

Natural Deduction 2

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Theorem

Theorem 3.3.1

If a formula A is deduced from an environment Γ (i.e., if $\Gamma \vdash A$) then A is a consequence of Γ ($\Gamma \models A$).

Every proof written in an environment Γ is correct !

Proof by induction :

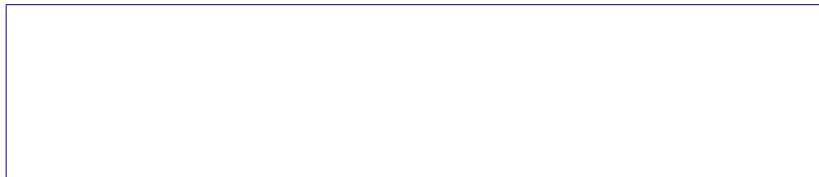
- ▶ Let Γ a set of formulae.
- ▶ Let P a proof of A in this environment.
- ▶ Let C_i the conclusion and H_i the context of i^{th} line of P .
- ▶ Let Γ, H_i the set of formulae of Γ and of the list H_i .

Show that for every k we have $\Gamma, H_k \models C_k$.

For the last line n of the proof : H_n is empty and $C_n = A$

Hence, $\Gamma \models A$.

Base case



Induction hypothesis

Suppose that for every line $i < k$ of the proof we have $\Gamma, H_i \models C_i$.

Let us show $\Gamma, H_k \models C_k$.

Three possible cases :

- ▶ Line k is $\langle \text{Suppose } C_k \rangle$.
- ▶ Line k is $\langle \text{Hence } C_k \rangle$.
- ▶ Line k is $\langle C_k \rangle$.

Line k is \llcorner Suppose $C_k \lrcorner$



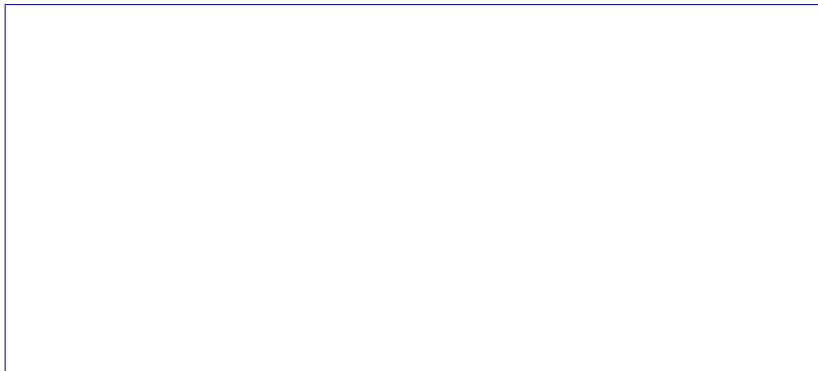
The line k is \llcorner Hence $C_k \lrcorner$



Line k is $\ll C_k \gg$

This formula is the conclusion of a rule of table 3.1, applied to its usable premises on the previous line or to the element of Γ .

We only consider the rule $\wedge I$, the other cases being similar.



Theorem

We prove the completeness of the rules only for formulas containing the following logic symbols : \perp , \wedge , \vee , \Rightarrow .

This is enough because additional symbols \top , \neg and \Leftrightarrow can be regarded as **abbreviations**.

Theorem 3.4.1

Let Γ be a finite set of formulae and A a formula, if $\Gamma \models A$ then $\Gamma \vdash A$.

Notations

We define a **literal** as a **variable** or an **implication between a variable and \perp** .

Let x be a variable, x and $x \Rightarrow \perp$ (which can be abbreviated as $\neg x$) are **complementary** literals.

Given Γ , a list of formulae, $s(\Gamma)$ denotes the set of formulae of Γ .

To simplify notations, we use the **comma** for **adding** an element at the beginning or at the end of the list and for **concatenating** two lists, which can be either lists of formulae or lists of proofs.

Measure

Then **measure** m of formulae and of lists of formulae is defined as :

- ▶ $m(\perp) = 0$,
- ▶ $m(x) = 1$ where x is a variable,
- ▶ $m_o(\Rightarrow) = 1$,
- ▶ $m_o(\wedge) = 1$,
- ▶ $m_o(\vee) = 2$,
- ▶ $m(A \circ B) = m(A) + m_o(\circ) + m(B)$,
- ▶ the measure of a list of formulae is the sum of the measures of the formulae of the list.

Since $\neg A$ is an abbreviation of $A \Rightarrow \perp$, we have :

$$m(\neg A) = m(A \Rightarrow \perp) = m(A) + 1.$$

For example, let $A = (a \vee \neg a)$. We have $m(\neg a) = 2$, $m(A) = 5$ and $m(A, (b \wedge b), A) = 13$.

Induction

We define $P(n)$ to be the following property :
given any list of formulae Γ and formula A such that the measure of Γ, A is n , we have **if $s(\Gamma) \models A$ then $s(\Gamma) \vdash A$.**

To show that $P(n)$ holds for every integer n , we use **< strong >** induction :

Suppose that for every $i < k$, $P(i)$ holds ; then show that $P(k)$ holds as well.

To this effect suppose moreover $m(\Gamma, A) = k$ and $s(\Gamma) \models A$, then show : **$s(\Gamma) \vdash A$.**

Decomposition

Idea : we decompose Γ, A in order to apply the induction hypothesis.

A is undecomposable if A is \perp or a variable and Γ is undecomposable if Γ is a list of literals or contain the formula \perp .

We study three cases :

Case 1 : **neither A , nor Γ is decomposable.**

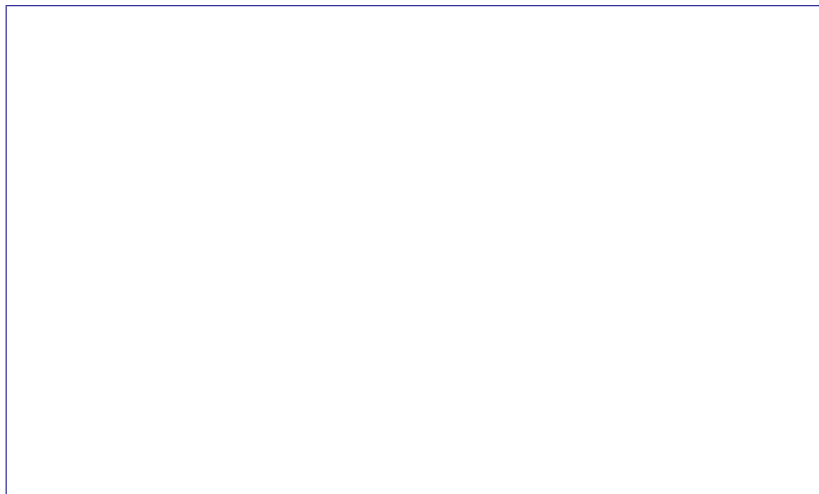
Case 2 : **A is decomposable.**

We decompose A in two sub-formulae B and C .

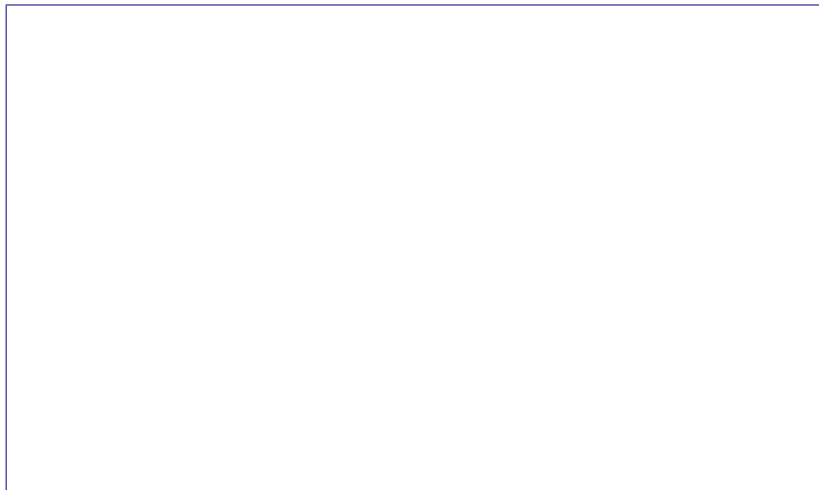
We obtain the following inequalities : $m(\Gamma, B) < m(\Gamma, A)$ and $m(\Gamma, C) < m(\Gamma, A)$.

Case 3 : **Γ is decomposable.** We permute Γ in order to obtain a list and a decomposable formula.

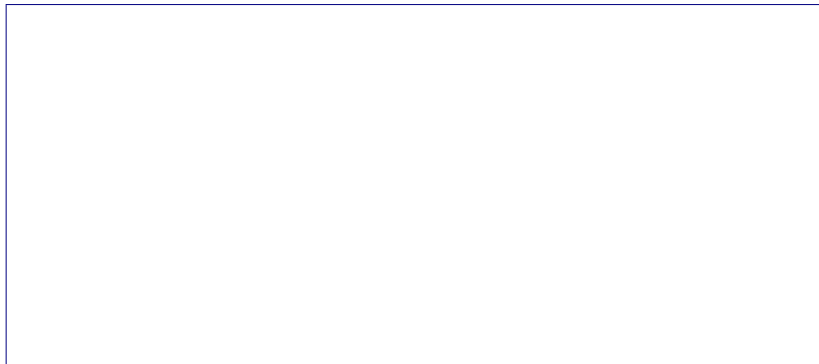
Case 1 : neither A , nor Γ are decomposable



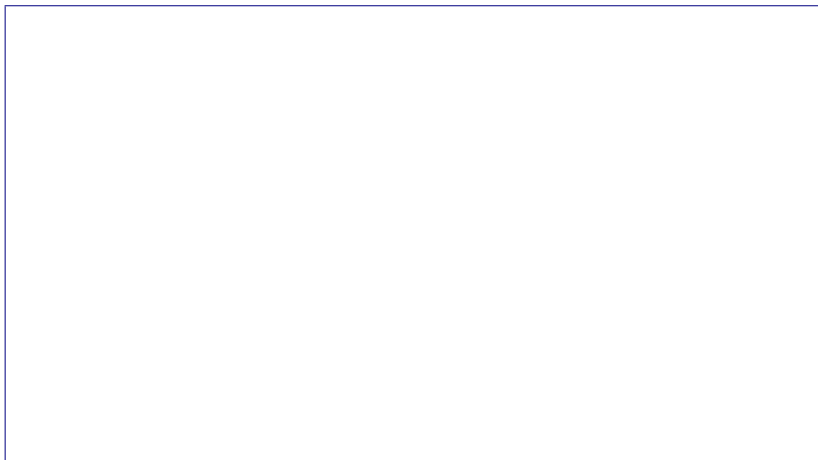
Case 2 : A is decomposable into B and C



Case 3 : Γ is decomposable



Γ is a **permutation** of the list $(B \wedge C), \Delta$



Remark 3.4.2

The proof of completeness is constructive, that is it provides a complete set of tactics to construct the proofs of a formula in an environment.

However, these tactics can lead to long proofs.

It is better then to use « optimised » tactics presented in section 3.2.

For example, to prove $B \vee C$:

- ▶ First try to prove B
- ▶ If failure, then try to prove C
- ▶ Otherwise, use tactic 10 (prove C under the hypothesis $\neg B$)

Proof tactics

We wish to prove A in the environment Γ

The 13 following tactics must be used in the following order !

Tactic 1

If $A \in \Gamma$ then the empty proof is obtained.

Tactic 2

If A is the consequence of a rule whose premises are in Γ , then the obtained proof is

$\ll A \gg$.

Tactic 3

If Γ contains a contradiction, that is a formula B and a formula $\neg B$, then the obtained proof is $\ll \perp, A \gg$.

Tactic 4

If A is $B \wedge C$ then :

- ▶ prove B : Let P the proof obtained for B ,
- ▶ prove C : Let Q the proof obtained for C .

The proof obtained for A is $\langle\langle P, Q, A \rangle\rangle$.

The proofs can fail (if it is asked to prove a formula that is unprovable in the given environment) : if the proof of B or C fails, it is the same for the proof of A . To simplify the remaining, we do not highlight the failure cases anymore, unless they must be followed by another proof.

Tactic 5

If A is $B \Rightarrow C$, then prove C under hypothesis B
(B is added to the environment).

Let P , the proof obtained for C .

The proof obtained for A is « Suppose B, P , Hence A ».

Tactic 6

If A is $B \vee C$, then prove B :

If P is the proof obtained for B , then $\langle P, A \rangle$ is the proof obtained for A .

If the proof of B fails then prove C :

If P is the proof obtained for C then $\langle P, A \rangle$ is the proof obtained for A .

If the proof of C fails, try the following rules.

Tactic 7

If $B \wedge C$ is in the environment, then prove A starting from formulae B , C , replacing $B \wedge C$ in the environment and let P the result of this proof.

Then $\langle B, C, P \rangle$ is a proof of A in the initial environment.

Tactic 8

If $B \vee C$ is in the environment, then :

- ▶ prove A in the environment where B replaces $B \vee C$: Let P the obtained proof,
- ▶ prove A in the environment where C replaces $B \vee C$: Let Q the obtained proof.

The proof of A is then \ll Suppose B, P , Hence $B \Rightarrow A$, Suppose C, Q , Hence $C \Rightarrow A, A \gg$.

Tactic 9

If $\neg(B \vee C)$ is in the environment, then

- ▶ we derive $\neg B$ by the proof $P4$ and
- ▶ $\neg C$ by the proof $P5$ (proofs requested in exercise 58).
- ▶ Let P the proof of A in the environment where $\neg B, \neg C$ replace the formula $\neg(B \vee C)$.

The proof of A is $\ll P4, P5, P \gg$.

Tactic 10

If A is $B \vee C$, then prove C under hypothesis $\neg B$: let P the obtained proof.

\ll Suppose $\neg B, P$, Hence $\neg B \Rightarrow C \gg$ is a proof of the formula $\neg B \Rightarrow C$ which is equivalent to A .

To obtain the proof of A , simply add the proof $P1$, requested in exercise 58 of A in the environment $\neg B \Rightarrow C$.

The proof obtained from A is therefore \ll Suppose $\neg B, P$, Hence $\neg B \Rightarrow C, P1 \gg$.

Tactic 11

If $\neg(B \wedge C)$ is in the environment, then we deduce from it $\neg B \vee \neg C$ by the proof $P3$ requested in exercise 58 then we reason case by case as follows :

- ▶ prove A in the environment where $\neg B$ replaces $\neg(B \wedge C)$: Let P the obtained proof,
- ▶ prove A in the environment where $\neg C$ replaces $\neg(B \wedge C)$: Let Q the obtained proof.

The proof of A is $\ll P3, \text{Suppose } \neg B, P, \text{Hence } \neg B \Rightarrow A, \text{Suppose } \neg C, Q, \text{Hence } \neg C \Rightarrow A, A \gg$.

Tactique 12

If $\neg(B \Rightarrow C)$ is in the environment, then

- ▶ we derive B by the proof $P6$,
- ▶ $\neg C$ by the proof $P7$ (proofs requested in exercise 58).
- ▶ Let P the proof of A in the environment where $B, \neg C$ replace the formula $\neg(B \Rightarrow C)$.

The proof of A is $\ll P6, P7, P \gg$.

Tactic 13

If $B \Rightarrow C$ is in the environment and if $C \neq \perp$, i.e. if $B \Rightarrow C$ is not $\neg B$, then,

we derive $\neg B \vee C$ in the environment $B \Rightarrow C$ by proof $P2$ from exercise 58, then we reason by cases :

- ▶ prove A in the environment where $\neg B$ replaces $B \Rightarrow C$: Let P the obtained proof,
- ▶ prove A in the environment where C replaces $B \Rightarrow C$: Let Q the obtained proof.

The proof of A is $\ll P2, \text{Suppose } \neg B, P, \text{Hence } \neg B \Rightarrow A, \text{Suppose } C, Q, \text{Hence } C \Rightarrow A, A \gg$.

Example

Proof of Peirce's formula :

$$((p \Rightarrow q) \Rightarrow p) \Rightarrow p$$

Proof plan

Tactic 5 is compulsory !

Proof Q :

Suppose $(p \Rightarrow q) \Rightarrow p$

Q_1 proof or p in the environment $(p \Rightarrow q) \Rightarrow p$

Hence $((p \Rightarrow q) \Rightarrow p) \Rightarrow p$

Proof Q_1 necessarily uses tactic 13. Hence this proof is written : In the environment $B \Rightarrow C$ where $B = p \Rightarrow q$, $C = p$.

Plan of the proof of Q_1

Proof Q_1 :

$Q_{11} = P_2$ where P_2 is the proof of $\neg B \vee C$ in the environment $B \Rightarrow C$, see exercise 58

Suppose $\neg B$

Q_{12} proof of $A = p$ in the environment $\neg B$

Hence $\neg B \Rightarrow A$

Suppose C

Q_{13} proof of $A = p$ in the environment C

Hence $C \Rightarrow A$

A

Proof of Q_1

Q_{13} is the empty proof, since $A = C = p$.

Q_{12} is the proof of $C = p$ in the environment $\neg(p \Rightarrow q)$.

Since $\neg A$ is an abbreviation of $A \Rightarrow \perp$, this proof is the proof P_6 requested in exercise 58, where $B = p$ and $C = q$.

By gluing pieces Q_1 , Q_{11} , Q_{12} , Q_{13} , we obtain the proof Q .

Below we show how to find the proof Q_{12} without using the tactics.

Proof of Q_{12}

The only rule, which does not lead to a deadlock, is the reduction ad absurdum. Hence this proof is of the form :

Proof Q_{12} of p in the environment $\neg(p \Rightarrow q)$ Suppose $\neg p$ <table border="1"> <tr> <td>Q_{121} proof of \perp in the environment $\neg(p \Rightarrow q), \neg p$</td> </tr> </table> Hence $\neg\neg p$ p	Q_{121} proof of \perp in the environment $\neg(p \Rightarrow q), \neg p$
Q_{121} proof of \perp in the environment $\neg(p \Rightarrow q), \neg p$	

To obtain a contradiction, hence a proof of \perp , $p \Rightarrow q$ must be derived. Hence the proof Q_{121} is :

Suppose p \perp q Hence $p \Rightarrow q$ \perp

Automated proofs

To automatically obtain the proofs in the system, one recommends to use the following software (implementing the 13 previous tactics) :

<http://teachinglogic.liglab.fr/DN/>

People who prefer proofs in the form of trees can use the following software :

<http://www-sop.inria.fr/marelle/Laurent.Thery/peanoware/Nd.html>

Conclusion : Next course

- ▶ First-order logic

Homework : solution using ND

$$(p \Rightarrow \neg j) \wedge (\neg p \Rightarrow j) \wedge (j \Rightarrow m) \Rightarrow m \vee p$$

Conclusion

Thank you for your attention.

Questions ?