Introduction to Interactive Proof of Software

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Lecture 7

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Structural induction

Induction on a inductive predicate

Induction on a inductive predicate

Well-founded induction

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A very natural generalisation of induction

On lists

$$\frac{P \text{ nil } \forall n \forall l, P l \Rightarrow P(n :: l)}{\forall l, P l}$$

Examples: stuttering list, associativity of append, reverse

On binary trees

$$\frac{P \text{ leaf } \forall n \forall t_l t_r, P t_l \Rightarrow P t_r \Rightarrow P (\text{Node } t_l n t_r)}{\forall t, P t}$$

Examples: number of keys and of leaves, algorithms on binary search trees

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Inductive even : nat -> Prop :=
 | E0 : even 0
 | E2: forall n:nat, even n -> even (S (S n)).

We expect the following induction principle:

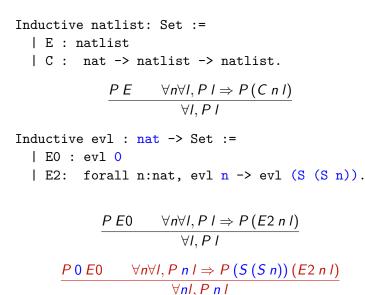
 $\frac{P \ 0 \qquad \forall n, even \ n \Rightarrow P \ n \Rightarrow P \ (S \ (S \ n))}{\forall n, even \ n \Rightarrow P \ n}$

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$$\frac{P \ 0 \ E0}{\forall n \forall I, P \ n \ I \Rightarrow P \ (S \ (S \ n)) \ (E2 \ n \ I)}{\forall nI, P \ n \ I}$$

Take for P a predicate which does not depend on its second argument: $P n I \stackrel{\text{def}}{=} Q n$

$$\frac{Q \ 0}{\forall n \ \forall (I : evl \ n), Q \ n \Rightarrow Q \ (S \ (S \ n))}} \\
\frac{Q \ 0}{\forall n(I : evl \ n), Q \ n} \\
\frac{Q \ 0}{\forall n, evl \ n \Rightarrow Q \ n \Rightarrow Q \ (S \ (S \ n))}}{\forall n, evl \ n \Rightarrow Q \ n}$$

Now, evl reads just even

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Functional interpretation

```
Inductive list : Set :=

| E : list

| C : nat -> list -> list.

\frac{P E \quad \forall n \forall I, P I \Rightarrow P(C n I)}{\forall I, P I}
```

Lists of consecutive even numbers typed according to the value of the expected next head Inductive ev] : nat -> Set := | EO : ev] O | E2: forall n:nat, $evl n \rightarrow evl (S (S n))$. $P E0 \quad \forall n \forall I, P I \Rightarrow P(E2 n I)$ $\forall I. P I$ $\forall n \forall I, P \mid n \mid I \Rightarrow P(S(S \mid n))(E2 \mid n \mid I)$ P 0 E0 $\forall n \mid P \mid n \mid$

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Booleans and inductively defined predicates

```
Fixpoint evenb (n:nat) : bool :=
  match n with
  | 0 => true
  | S O => false
  | S (S n') => evenb n'
  end.
Inductive even : nat -> Prop :=
  | EO : even O
  | E2 : \forall n, even n -> even (S (S n)).
Theorem even evenb : \forall n, even n -> evenb n = true.
By induction on the structure of the proof of even n
Theorem evenb even : \forall n, evenb n = true -> even n.
By induction on n
```

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```
Theorem even_evenb :
\forall n, even n -> evenb n = true.
```

By induction on the structure of the proof of even n Don't have to bother about odd numbers

```
Theorem evenb_even :
\forall n, evenb n = true -> even n.
```

By induction on *n*: need for strengthening and discrimination.

Inversion

lssue: getting the possible ways of constructing a hypothesis
Easier for evenb than for even, see even_inversion.v
This issue cannot be avoided for non-deterministic relations

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Stronger induction principles

$$\frac{P0 \qquad P1 \qquad \forall n, Pn \land P(Sn) \Rightarrow P(S(Sn))}{\forall n, Pn}$$

$$\frac{P \ 0 \qquad \forall n, (\forall m, m \le n \Rightarrow P \ m) \Rightarrow P \ (S \ n)}{\forall n, P \ n}$$

By (basic) induction on $Q \ n \stackrel{\text{def}}{=} \forall m, m \leq n \Rightarrow P \ m$ Rephrasing

$$\frac{\forall n, (\forall m, m < n \Rightarrow P m) \Rightarrow P n}{\forall n, P n}$$

Well-founded induction on (*nat*, <)

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Material:

- ► S: a set, called the domain of the induction
- ► *R*: a relation on *S*
- R is well-founded (see below)

Then we have the following induction principle:

 $\frac{\forall x, (\forall y, R \ y \ x \Rightarrow P \ y) \Rightarrow P \ x}{\forall x, P \ x}$

Two definitions on well-founded (equivalent in classical logic)

- any decreasing chain eventually stops
- all elements of S are accessible

An element is accessible $\stackrel{\text{def}}{=}$ all its predecessors are accessible

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Theorem of chocolate tablets

Statement

Let us take a tablet containing n tiles and cut it into pieces along grooves

How many shots are needed for reducing the tablet into tiles?

Answer

n-1

It does not depend on successive choices of grooves!

Proof

By well-founded induction on (nat, <)

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Construction of well-founded relations

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Well-founded induction

E.g. the lexicographic ordering of two well-founded relations is well-founded.