Natural Deduction 1

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Last course

- Davis-Putnam
- Complete Strategy

Homework : solution using Davis-Putnam

- (H1) : $p \Rightarrow \neg j \equiv \neg p \vee \neg j$
- (H2) : $\neg p \Rightarrow j \equiv p \vee j$
- (H3) : $j \Rightarrow m \equiv \neg j \vee m$
- (¬ C) : $\neg m \wedge \neg p$

Clauses : $\{\neg p \vee \neg j, p \vee j, \neg j \vee m, \neg m, \neg p\}$

$$\neg p \vee \neg j, p \vee j, \neg j \vee m, \neg m, \neg p$$

Homework : solution using complete strategy

Δ_{i+1}

- Construct all the resolvents of Δ_i and of $\Delta_i \cup \Theta_i$
- Reduce this set
- Remove the new resolvents including a clause of $\Delta_i \cup \Theta_i$

Θ_{i+1} : Remove from $\Delta_i \cup \Theta_i$ the clauses including a clause of Δ_{i+1}.

$$\neg p \vee \neg j, p \vee j, \neg j \vee m, \neg m, \neg p$$

Plan

Introduction

Rules

Natural deduction proofs

Intuition

When we write proofs in math courses,

when we decompose a reasoning in elementary obvious steps,

we practice natural deduction.

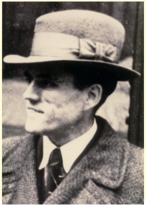
The natural deduction ND

History : In 1934 **Gerhard Gentzen**

Introduced two models of ND for classical logic :

- **NK** : a proof is a tree of formulas.
- **LK** : a proof is a tree of sequents.

NJ et **LJ** for intuitionistic logic.



Here, yet another presentation of natural deduction.

Resolution *vs.* Natural deduction

A proof by **resolution** is a **list of clauses**.

In **natural deduction**, during a proof,
we can add and remove hypotheses,

Hence a more complex definition of a proof.

Abbreviations

Negation and equivalence are abbreviations defined as :

- $\top$ abbreviates to $\bot \Rightarrow \bot$.
- $\neg A$ abbreviates to $A \Rightarrow \bot$.
- $A \Leftrightarrow B$ abbreviates to $(A \Rightarrow B) \wedge (B \Rightarrow A)$.

Two formulae are considered to be equal, if the formulas obtained by removing the abbreviations are identical.

For example, the formulae $\neg\neg a$, $\neg a \Rightarrow \bot$ and $(a \Rightarrow \bot) \Rightarrow \bot$ are equal.

Two equal formulae are equivalent !

Rule

Definition 3.1.1

A rule consists of formulas called **premises** (sometimes called hypotheses) $H_1, \ldots, H_n$ and of a unique **conclusion**.
Premises are written above a line and the conclusion below this line.
The name of the rule is written at the same level as the line.

$$\frac{H_1 \ldots H_n}{C}\ R$$

Fundamental rule of Natural Deduction

Implies-introduction :

In order to prove $A \Rightarrow B$,
just derive B with the additional hypothesis A.

Classification of rules

- **Introduction rules** for introducing a connective in the conclusion.
- **Elimination rules** for removing a connective from the premises.
- $+$ **two special rules**

Rules (system NK of Gentzen)

Table 3.1

Introduction		Élimination	
$\dfrac{\begin{matrix}[A]\\ \vdots \\ B\end{matrix}}{A\Rightarrow B}$	$\Rightarrow I$	$\dfrac{A \quad A\Rightarrow B}{B}$	$\Rightarrow E$
$\dfrac{A \quad B}{A\wedge B}$	$\wedge I$	$\dfrac{A\wedge B}{A}$	$\wedge E1$
		$\dfrac{A\wedge B}{B}$	$\wedge E2$
$\dfrac{A}{A\vee B}$	$\vee I1$	$\dfrac{A\vee B \quad A\Rightarrow C \quad B\Rightarrow C}{C}$	$\vee E$
$\dfrac{A}{B\vee A}$	$\vee I2$		
Rule about false			
$\dfrac{\bot}{A}$ Efq			
Reductio ad absurdum			
$\dfrac{\neg\neg A}{A}$ RAA			

[A] means that A is a hypothesis

Principle

- A proof is a sequence of lines ; lines are derived from previous lines using the rules.
- An additional hypothesis A is assumed by the line : `Suppose` A.
- The last hypothesis is removed A by the line : `Hence` $A \Rightarrow B$.
- This line is the rule of implies-introduction.

$$\frac{\begin{array}{c}[A]\\ \vdots \\ B\end{array}}{A \Rightarrow B} \Rightarrow I$$

Proof line

Definition 3.1.2

A proof line is of one of the three following forms :

- `Suppose` formula,
- formula,
- `Hence` formula.

Examples :

- `Suppose` $A \wedge B$
- A
- `Hence` $A \wedge B \Rightarrow A$

Scratch proof

Definition 3.1.3

A scratch proof is a sequence of lines such that, in every prefix of the sequence, the number of lines starting with the word `Suppose` is at least equal to the number of lines starting with the word `Hence`.

Example 3.1.4

number	line
1	Suppose a
2	$a \vee b$
3	Hence $a \Rightarrow a \vee b$
4	Hence $\neg a$
5	Suppose b

Scratch proof : examples

Where are the scratches ?

num	line
1	Suppose $a \wedge b$
2	b
3	$b \vee c$
4	Hence $a \wedge b \Rightarrow b \vee c$
5	Hence $\neg a$
6	Suppose b

num	line
1	Suppose a
2	$a \vee b$
3	Hence $a \Rightarrow a \vee b$
4	Suppose b

num	line
1	Suppose a
2	$a \vee b$
3	Hence $a \Rightarrow a \vee b$
4	Suppose b
5	Hence $\neg a$

Context (1/2)

- Each line of a scratch proof contains a context
- A context is the sequence of hypotheses previously introduced in lines `Suppose` and not removed in lines `Hence`.

Example 3.1.6 :

context	number	line	rule
1	1	Suppose a	
1,2	2	Suppose b	
1,2	3	$a \wedge b$	$\wedge$I 1,2
1	4	Hence $b \Rightarrow a \wedge b$	$\Rightarrow$I 2,3
1,5	5	Suppose e	

Context (2/2)

Definition 3.1.5

Lines of a scratch proof are numbered from 1 to n. For i between 1 and n, the list of formulae Γ_i is the context of the line i. The list Γ_0 is empty and the lists of formulae Γ_i are defined as :

- If the line i is « `Suppose` A », then $\Gamma_i = \Gamma_{i-1}, i$.
- If the line i is « A » then $\Gamma_i = \Gamma_{i-1}$
- If the line i is « `Hence` A » then Γ_i is obtained by eliminated the last formula of Γ_{i-1}

The list Γ_i is the context of the line i.

The context of a formula represents the hypotheses from which it has been derived.

Example of context

Give the **context** of the following proof scratch :

context	number	line
1	1	Suppose a
1	2	$a \vee b$
	3	Hence $a \Rightarrow a \vee b$
4	4	Suppose b
	5	Hence b

Usable formulae (1/3)

Definition 3.1.7

- A formula appearing on a line of a scratch proof is the conclusion of the line.
- The conclusion of a line is usable as long as its context (i.e., the context from which it has been derived) is present.

Usable formulae (2/3)

Example 3.1.8

Usable formulae (3/3)

Give the lines for which lines 1 and 3 are **usable** in the following example :

context	number	line
1	1	`Suppose` *a*
1,2	2	`Suppose` *b*
1,2	3	*c*
1	4	`Hence` *d*
1,5	5	`Suppose` *e*

Definition of a Proof

Definition 3.1.9

Let Γ a set of formulae, a proof in the environment Γ is a scratch proof having the following properties :

1. For every line $\ll$ `Hence` $A \gg$, the formula A is $B \Rightarrow C$, where B is the last formula from the context of the previous line and where C is a usable formula for the previous line or is a member of the environment Γ.
2. For every line $\ll A \gg$, the formula A is the conclusion of a rule (other than the rule of implies-introduction) whose premises are usable on the previous line, or are members of the environment Γ.

Proof of formulae

Definition 3.1.10

A proof of the formulae A within the environment Γ is :

- either the empty proof when A is element of Γ,
- or a proof whose last line is A with an empty context.

We note :

- $\Gamma \vdash A$ the fact that there is a proof of A within the environment Γ,
- $\Gamma \vdash P : A$ the fact that P is a proof of A within the environment Γ.
- When the environment is empty, we abbreviate $\emptyset \vdash A$ by $\vdash A$.
- When we ask for a proof of a formula without indicating the environment, we suppose that $\Gamma = \emptyset$.

First Example (exemple 3.1.11)

Let us prove $(a \Rightarrow b) \Rightarrow (\neg b \Rightarrow \neg a)$.

Proofs with abbreviations *vs.* without abbreviations

Tree (example 3.1.11)

$$\dfrac{\dfrac{\dfrac{\dfrac{(2)\cancel{\neg b} \qquad \dfrac{(1)\cancel{a \Rightarrow b} \quad (3)\cancel{a}}{(4)b} \Rightarrow E}{(5)\bot} \Rightarrow E}{(6)\neg a} \Rightarrow I[3]}{(7)\neg b \Rightarrow \neg a} \Rightarrow I[2]}{(8)(a \Rightarrow b) \Rightarrow (\neg b \Rightarrow \neg a)} \Rightarrow I[1]$$

context	number	proof	justification
1	1	Suppose $a \Rightarrow b$	
1,2	2	Suppose $\neg b$	
1,2,3	3	Suppose a	
1,2,3	4	b	$\Rightarrow E$ 1, 3
1,2,3	5	$\bot$	$\Rightarrow E$ 2, 4
1,2	6	Hence $\neg a$	$\Rightarrow I$ 3, 5
1	7	Hence $\neg b \Rightarrow \neg a$	$\Rightarrow I$ 2, 6
	8	Hence $(a \Rightarrow b) \Rightarrow (\neg b \Rightarrow \neg a)$	$\Rightarrow I$ 1,7

Second Example

Prove $a \wedge \neg a \Rightarrow b$.

context	number	proof	justification

context	number	proof	justification
1	1	Suppose $a \wedge \neg a$	

context	number	proof	justification
1	1	Suppose $a \wedge \neg a$	
1	2	a	$\wedge E1$ 1

context	number	proof	justification
1	1	Suppose $a \wedge \neg a$	
1	2	a	$\wedge E1$ 1
1	3	$\neg a$	$\wedge E2$ 1

context	number	proof	justification
1	1	Suppose $a \wedge \neg a$	

Proofs with abbreviations *vs.* without abbreviation (2/2)

contexte	number	proof with abbreviation	proof without abbreviation	justification
1	1	Supposons $a \land \neg a$	Supposons $a \land (a \Rightarrow \bot)$	
1	2	a	a	$\land E1$ 1
1	3	$\neg a$	$a \Rightarrow \bot$	$\land E2$ 1
1	4	$\bot$	$\bot$	$\Rightarrow E$ 2,3
1	5	b	b	$Efq4$
	6	Donc $a \land \neg a \Rightarrow b$	Donc $a \land (a \Rightarrow \bot) \Rightarrow b$	$\Rightarrow I$ 1,5

Third Example

Prove $\neg A$ in the environment $\neg(A \vee B)$

environment			
reference		formula	
i		$\neg(A \vee B)$	
context	number	proof	justification

environment			
reference		formula	
i		$\neg(A \vee B)$	
context	number	proof	justification
1	1	Suppose A	

environment			
reference		formula	
i		$\neg(A \vee B)$	
context	number	proof	justification
1	1	Suppose A	
1	2	$A \vee B$	$\vee I1$ 1

Forth exemple (example 3.1.12)

Prove $\neg A \vee B$ in the environment $A \Rightarrow B$.

Tree (example 3.1.12)

Give the tree representation of the previous proof :

Conclusion

Thank you for your attention.

Questions ?