Propositional Resolution

Second Part: Algorithms

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Last course

- Resolution

Reminder

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(1) $\Rightarrow$ (2)

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Correctness

(1) $\Rightarrow$ (2)

Completeness

(2) $\Rightarrow$ (1)

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(2) $A \models B$

B is consequence of A.
Every model of A is also a model of B

Correctness

(1) $\Rightarrow$ (2)

Completeness

(2) $\Rightarrow$ (1)

Resolution

Correct and complete

Overview

Overview

Homework : solution

- (H1) : $p \Rightarrow \neg j$, rewritten as $\neg p \vee \neg j$
- (H2) : $\neg p \Rightarrow j$, rewritten as $p \vee j$
- (H3) : $j \Rightarrow m$, rewritten as $\neg j \vee m$
- ($\neg$ C) : $\neg m \wedge \neg p$

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Clauses : $\{\neg p \vee \neg j,\ p \vee j,\ \neg j \vee m,\ \neg m,\ \neg p\}$

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Clauses : $\{\neg p \vee \neg j, p \vee j, \neg j \vee m, \neg m, \neg p\}$

$$\cfrac{\cfrac{\cfrac{p \vee j \qquad \neg j \vee m}{p \vee m} \qquad \neg m}{p} \qquad \neg p}{\bot}$$

Presentation of the two algorithms

How to ≪ systematically ≫ decide whether Γ is inconsistent or not ?

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- Complete strategy
  Construction of ALL the deductible clauses (resolvents) from Γ

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  ≪ Intelligent ≫ traversal of the possible assignments of Γ

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  Construction of ALL the deductible clauses (resolvents) from Γ
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  ≪ Intelligent ≫ traversal of the possible assignments of Γ

Remark

Exponential solutions in time in the worst case.

Overview

Exponential complexity

Consider that two clauses having the same set of literals are equal.

If the length of $s(\Gamma) = n$, then we have at most 2^n clauses deduced from Γ.

Reduction of a set of clauses

In order to accelerate the algorithm, we reduce the set of clauses.

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How to proceed with reduction ?

Remove the valid clauses and the clauses containing another clause of the set.

A set of clauses is reduced if it is not reducible anymore.

Reduced set of clauses

Definition 2.1.26

A set of clauses is reduced if it does not contain any valid clause and none of the clauses is included in another clause.

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Example 2.1.27

The reduction of the set of clauses
$\{p \lor q \lor \neg p,\ p \lor r,\ p \lor r \lor \neg s,\ r \lor q\}$ gives the reduced set :

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The reduction of the set of clauses
$\{p \vee q \vee \neg p, p \vee r, p \vee r \vee \neg s, r \vee q\}$ gives the reduced set :

$\{p \vee r, r \vee q\}$.

Justification

Property 2.1.28

A set of clauses E is equivalent to the reduced set of clauses obtained from E.

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Proof.

- Removing $\top$: $x \wedge \top \equiv x$

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Proof.

- Removing $\top$: $x \wedge \top \equiv x$
- Removing a clause including another clause : $x \wedge (x \vee y) \equiv x$

□

Result of the algorithm : minimum deduction clauses

Definition 2.1.29

Let Γ a set of clauses. A minimum clause for the deduction from Γ is a non-valid clause deduced from Γ and *strictly* not containing any clause deduced from Γ.

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Let us consider the set of clauses $\Gamma = \{a \vee \neg b, b \vee c \vee d\}$ the clause $a \vee c \vee d$ is a minimum deduction clause.

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Let Γ a set of clauses. A minimum clause for the deduction from Γ is a non-valid clause deduced from Γ and *strictly* not containing any clause deduced from Γ.

Example 2.1.30

Let us consider the set of clauses $\Gamma = \{a \vee \neg b, b \vee c \vee d\}$ the clause $a \vee c \vee d$ is a minimum deduction clause.

However, if we add the clause $\neg a \vee c$ to Γ then $a \vee c \vee d$ is not a minimum clause since we can deduce $c \vee d$ which is included in the clause $a \vee c \vee d$.

Property

Property 2.1.31

Let Θ the set of minimum deduction clauses for the set of clauses Γ.
The set Γ is unsatisfiable if and only if $\bot \in \Theta$.

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Let Θ the set of minimum deduction clauses for the set of clauses Γ.
The set Γ is unsatisfiable if and only if $\perp \in \Theta$.

Proof.

- Suppose $\perp \in \Theta$, then $\Gamma \vdash \perp$, hence by resolution correctness, Γ is unsatisfiable.
- Suppose Γ unsatisfiable, by resolution completeness, $\Gamma \vdash \perp$. Consequently $\perp$ is minimum clause for the deduction of Γ, therefore $\perp \in \Theta$.

□

Interpretation

When the following algorithm terminates :

- $\bot \in \Theta_k$: Γ is unsatisfiable
- $\bot \notin \Theta_k$: Γ is satisfiable, but what does Θ_k represent ?

Θ_k = minimum clauses for the consequence

Definition 2.1.32

Let Γ a set of clauses. A minimum clause for the consequence of Γ is a non valid consequence clause of Γ *strictly* not containing any consequence clause of Γ.

$\Theta_k =$ minimum clauses for the consequence

Definition 2.1.32

Let Γ a set of clauses. A minimum clause for the consequence of Γ is a non valid consequence clause of Γ *strictly* not containing any consequence clause of Γ.

Theorem 2.1.35

Let Γ a set of clauses. A clause is minimum for the deduction of Γ if and only if it is minimum for the consequence of Γ.

Proofs given in the course support.

Example

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Consider the set of clauses $\Gamma = \{a \vee d, \neg a \vee b, \neg b \vee c\}$. The clause $d \vee c$ is minimum for the consequence of Γ.

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Consequence : $d \vee c$ is a consequence of Γ since in all model of Γ, either d is true or c is true.

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Consequence : $d \vee c$ is a consequence of Γ since in all model of Γ, either d is true or c is true.

Minimality : There exist models of Γ which are not models of d (respectively c) : $a \mapsto 1$, $d \mapsto 0$, $c \mapsto 1$ and $b \mapsto 1$ (respectively $a \mapsto 0$, $d \mapsto 1$, $c \mapsto 0$ and $b \mapsto 0$).

Principle of the algorithm : Construct all the clauses deduced from Γ

Following the height of the proof trees.

Algorithm

For any integer i
While it is possible to construct new clauses
Construct the reduced set of all the clauses having a proof tree of height at most i.

Principle of the algorithm : Construct all the clauses deduced from Γ

Following the height of the proof trees.

Algorithm

For any integer i
While it is possible to construct new clauses
Construct the reduced set of all the clauses having a proof tree of height at most i.

In practice :
Maintain two sequences of the sets of clauses, $\Delta_{i(i \geq 0)}$ and $\Theta_{i(i \geq 0)}$

Two sequences of sets of clauses

$\Delta_{i (i \geq 0)}$

Clauses deduced from Γ by a proof of height i, after clauses removal :

- valid clauses
- clauses including another clause of the proof of height at most i.

Δ_0 is obtained by reducing Γ

Two sequences of sets of clauses

$\Theta_{i(i \geq 0)}$

Clauses deduced from Γ by a proof of height less than i after clauses removal :

- valid clauses
- clauses including another clause of the proof of height at most i.

Θ_0 is the empty set.

Details of the method

If $\Delta_k = \emptyset$, stop the construction :

- $k-1$ is then the maximum height of a proof
- Θ_k is the reduced set of the clauses deduced from Γ

Construction of the sequences $\Delta_{i(i \geq 0)}$ and $\Theta_{i(i \geq 0)}$

Δ_{i+1}

- Construct all the resolvents of Δ_i and $\Delta_i \cup \Theta_i$
- Reduce this set
- Remove the new resolvents including a clause of $\Delta_i \cup \Theta_i$

Θ_{i+1}

Remove from $\Delta_i \cup \Theta_i$ the clauses which include one of the clauses of Δ_{i+1}.

Example 2.2.1

Let $\Gamma = \{a \vee b \vee \neg a, a \vee b, a \vee b \vee c, a \vee \neg b, \neg a \vee b, \neg a \vee \neg b\}$

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0	$a \vee b,\ a \vee \neg b,$ $\neg a \vee b,\ \neg a \vee \neg b$	$\emptyset$		

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0	$a \vee b,\ a \vee \neg b,$ $\neg a \vee b,\ \neg a \vee \neg b$	$\emptyset$	$a \vee b,\ a \vee \neg b,$ $\neg a \vee b,\ \neg a \vee \neg b$	$b,\ b \vee \neg b,\ a,$ $a \vee \neg a,\ \neg a,\ \neg b$
1	$a,\ b,\ \neg b,\ \neg a$			

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1	a, b, $\neg b$, $\neg a$	$\emptyset$	a, b, $\neg b$, $\neg a$	

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0	$a \vee b,\ a \vee \neg b,$ $\neg a \vee b,\ \neg a \vee \neg b$	$\emptyset$	$a \vee b,\ a \vee \neg b,$ $\neg a \vee b,\ \neg a \vee \neg b$	$b,\ b \vee \neg b,\ a,$ $a \vee \neg a,\ \neg a,\ \neg b$
1	$a,\ b,\ \neg b,\ \neg a$	$\emptyset$	$a,\ b,\ \neg b,\ \neg a$	$\perp$

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Let $\Gamma = \{a \vee b \vee \neg a,\ a \vee b,\ a \vee b \vee c,\ a \vee \neg b,\ \neg a \vee b,\ \neg a \vee \neg b\}$

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0	$a \vee b$, $a \vee \neg b$, $\neg a \vee b$, $\neg a \vee \neg b$	$\emptyset$	$a \vee b$, $a \vee \neg b$, $\neg a \vee b$, $\neg a \vee \neg b$	b, $b \vee \neg b$, a, $a \vee \neg a$, $\neg a$, $\neg b$
1	a, b, $\neg b$, $\neg a$	$\emptyset$	a, b, $\neg b$, $\neg a$	$\bot$
2				

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0	$a \vee b,\ a \vee \neg b,$ $\neg a \vee b,\ \neg a \vee \neg b$	$\emptyset$	$a \vee b,\ a \vee \neg b,$ $\neg a \vee b,\ \neg a \vee \neg b$	$b,\ b \vee \neg b,\ a,$ $a \vee \neg a,\ \neg a,\ \neg b$
1	$a,\ b,\ \neg b,\ \neg a$	$\emptyset$	$a,\ b,\ \neg b,\ \neg a$	$\perp$
2	$\perp$			

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Let $\Gamma = \{a \vee b \vee \neg a,\ a \vee b,\ a \vee b \vee c,\ a \vee \neg b,\ \neg a \vee b,\ \neg a \vee \neg b\}$

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0	$a \vee b,\ a \vee \neg b,$ $\neg a \vee b,\ \neg a \vee \neg b$	$\emptyset$	$a \vee b,\ a \vee \neg b,$ $\neg a \vee b,\ \neg a \vee \neg b$	$b,\ b \vee \neg b,\ a,$ $a \vee \neg a,\ \neg a,\ \neg b$
1	$a,\ b,\ \neg b,\ \neg a$	$\emptyset$	$a,\ b,\ \neg b,\ \neg a$	$\perp$
2	$\perp$	$\emptyset$		

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Let $\Gamma = \{a \vee b \vee \neg a,\ a \vee b,\ a \vee b \vee c,\ a \vee \neg b,\ \neg a \vee b,\ \neg a \vee \neg b\}$

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0	$a \vee b$, $a \vee \neg b$, $\neg a \vee b$, $\neg a \vee \neg b$	$\emptyset$	$a \vee b$, $a \vee \neg b$, $\neg a \vee b$, $\neg a \vee \neg b$	b, $b \vee \neg b$, a, $a \vee \neg a$, $\neg a$, $\neg b$
1	a, b, $\neg b$, $\neg a$	$\emptyset$	a, b, $\neg b$, $\neg a$	$\bot$
2	$\bot$	$\emptyset$	$\bot$	

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Let $\Gamma = \{a \vee b \vee \neg a,\ a \vee b,\ a \vee b \vee c,\ a \vee \neg b,\ \neg a \vee b,\ \neg a \vee \neg b\}$

i	Δ_i	Θ_i	$\Delta_i \cup \Theta_i$	Resolvents of Δ_i and $\Delta_i \cup \Theta_i$
0	$a \vee b,\ a \vee \neg b,$ $\neg a \vee b,\ \neg a \vee \neg b$	$\emptyset$	$a \vee b,\ a \vee \neg b,$ $\neg a \vee b,\ \neg a \vee \neg b$	$b,\ b \vee \neg b,\ a,$ $a \vee \neg a,\ \neg a,\ \neg b$
1	$a,\ b,\ \neg b,\ \neg a$	$\emptyset$	$a,\ b,\ \neg b,\ \neg a$	$\perp$
2	$\perp$	$\emptyset$	$\perp$	$\emptyset$

Example 2.2.1 (contd.)

i	Δ_i	Θ_i	$\Delta_i \cup \Theta_i$	Resolvents of Δ_i and $\Delta_i \cup \Theta_i$
0	$a \vee b$, $a \vee \neg b$, $\neg a \vee b$, $\neg a \vee \neg b$	$\emptyset$	$a \vee b$, $a \vee \neg b$, $\neg a \vee b$, $\neg a \vee \neg b$	b, $b \vee \neg b$, a, $a \vee \neg a$, $\neg a$, $\neg b$
1	a, b, $\neg b$, $\neg a$	$\emptyset$	a, b, $\neg b$, $\neg a$	$\perp$
2	$\perp$	$\emptyset$	$\perp$	$\emptyset$
3				

Example 2.2.1 (contd.)

i	Δ_i	Θ_i	$\Delta_i \cup \Theta_i$	Resolvents of Δ_i and $\Delta_i \cup \Theta_i$
0	$a \vee b$, $a \vee \neg b$, $\neg a \vee b$, $\neg a \vee \neg b$	$\emptyset$	$a \vee b$, $a \vee \neg b$, $\neg a \vee b$, $\neg a \vee \neg b$	b, $b \vee \neg b$, a, $a \vee \neg a$, $\neg a$, $\neg b$
1	a, b, $\neg b$, $\neg a$	$\emptyset$	a, b, $\neg b$, $\neg a$	$\perp$
2	$\perp$	$\emptyset$	$\perp$	$\emptyset$
3	$\emptyset$	$\perp$		

Example 2.2.2

$$\{a, c, \neg a \vee \neg b, \neg c \vee e\}$$

Example 2.2.2

$$\{a, c, \neg a \vee \neg b, \neg c \vee e\}$$

i	Δ_i	Θ_i	$\Delta_i \cup \Theta_i$	$\Delta_i \sharp (\Delta_i \cup \Theta_i)$
0	$a, c, \neg a \vee \neg b, \neg c \vee e$	$\emptyset$	$a, c, \neg a \vee \neg b, \neg c \vee e$	$e, \neg b$
1	$e, \neg b$	a, c	$a, \neg b, c, e$	$\emptyset$
2	$\emptyset$	$a, \neg b, c, e$		

Termination of the algorithm : idea

There are at most 2^n clauses deduced from Γ.

$\Delta_{i(i \geq 0)}$ contains only clauses deduced from Γ

$\Delta_{i(i \geq 0)}$ are mutually disjoint (To demonstrate)

Hence there are at most $2^n + 1$ sets, therefore $k \leq 2^n + 1$

$\Delta_{i(i \geq 0)}$ are mutually disjoint

Property 2.2.3

Let $i \leq k$. Any clause of $\bigcup_{j \leq i} \Delta_j$ contains a clause of $\Delta_i \cup \Theta_i$.

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Let $i \leq k$. Any clause of $\bigcup_{j \leq i} \Delta_j$ contains a clause of $\Delta_i \cup \Theta_i$.

Proof.

By induction.

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Let $i \leq k$. Any clause of $\bigcup_{j \leq i} \Delta_j$ contains a clause of $\Delta_i \cup \Theta_i$.

Proof.

By induction.

- For $i = 0$ the property is trivial since $\Theta_0 = \emptyset$.

□

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Let $i \leq k$. Any clause of $\bigcup_{j \leq i} \Delta_j$ contains a clause of $\Delta_i \cup \Theta_i$.

Proof.

By induction.

- For $i = 0$ the property is trivial since $\Theta_0 = \emptyset$.
- Suppose the property true for i, let us show that it is also true for $i + 1$. Let $C \in \bigcup_{j \leq i+1} \Delta_j$. Let us show that C contains a clause of $\Delta_{i+1} \cup \Theta_{i+1}$. We examine all the possible cases for C.

□

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Proof.

By induction.

- For $i = 0$ the property is trivial since $\Theta_0 = \emptyset$.
- Suppose the property true for i, let us show that it is also true for $i + 1$. Let $C \in \bigcup_{j \leq i+1} \Delta_j$. Let us show that C contains a clause of $\Delta_{i+1} \cup \Theta_{i+1}$. We examine all the possible cases for C.
    1. $C \in \Delta_{i+1}$.
    2. $C \in \bigcup_{j \leq i} \Delta_j$.

□

$\Delta_{i(i \geq 0)}$ are mutually disjoint

Property 2.2.3

Let $i \leq k$. Any clause of $\bigcup_{j \leq i} \Delta_j$ contains a clause of $\Delta_i \cup \Theta_i$.

Proof.

By induction.

- For $i = 0$ the property is trivial since $\Theta_0 = \emptyset$.
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    1. $C \in \Delta_{i+1}$. Hence C contains a clause of $\Delta_{i+1} \cup \Theta_{i+1}$.
    2. $C \in \bigcup_{j \leq i} \Delta_j$.

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  1. $C \in \Delta_{i+1}$. Hence C contains a clause of $\Delta_{i+1} \cup \Theta_{i+1}$.
  2. $C \in \bigcup_{j \leq i} \Delta_j$. By induction hypothesis, C contains a clause $D \in \Delta_i \cup \Theta_i$.

□

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  1. $C \in \Delta_{i+1}$. Hence C contains a clause of $\Delta_{i+1} \cup \Theta_{i+1}$.
  2. $C \in \bigcup_{j \leq i} \Delta_j$. By induction hypothesis, C contains a clause $D \in \Delta_i \cup \Theta_i$. We distinguish two situations for D.
     - 2.1 $D \in \Theta_{i+1}$.
     - 2.2 $D \notin \Theta_{i+1}$.

□

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- For $i = 0$ the property is trivial since $\Theta_0 = \emptyset$.
- Suppose the property true for i, let us show that it is also true for $i+1$. Let $C \in \bigcup_{j \leq i+1} \Delta_j$. Let us show that C contains a clause of $\Delta_{i+1} \cup \Theta_{i+1}$. We examine all the possible cases for C.
  1. $C \in \Delta_{i+1}$. Hence C contains a clause of $\Delta_{i+1} \cup \Theta_{i+1}$.
  2. $C \in \bigcup_{j \leq i} \Delta_j$. By induction hypothesis, C contains a clause $D \in \Delta_i \cup \Theta_i$. We distinguish two situations for D.
     - 2.1 $D \in \Theta_{i+1}$. Hence C contains a clause of $\Delta_{i+1} \cup \Theta_{i+1}$.
     - 2.2 $D \notin \Theta_{i+1}$.

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    2. $C \in \bigcup_{j \leq i} \Delta_j$. By induction hypothesis, C contains a clause $D \in \Delta_i \cup \Theta_i$. We distinguish two situations for D.
        - 2.1 $D \in \Theta_{i+1}$. Hence C contains a clause of $\Delta_{i+1} \cup \Theta_{i+1}$.
        - 2.2 $D \notin \Theta_{i+1}$. By construction of Θ_{i+1}, since $D \in \Delta_i \cup \Theta_i$ and $D \notin \Theta_{i+1}$, it means that D contains a clause of Δ_{i+1}. Or C contains D, hence C also contains a clause of $\Delta_{i+1} \cup \Theta_{i+1}$.

□

$\Delta_{i(i \geq 0)}$ are mutually disjoint

Property 2.2.4

For all $i \leq k$, the sets Δ_i are mutually disjoint.

Proof.

We perform an induction on the sets Δ_j with $0 \leq j \leq i$ and $i \leq k$.

$\Delta_{i(i \geq 0)}$ are mutually disjoint

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For all $i \leq k$, the sets Δ_i are mutually disjoint.

Proof.

We perform an induction on the sets Δ_j with $0 \leq j \leq i$ and $i \leq k$.

The base case (basis) : If $i = 0$, there is only one set, hence the property is verified.

$\Delta_{i(i \geq 0)}$ are mutually disjoint

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Proof.

We perform an induction on the sets Δ_j with $0 \leq j \leq i$ and $i \leq k$.

The base case (basis) : If $i = 0$, there is only one set, hence the property is verified.

Inductive step : Let $i < k$. Suppose that all the sets Δ_j where $j \leq i$ are mutually disjoint.

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Inductive step : Let $i < k$. Suppose that all the sets Δ_j where $j \leq i$ are mutually disjoint. Let us show that Δ_{i+1} is disjoint with respect to these sets.

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Inductive step : Let $i < k$. Suppose that all the sets Δ_j where $j \leq i$ are mutually disjoint. Let us show that Δ_{i+1} is disjoint with respect to these sets. Let $C \in \Delta_{i+1}$. Suppose, on the contrary, that $C \in \bigcup_{j \leq i} \Delta_j$.

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Inductive step : Let $i < k$. Suppose that all the sets Δ_j where $j \leq i$ are mutually disjoint. Let us show that Δ_{i+1} is disjoint with respect to these sets. Let $C \in \Delta_{i+1}$. Suppose, on the contrary, that $C \in \bigcup_{j \leq i} \Delta_j$. According to the previous property, C includes a clause of $\Delta_i \cup \Theta_i$.

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We perform an induction on the sets Δ_j with $0 \leq j \leq i$ and $i \leq k$.

The base case (basis) : If $i = 0$, there is only one set, hence the property is verified.

Inductive step : Let $i < k$. Suppose that all the sets Δ_j where $j \leq i$ are mutually disjoint. Let us show that Δ_{i+1} is disjoint with respect to these sets. Let $C \in \Delta_{i+1}$. Suppose, on the contrary, that $C \in \bigcup_{j\leq i} \Delta_j$. According to the previous property, C includes a clause of $\Delta_i \cup \Theta_i$. Hence by construction of Δ_{i+1}, $C \notin \Delta_{i+1}$, contradiction. Consequently, $C \notin \bigcup_{j\leq i} \Delta_j$.

□

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We perform an induction on the sets Δ_j with $0 \leq j \leq i$ and $i \leq k$.

The base case (basis) : If $i = 0$, there is only one set, hence the property is verified.

Inductive step : Let $i < k$. Suppose that all the sets Δ_j where $j \leq i$ are mutually disjoint. Let us show that Δ_{i+1} is disjoint with respect to these sets. Let $C \in \Delta_{i+1}$. Suppose, on the contrary, that $C \in \bigcup_{j \leq i} \Delta_j$. According to the previous property, C includes a clause of $\Delta_i \cup \Theta_i$. Hence by construction of Δ_{i+1}, $C \notin \Delta_{i+1}$, contradiction. Consequently, $C \notin \bigcup_{j \leq i} \Delta_j$.

□

Hence, the algorithm terminates.

Result of the algorithm

- Γ and Θ_k are equivalent
- $\Theta_k =$ set of minimum deduction clauses.

Γ and Θ_k are equivalent

Property 2.2.5

For all $i < k$, the sets $\Delta_i \cup \Theta_i$ and $\Delta_{i+1} \cup \Theta_{i+1}$ are equivalent.

Proof.

□

Γ and Θ_k are equivalent

Property 2.2.5

For all $i < k$, the sets $\Delta_i \cup \Theta_i$ and $\Delta_{i+1} \cup \Theta_{i+1}$ are equivalent.

Proof.

1. Any clause of $\Delta_{i+1} \cup \Theta_{i+1}$ is a consequence of $\Delta_i \cup \Theta_i$.
2. Any clause of $\Delta_i \cup \Theta_i$ is a consequence of $\Delta_{i+1} \cup \Theta_{i+1}$.

□

Γ and Θ_k are equivalent

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For all $i < k$, the sets $\Delta_i \cup \Theta_i$ and $\Delta_{i+1} \cup \Theta_{i+1}$ are equivalent.

Proof.

1. Any clause of $\Delta_{i+1} \cup \Theta_{i+1}$ is a consequence of $\Delta_i \cup \Theta_i$. Any clause of $\Delta_{i+1} \cup \Theta_{i+1}$ is an element of $\Delta_i \cup \Theta_i$ or a resolvent of two elements of this set, therefore it is a consequence of this set.
2. Any clause of $\Delta_i \cup \Theta_i$ is a consequence of $\Delta_{i+1} \cup \Theta_{i+1}$.

□

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2. Any clause of $\Delta_i \cup \Theta_i$ is a consequence of $\Delta_{i+1} \cup \Theta_{i+1}$. Let $C \in \Delta_i \cup \Theta_i$.

□

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For all $i < k$, the sets $\Delta_i \cup \Theta_i$ and $\Delta_{i+1} \cup \Theta_{i+1}$ are equivalent.

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2. Any clause of $\Delta_i \cup \Theta_i$ is a consequence of $\Delta_{i+1} \cup \Theta_{i+1}$. Let $C \in \Delta_i \cup \Theta_i$. We distinguish two possible cases :
   2.1 $C \in \Theta_{i+1}$
   2.2 $C \notin \Theta_{i+1}$

□

Γ and Θ_k are equivalent

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For all $i < k$, the sets $\Delta_i \cup \Theta_i$ and $\Delta_{i+1} \cup \Theta_{i+1}$ are equivalent.

Proof.

1. Any clause of $\Delta_{i+1} \cup \Theta_{i+1}$ is a consequence of $\Delta_i \cup \Theta_i$. Any clause of $\Delta_{i+1} \cup \Theta_{i+1}$ is an element of $\Delta_i \cup \Theta_i$ or a resolvent of two elements of this set, therefore it is a consequence of this set.
2. Any clause of $\Delta_i \cup \Theta_i$ is a consequence of $\Delta_{i+1} \cup \Theta_{i+1}$. Let $C \in \Delta_i \cup \Theta_i$. We distinguish two possible cases :
   2.1 $C \in \Theta_{i+1}$, thus C is a consequence of $\Delta_{i+1} \cup \Theta_{i+1}$.
   2.2 $C \notin \Theta_{i+1}$

□

Γ and Θ_k are equivalent

Property 2.2.5

For all $i < k$, the sets $\Delta_i \cup \Theta_i$ and $\Delta_{i+1} \cup \Theta_{i+1}$ are equivalent.

Proof.

1. Any clause of $\Delta_{i+1} \cup \Theta_{i+1}$ is a consequence of $\Delta_i \cup \Theta_i$. Any clause of $\Delta_{i+1} \cup \Theta_{i+1}$ is an element of $\Delta_i \cup \Theta_i$ or a resolvent of two elements of this set, therefore it is a consequence of this set.
2. Any clause of $\Delta_i \cup \Theta_i$ is a consequence of $\Delta_{i+1} \cup \Theta_{i+1}$. Let $C \in \Delta_i \cup \Theta_i$. We distinguish two possible cases :
   - 2.1 $C \in \Theta_{i+1}$, thus C is a consequence of $\Delta_{i+1} \cup \Theta_{i+1}$.
   - 2.2 $C \notin \Theta_{i+1}$, thus C contains a clause of Δ_{i+1} hence is a consequence of $\Delta_{i+1} \cup \Theta_{i+1}$.

□

Γ and Θ_k are equivalent

Property 2.2.6

The sets Γ and Θ_k are equivalent.

Proof.

□

Γ and Θ_k are equivalent

Property 2.2.6

The sets Γ and Θ_k are equivalent.

Proof.

- Δ_0 is the set obtained by reduction of Γ, according to property 2.1.28, these two sets are equivalent.

□

Γ and Θ_k are equivalent

Property 2.2.6

The sets Γ and Θ_k are equivalent.

Proof.

- Δ_0 is the set obtained by reduction of Γ, according to property 2.1.28, these two sets are equivalent.
- Since Θ_0 is empty, Γ is equivalent to $\Delta_0 \cup \Theta_0$.

□

Γ and Θ_k are equivalent

Property 2.2.6

The sets Γ and Θ_k are equivalent.

Proof.

- Δ_0 is the set obtained by reduction of Γ, according to property 2.1.28, these two sets are equivalent.
- Since Θ_0 is empty, Γ is equivalent to $\Delta_0 \cup \Theta_0$.
- According to property 2.2.5 and by induction, $\Delta_0 \cup \Theta_0$ is equivalent to the set of clauses $\Delta_k \cup \Theta_k$.

□

Γ and Θ_k are equivalent

Property 2.2.6

The sets Γ and Θ_k are equivalent.

Proof.

- Δ_0 is the set obtained by reduction of Γ, according to property 2.1.28, these two sets are equivalent.
- Since Θ_0 is empty, Γ is equivalent to $\Delta_0 \cup \Theta_0$.
- According to property 2.2.5 and by induction, $\Delta_0 \cup \Theta_0$ is equivalent to the set of clauses $\Delta_k \cup \Theta_k$.
- Since the algorithm terminates when Δ_k is the empty set, the sets Γ and Θ_k are equivalent.

$\square$

Θ_k = set of minimum deduction clauses

Property 2.2.13

Θ_k is the set of minimum deduction clauses of Γ.

Proof.

Cf. Course support (Poly) □

Θ_k = set of minimum deduction clauses

Property 2.2.13

Θ_k is the set of minimum deduction clauses of Γ.

Proof.

Cf. Course support (Poly) □

Example from 1.6.2 : $maj(x, y, z) = (x \vee y \vee z) \wedge (x \vee y \vee \neg z) \wedge (x \vee \neg y \vee z) \wedge (\neg x \vee y \vee z)$.

	Δ_0	Θ_0
1	$x \vee y \vee z$	
2	$x \vee y \vee \neg z$	
3	$x \vee \neg y \vee z$	
4	$\neg x \vee y \vee z$	

	Δ_1		Θ_1
5	$x \vee y$	resolvent of 1, 2	
6	$x \vee z$	resolvent of 1, 3	
7	$y \vee z$	resolvent of 1, 4	

Δ_2 is empty and $\Theta_2 = \Delta_1$.

Consequently $maj(x, y, z) = (x \vee y)(x \vee z)(y \vee z)$.

Overview

History

The Davis-Putnam-Logemann-Loveland (DPLL) Algorithm

- Introduced by Martin Davis and Hilary Putnam in 1960, then refined by Martin Davis, George Logemann and Donald Loveland in 1962
- Indicates if a set of clauses is satisfiable.
- Basis for (most efficient) complete SAT-solvers such as **chaff, zchaff and satz**.

Principle I

Two types of formulae transformation :

1. preserving the truth value : transforming a formula into an equivalent formula
   - reduction
2. preserving the satisfiability only : transforming a satisfiable formula into another satisfiable formula
   - removal of clauses containing isolated literals
   - unit resolution

DPLL is efficient since it uses these two transformations.

Principle II

≪ Branching/Backtracking ≫ (splitting rule)

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- Continue the algorithm recursively.

Principle II

« Branching/Backtracking » (splitting rule)

- **Branching** : After simplification, assign to **true** a heuristically chosen variable (branching literal).
- Continue the algorithm recursively.
- **Backtracking** : If we arrive to a contradiction, we return to the last choice, and we « branch » by assigning **false** to the chosen variable.

Removal of clauses having isolated literals.

Definition 2.3.1 Isolated literal L

If none of the clauses of Γ contains L^c.

Lemme 2.3.2

Removing clauses with an isolated literal preserves the satisfiability.

Proof : see exercise 48.

Example 2.3.3

Let Γ the set of clauses

(1) $p \vee q \vee r$

(2) $\neg q \vee \neg r$

(3) $q \vee s$

(4) $\neg s \vee t$

Simplify Γ by removing clauses having isolated literals.

Example 2.3.3

Let Γ the set of clauses

(1) $p \vee q \vee r$

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Simplify Γ by removing clauses having isolated literals.

The literals p and t are isolated.
We therefore obtain

(2) $\neg q \vee \neg r$

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Example 2.3.3

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(2) $\neg q \vee \neg r$

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The literals $\neg r$ and s are isolated.

Example 2.3.3

Let Γ the set of clauses

(1) $p \vee q \vee r$

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Simplify Γ by removing clauses having isolated literals.

The literals p and t are isolated.
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The literals $\neg r$ and s are isolated.
We obtain the empty set.

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Let Γ the set of clauses

(1) $p \vee q \vee r$

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Simplify Γ by removing clauses having isolated literals.

The literals p and t are isolated.
We therefore obtain

(2) $\neg q \vee \neg r$

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The literals $\neg r$ and s are isolated.
We obtain the empty set.
According to lemma 2.3.2, Γ has a model.

Example 2.3.3

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Simplify Γ by removing clauses having isolated literals.

The literals p and t are isolated.
We therefore obtain

(2) $\neg q \vee \neg r$

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The literals $\neg r$ and s are isolated.
We obtain the empty set.
According to lemma 2.3.2, Γ has a model.
But there is a counter-model, *e.g.* $p \mapsto 0$, $q \mapsto 0$, $r \mapsto 0$!!!

Unit resolution

Definition 2.3.4

A unit clause is a clause which contains only one literal.

Unit resolution

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A unit clause is a clause which contains only one literal.

Lemma 2.3.5

Let L the set of literals of the unit clauses of Γ. Let Θ the set of clauses obtained starting from Γ, as follows

- if L contains two complementary literals, then $\Theta = \{\bot\}$.
- else Θ is obtained as follows
  - removing the clauses containing an element of L
  - in the remaining clauses, remove the complementary literals of the elements of L

Γ has a model if and only if Θ has a model.

Proof : The proof is requested in exercise 49.

Example 2.3.6 Unit resolution

Simplify the following sets of clauses by unit resolution :

- **Let Γ the set of clauses :** $p \vee q$, $\neg p$, $\neg q$

Example 2.3.6 Unit resolution

Simplify the following sets of clauses by unit resolution :

- **Let Γ the set of clauses :** $p \vee q$, $\neg p$, $\neg q$

  $\perp$ by unit resolution, hence Γ has no model.

- **Let Γ the set of clauses :** $a \vee b \vee \neg d$, $\neg a \vee c \vee \neg d$, $\neg b$, d, $\neg c$**.**

Example 2.3.6 Unit resolution

Simplify the following sets of clauses by unit resolution :

- **Let Γ the set of clauses :** $p \vee q, \neg p, \neg q$

$\bot$ by unit resolution, hence Γ has no model.

- **Let Γ the set of clauses :** $a \vee b \vee \neg d, \neg a \vee c \vee \neg d, \neg b, d, \neg c$**.**

1. $a, \neg a$.

Example 2.3.6 Unit resolution

Simplify the following sets of clauses by unit resolution :

- **Let Γ the set of clauses :** $p \vee q, \neg p, \neg q$

  $\perp$ by unit resolution, hence Γ has no model.

- **Let Γ the set of clauses :** $a \vee b \vee \neg d, \neg a \vee c \vee \neg d, \neg b, d, \neg c$.

  1. $a, \neg a$.
  2. Empty clause.

Example 2.3.6 Unit resolution

Simplify the following sets of clauses by unit resolution :

- **Let Γ the set of clauses :** $p \vee q, \neg p, \neg q$

$\perp$ by unit resolution, hence Γ has no model.

- **Let Γ the set of clauses :** $a \vee b \vee \neg d, \neg a \vee c \vee \neg d, \neg b, d, \neg c$**.**

1. $a, \neg a$.
2. Empty clause.

hence Γ has no model.

- **Let Γ' the set of clauses :** $p, q, p \vee r, \neg p \vee r, q \vee \neg r, \neg q \vee s$**.**

Example 2.3.6 Unit resolution

Simplify the following sets of clauses by unit resolution :

- **Let Γ the set of clauses :** $p \vee q$, $\neg p$, $\neg q$

$\perp$ by unit resolution, hence Γ has no model.

- **Let Γ the set of clauses :** $a \vee b \vee \neg d$, $\neg a \vee c \vee \neg d$, $\neg b$, d, $\neg c$**.**

1. a, $\neg a$.
2. Empty clause.

hence Γ has no model.

- **Let Γ' the set of clauses :** p, q, $p \vee r$, $\neg p \vee r$, $q \vee \neg r$, $\neg q \vee s$**.**

By unit resolution, we obtain : r, s.
This set of clauses has a model, hence Γ' has a model.

Removal of valid clauses

Lemma 2.3.7

Let Θ the set of clauses obtained by removing the valid clauses of Γ.

Γ has a model iff Θ has a model.

Proof.

Removal of valid clauses

Lemma 2.3.7

Let Θ the set of clauses obtained by removing the valid clauses of Γ.

Γ has a model iff Θ has a model.

Proof.

- Suppose that Γ has a model v, since Θ is a subset of clauses of Γ, v is also model of Θ. Hence Θ has a model.

Removal of valid clauses

Lemma 2.3.7

Let Θ the set of clauses obtained by removing the valid clauses of Γ.

$$\Gamma \text{ has a model iff } \Theta \text{ has a model.}$$

Proof.

- Suppose that Γ has a model v, since Θ is a subset of clauses of Γ, v is also model of Θ. Hence Θ has a model.
- Suppose that Θ has a model v. Let v' a truth assignment of Γ so that $v'(x) = v(x)$ for all variable x belonging to both Γ and Θ. Let C a clause of Γ. If C is also a clause of Θ, then v' is a model of C since v and v' give the same value to C. If C is not a clause of Θ, then C is valid, consequently all truth assignment, v' in particular, is model of C. Hence Γ has a model : v'.

□

The DPLL Algorithm(figure 2.1)

bool function Algo_DPLL(Γ : set of clauses)
0 Remove the valid clauses of Γ.
If $\Gamma = \emptyset$, return (`true`).
Else return (`DPLL`(Γ))

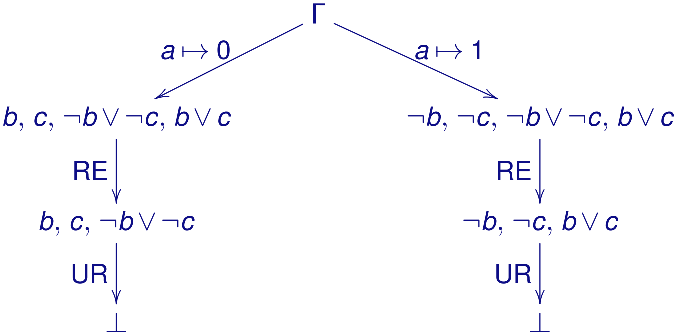
bool function `DPLL`(Γ : non-valid set of clauses)
The function returns true if and only if Γ is satisfiable
1 **If** $\perp \in \Gamma$, return(`false`).
If $\Gamma = \emptyset$, return (`true`).
2 Reduce Γ : simply remove any clause containing *another* clause.
3 Remove from Γ the clauses containing isolated literals (cf. paragraph 2.3.1).
If the set Γ has been modified, goto 1.
4 Apply to Γ the unit resolution (cf paragraph 2.3.2).
If the set Γ has been modified, goto 1.
5 Select x, an arbitrary variable of Γ
return (`DPLL`($\Gamma[x := 0]$) or then `DPLL`($\Gamma[x := 1]$))

Example 2.3.8

Let Γ the set of clauses :
$\neg a \vee \neg b$, $a \vee b$, $\neg a \vee \neg c$, $a \vee c$, $\neg b \vee \neg c$, $b \vee c$.

Example 2.3.8

Let Γ the set of clauses :
$\neg a \vee \neg b$, $a \vee b$, $\neg a \vee \neg c$, $a \vee c$, $\neg b \vee \neg c$, $b \vee c$.



Since all leaves contain the empty clause, the set Γ is unsatisfiable.

Example 2.3.8

Let Γ the set of clauses : $\neg p \vee \neg q$, $\neg p \vee s$, $p \vee q$, $\neg p \vee \neg s$.

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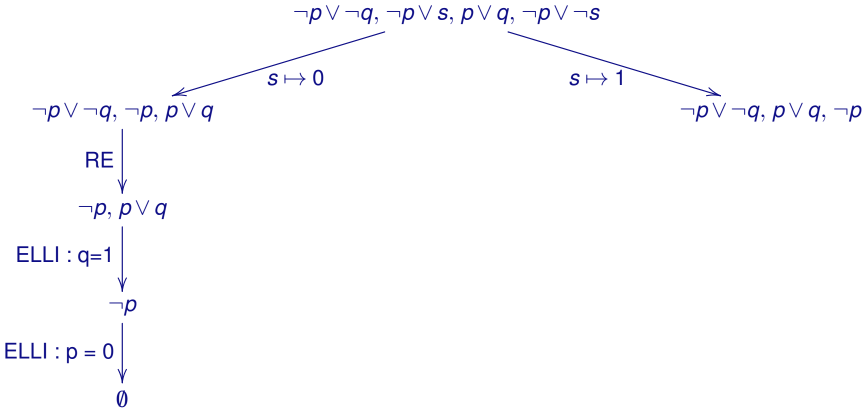

Since one leaf contains the empty clause, the set Γ is satisfiable. It is useless to continue the construction of the right branch.

Theorems 2.3.9 et 2.3.10

The algorithm Algo_DPLL is correct and terminates.

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The algorithm Algo_DPLL is correct and terminates.

Termination proof

- Step 0 is only executed once.
- Iteration in 1 : the number of clauses strictly decreases, hence termination.
- Recursivity in 5 : the number of variables strictly decreases, hence termination.

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- Recursivity in 5 : the number of variables strictly decreases, hence termination.

Reminder of property 2.1.21 : Γ has a model iff $\Gamma[x := 0]$ is satisfiable or $\Gamma[x := 1]$ is satisfiable.

Correctness proof

Invariant : the current value of Γ has a model iff Γ has a model.

Verified at step 0, 1 and 5, hence correct answers. Suppose the recursive calls are correct :

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- if DPLL($\Gamma[x := 0]$) is true , then by induction $\Gamma[x := 0]$ is satisfiable, hence Γ is satisfiable, according to property 2.1.21. which corresponds to the true value of DPLL(Γ).
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  - Suppose that DPLL($\Gamma[x := 1]$) is false

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  - Suppose that DPLL($\Gamma[x := 1]$) is true, then by induction $\Gamma[x := 1]$ is satisfiable, hence Γ is satisfiable, which corresponds to the true value of DPLL(Γ).
  - Suppose that DPLL($\Gamma[x := 1]$) is false, then by induction $\Gamma[x := 1]$ is unsatisfiable. Hence Γ is unsatisfiable, which corresponds to the false value of DPLL(Γ).

Remarks 2.3.11 and 2.3.12

- **Forgetting simplifications :** DPLL stays correct if we forget the reduction (2), the removal of the isolated literals (3) and/or the unit reduction (4).

Remarks 2.3.11 and 2.3.12

- **Forgetting simplifications :** DPLL stays correct if we forget the reduction (2), the removal of the isolated literals (3) and/or the unit reduction (4).
- **Choice of the variable (branching literal) :**
  - A good choice for the variable x from step (5), is to choose the variable that appears most often.
  - A better choice is to choose the variable which will lead to the most of simplifications

Cf. Sub-section 2.3.5, for the principal branching heuristics

Overview

Planning of the Semester

TODAY

- Propositional logic
- Propositional resolution *
- Propositional natural deduction
- First order logic

MIDTERM EXAM

- Basis for the automated proof (≪ first order resolution ≫)
- First order natural deduction

EXAM

Conclusion : Next course

- Natural deduction

Conclusion

Thank you for your attention.

Questions ?