Introduction to Interactive Proof of Software

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Lecture 2

Outline

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Definition by cases on an enumerated type

Question

Give a `color` for each possible value in `rgb`

$$\frac{}{\texttt{rgb}}\,\texttt{Rf} \qquad \frac{}{\texttt{rgb}}\,\texttt{Gf} \qquad \frac{}{\texttt{rgb}}\,\texttt{Bf}$$

Example

`Rf` maps to `Red`, `Gf` maps to `Green`, `Bf` maps to `Blue`

$$\frac{\overbrace{\texttt{rgb}}^{r} \qquad \dfrac{}{\texttt{color}}\,\texttt{Red} \qquad \dfrac{}{\texttt{color}}\,\texttt{Green} \qquad \dfrac{}{\texttt{color}}\,\texttt{Blue}}{\texttt{color}}\,\texttt{case}$$

Definition by cases on an enumerated type

Question

Give a `rgb` for each possible value in `rgb`

$$\frac{}{\texttt{rgb}}\ \texttt{Rf} \qquad \frac{}{\texttt{rgb}}\ \texttt{Gf} \qquad \frac{}{\texttt{rgb}}\ \texttt{Bf}$$

Example

`Rf` maps to `Bf`, `Gf` maps to `Gf`, `Bf` maps to `Rf`

$$\frac{\overbrace{\texttt{rgb}}^{r} \qquad \frac{}{\texttt{rgb}}\ \texttt{Bf} \qquad \frac{}{\texttt{rgb}}\ \texttt{Gf} \qquad \frac{}{\texttt{rgb}}\ \texttt{Rf}}{\texttt{rgb}}\ \texttt{case}$$

Building block

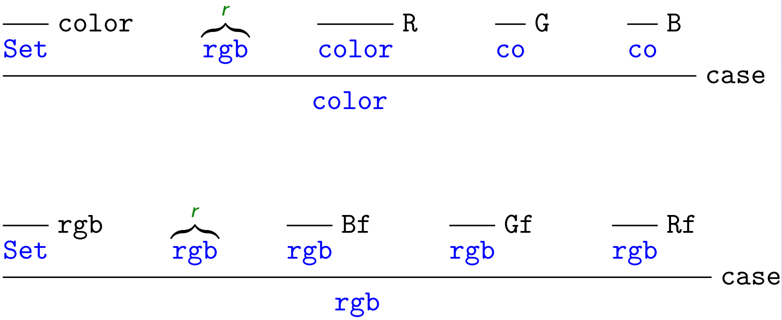
In this presentation, the order of contructors matters:
`Rf`, `Gf`, `Bf`

The case construct is driven by 2 parameters

- the type of the value to be analyzed
  each enumerated type (e.g. `rgb`) comes automatically with its case construct, which should actually be written, e.g. $\texttt{case}_{\texttt{rgb}}$
- the type of the result

$$\frac{\overbrace{\texttt{Set}}^{A} \quad \overbrace{\texttt{rgb}}^{r} \quad \overbrace{\texttt{A}}^{x_1} \quad \overbrace{\texttt{A}}^{x_2} \quad \overbrace{\texttt{A}}^{x_3}}{\texttt{A}}\ \texttt{case}$$

Correct version of previous examples



Definition by cases on a general inductive type

To be introduced below...

Outline

From graphical presentation to Coq syntax

$$\frac{}{\texttt{color}}\ \texttt{Red} \qquad \frac{}{\texttt{color}}\ \texttt{Orange} \qquad \frac{}{\texttt{color}}\ \texttt{Yellow} \qquad \frac{}{\texttt{color}}\ \texttt{Green}$$

$$\frac{}{\texttt{color}}\ \texttt{Blue} \qquad \frac{}{\texttt{color}}\ \texttt{Indigo} \qquad \frac{}{\texttt{color}}\ \texttt{Violet}$$

And simultaneously $\quad \dfrac{}{\texttt{Set}}\ \texttt{color}$

```
Inductive color: Set :=
  | Red : color     | Orange : color  | Yellow : color
  | Green : color   | Blue : color    | Indigo : color
  | Violet : color
.
```

Coq syntax of tuple4

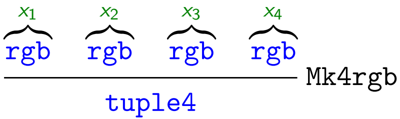
Making a 4-tuple of rgb

$$\frac{\overbrace{\texttt{rgb}}^{x_1} \quad \overbrace{\texttt{rgb}}^{x_2} \quad \overbrace{\texttt{rgb}}^{x_3} \quad \overbrace{\texttt{rgb}}^{x_4}}{\texttt{tuple4}}\ \texttt{Mk4rgb}$$

```
Inductive tuple4 : Set :=
  | Mk4rgb :
      forall x1: rgb, forall x2: rgb,
      forall x3: rgb, forall x4: rgb, tuple4
```

Coq syntax of tuple4, shorthand

Making a 4-tuple of rgb




```
Inductive tuple4 : Set :=
  | Mk4rgb : forall x1 x2 x3 x4: rgb, tuple4.
```

More general 4-tuples: several constructors

$$\frac{\overbrace{\texttt{rgb}}^{x_1}\quad\overbrace{\texttt{rgb}}^{x_2}\quad\overbrace{\texttt{rgb}}^{x_3}\quad\overbrace{\texttt{rgb}}^{x_4}}{\texttt{tuple4}}\ \texttt{Mk4rgb}$$

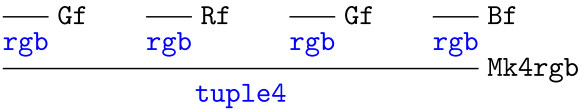
$$\frac{\overbrace{\texttt{color}}^{x_1}\quad\overbrace{\texttt{color}}^{x_2}\quad\overbrace{\texttt{color}}^{x_3}\quad\overbrace{\texttt{color}}^{x_4}}{\texttt{tuple4}}\ \texttt{Mk4co}$$

$$\frac{\overbrace{\texttt{tuple4}}^{x_1}\quad\overbrace{\texttt{tuple4}}^{x_2}\quad\overbrace{\texttt{tuple4}}^{x_3}\quad\overbrace{\texttt{tuple4}}^{x_4}}{\texttt{tuple4}}\ \texttt{Mk4t4}$$

```
Inductive tuple4 : Set :=
  | Mk4rgb : forall x1 x2 x3 x4: rgb, tuple4
  | Mk4color : forall x1 x2 x3 x4: color, tuple4
  | Mk4t4 : forall x1 x2 x3 x4: tuple4, tuple4
.
```

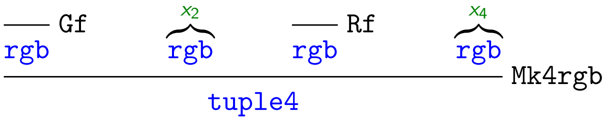
Pluging trees: the interactive way

A concrete 4-tuple of rgb




```
Definition t1: tuple4.
apply Mk4rgb.
  apply Gf.
  apply Rf.
  apply Gf.
  apply Bf.
Defined.
```

Open trees



In Coq

```
Section a_tuple_with_variable.

Variable x2: rgb.
Variable x4: rgb.

Definition t4 etc.

End a_tuple_with_variable.
```

Exercises

Write trees for examples of 4-tuples of 4-tuples using `tuple4`.
Some of them, closed, some of them open
E.g. $\langle\langle \mathtt{R}, \mathtt{Y}, \mathtt{B}, \mathtt{B}\rangle, \langle \mathtt{B}, \mathtt{O}, x_4, \mathtt{R}\rangle, \langle x_7, x_7, x_7, \mathtt{V}\rangle, \langle \mathtt{V}, \mathtt{Y}, \mathtt{O}, \mathtt{R}\rangle\rangle$

Definition by cases on an enumerated type

Rf maps to Red, Gf maps to Green, Bf maps to Blue

$$\dfrac{\dfrac{}{\texttt{Set}}\texttt{color} \qquad \overbrace{\texttt{rgb}}^{r} \qquad \dfrac{}{\texttt{color}}\texttt{R} \qquad \dfrac{}{\texttt{co}}\texttt{G} \qquad \dfrac{}{\texttt{co}}\texttt{B}}{\texttt{color}}\ \texttt{case}$$

```
Definition color_of_r: color.
destruct r.
  apply Red.
  apply Green.
  apply Blue.
Defined.
```

General case

$$\frac{\overbrace{\texttt{Set}}^{A}\qquad \overbrace{\texttt{rgb}}^{r}\qquad \overbrace{\texttt{A}}^{x_1}\qquad \overbrace{\texttt{A}}^{x_2}\qquad \overbrace{\texttt{A}}^{x_3}}{\texttt{A}}\ \texttt{case}$$

```
Definition A_of_r: color.
destruct r.
  apply x1.
  apply x2.
  apply x3.
Defined.
```

Direct definition

```
Definition color_of_r : color :=
  match r with
  | Rf => Red
  | Gf => Green
  | Bf => Blue
  end.
```

Case analysis on a tree

See interactive session

Outline

Case analysis on a constant

```
Definition color_of_Bf : color :=
  match Bf with
  | Rf => Red
  | Gf => Green
  | Bf => Blue
  end.
```

Reduction

```
match Bf with
| Rf => Red
| Gf => Green
| Bf => Blue
end.
```

Reduces to

```
Blue
```

Outline

Functions

```
Definition color_of : forall (r: rgb), color :=
  fun (r: rgb) =>
  match r with
  | Rf => Red
  | Gf => Green
  | Bf => Blue
  end.
```

Application: by juxtaposition without parenthesis

```
color_of Bf
```

Parentheses can be used for grouping

More functions

```
Definition Set_of : forall (r: rgb), Set :=
  fun (r: rgb) =>
  match r with
  | Rf => rgb
  | Gf => color
  | Bf => tuple4
  end.

Definition funny : forall (r: rgb), Set_of r :=
  fun (r: rgb) =>
  match r with
  | Rf => Red
  | Gf => Green
  | Bf => Blue
  end.
```

Interactive definition of a function

Use intro

```
Definition interactive_color_of :
   forall (r: rgb), color.
intro r.
destruct r.
  apply Red.
  apply Green.
  apply Blue.
Defined.
```

Outline

Two kinds of trees: first kind

$$t1 := \left\{ \dfrac{\dfrac{}{\texttt{rgb}}\,\texttt{Gf} \qquad \dfrac{}{\texttt{rgb}}\,\texttt{Rf} \qquad \overbrace{\texttt{rgb}}^{r} \qquad \dfrac{}{\texttt{rgb}}\,\texttt{Bf}}{\texttt{tuple4}}\;\texttt{Mk4rgb} \right.$$

t2:= ..., t3 := ... t4 := ...
(similar to t1, using constructors Mk4co, Mk4rgb, Mk4rt4 and variables only)

$$\dfrac{\dfrac{\overbrace{\texttt{rgb}}^{r}}{\texttt{tuple4}}\,\texttt{t1} \qquad \dfrac{}{\texttt{tuple4}}\,\texttt{t2} \qquad \dfrac{}{\texttt{tuple4}}\,\texttt{t3} \qquad \dfrac{}{\texttt{tuple4}}\,\texttt{t4}}{\texttt{tuple4}}\;\texttt{Mk4t4}$$

Similar to the usual data structures in programming

Two kinds of trees: second kind (with case)

$$\dfrac{\dfrac{}{\texttt{Set}}\ \texttt{color} \qquad \overbrace{\texttt{rgb}}^{r} \qquad \dfrac{}{\texttt{color}}\ \texttt{R} \qquad \dfrac{}{\texttt{co}}\ \texttt{G} \qquad \dfrac{}{\texttt{co}}\ \texttt{B}}{\texttt{color}}\ \texttt{case}$$

With $\texttt{co} := \texttt{color}$

$$\dfrac{\dfrac{}{\texttt{Set}}\ \texttt{tu4} \qquad \overbrace{\texttt{rgb}}^{r} \qquad \dfrac{}{\texttt{tu4}}\ \texttt{t2} \qquad \dfrac{}{\texttt{tu4}}\ \texttt{t4} \qquad \dfrac{}{\texttt{tu4}}\ \texttt{t1}}{\texttt{tu4}}\ \texttt{case}$$

With $\texttt{tu4} := \texttt{tuple4}$

Here, case looks strange: the usual intuition associates it to control, not to data

Explanation

Inside Coq

The internal representation *all* trees is really what you expect. Rules labelled with `case` are implemented by a node pointing to all branches representing the subtrees on top of the corresponding line (5 of them in the previous examples).

To some extent, `case` can be seen as a primitive (and very flexible) constructor.

Evolution

The intuitive idea of control behind `case` can be understood as the fate of the corresponding node: when a constant, e.g., `Bf` will be plugged to the key argument (`r:rgb` in our examples), then this part of the tree will be reduced to the corresponding subtree (here: the rightmost, i.e., respectively `B` and `t1` on the 2 previous examples).

Reduction

Things will be come clear after the introduction of the notion of reduction in lecture 03.

Type Theory (the mathematical foundation of Coq) relies on 3 tightly coupled notions, which only make sense when they are together:

- constructors of an inductive type
- case analysis on an inductive type
- reduction

Last remark

The idea of plugging a tree making a given type into an input (having the same type) of another tree is completely uniform.
Hence, a `case` can be embedded in a tree.

$$c1 := \left\{ \dfrac{\dfrac{}{\mathtt{Set}}\,\mathtt{co} \qquad \overbrace{\mathtt{rgb}}^{r} \qquad \dfrac{}{\mathtt{co}}\,\mathtt{R} \qquad \dfrac{}{\mathtt{co}}\,\mathtt{G} \qquad \dfrac{}{\mathtt{co}}\,\mathtt{B}}{\mathtt{color}}\,\mathtt{case} \right.$$

$$\dfrac{\dfrac{}{\mathtt{color}}\,\mathtt{G} \qquad \dfrac{\overbrace{\mathtt{rgb}}^{r}}{\mathtt{color}}\,\mathtt{c1} \qquad \dfrac{}{\mathtt{color}}\,\mathtt{B} \qquad \dfrac{}{\mathtt{color}}\,\mathtt{R}}{\mathtt{tuple4}}\,\mathtt{Mk4co}$$