of useless variable
- algorithms in constraint-only can benefit libraries in DD: the costly Chernikova's conversion can be delayed

Will someday Polyhedra be usable?

The field of polyhedra still makes progress

Nice result by [Singh et al., 2017] based on [Halbwachs et al., 2006] to cope with the hypercube phenomenon:

$$\begin{aligned} \mathcal{H} &= \mathrm{DD}(\textstyle\bigwedge_{i=1}^{n} -1 \leq x_i \leq 1) \\ &= (2 \times n \text{ constraints}, 2^n \text{ generators}) \end{aligned}$$

- The Fast Polyhedra Abstract Domain automatically splits polyhedra into a cartesian product during the analysis.
- Fast polyhedra almost behave like intervals when variables are not related.

Cartesian product of polyhedra [Halbwachs+1996, Singh+2015]

$$\mathcal{H} = P_1 \times \ldots \times P_n \text{ where } P_i = (\ \{-1 \leq x_i \leq 1\}\ ,\ \{-1, 1\}\ )$$

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Publications related to the VPL

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on the algorithmic side of the VPL

SAS'2013 *Efficient generation of correctness certificates for the abstract domain of polyhedra*

VMCAI'2016 *Polyhedral Approximation of Multivariate Polynomials using Handelman's Theorem*

VMCAI'2017 *Efficient Elimination of Redundancies in Polyhedra using Raytracing*

SAS'2017 *Scalable Minimizing-Operators on Polyhedra via Parametric Linear Programming*

on the COQ side of the VPL

ITP'2015 *Refinement to Certify Abstract Interpretations, Illustrated on Linearization for Polyhedra*

VSTTE'2014 *A Certifying Frontend for (Sub)polyhedral Abstract Domains*