Introduction to trees and proofs

Jean-François Monin

2014-06-04

Outline

Outline

Formal methods and Coq

Trees

Decomposing, case analysis

Functions and implication

Formal Methods

- Prove that some piece of software behaves accordingly to a given specification
- Boils down to theorem proving: programs and specifications are represented by logical formulas
- Hand waving not allowed
- Several complementary approaches and tools

In summary

- Describe a model
- Explain it
- Reason about it
- Be clean and precise

Use math and logic

Some industrial uses

Spacecrafts, airplanes (Airbus, Boeing)

Microsoft
Intel
French railways
Telecom operators
Nuclear power plants
Banks
Cryptography

Coq

A language

- Logic formulas
- Proofs
- Programs

A software proof assistant

Very secure by architectural design

Outline

Formal models, proofs and programs

Based on a common structure: **trees**

- Can be implemented in many ways by software (pointers, arrays, ...)
- This talk: use an intuitive and graphical presentation of trees

Developed by the same activity

which can be

- explicit: using an appropriate syntax for trees
- interactive: using commands for building trees step by step

Make a strong use of **types**

- Everything has a type, even types have a type
- Computations can be carried out on types as well

Basic building blocks for trees: **rules**

$$\frac{\mathit{input}_1 \quad \mathit{input}_2 \quad \dots \quad \mathit{input}_n}{\mathit{output}}\ \mathit{howto}$$

$\mathit{input}_1, \mathit{input}_2 \dots, \mathit{input}_n$ and output are **types**

howto explains how to make an output, given $\mathit{input}_1, \mathit{input}_2 \dots, \mathit{input}_n$

Proofs

Same constructs with another reading

Basic building blocks for proof trees: **rules**

$$\frac{\mathit{input}_1 \quad \mathit{input}_2 \quad \dots \quad \mathit{input}_n}{\mathit{output}}\ \mathit{howto}$$

$\mathit{input}_1, \mathit{input}_2 \dots, \mathit{input}_n$ and output are **propositions**

$\mathit{input}_1, \mathit{input}_2 \dots, \mathit{input}_n$ are **hypotheses**
output is the **conclusion**

howto justifies how to make a proof of output, given proofs of $\mathit{input}_1, \mathit{input}_2 \dots, \mathit{input}_n$

Combining rules

Basic building blocks can be combined

Just plug outputs to identical inputs

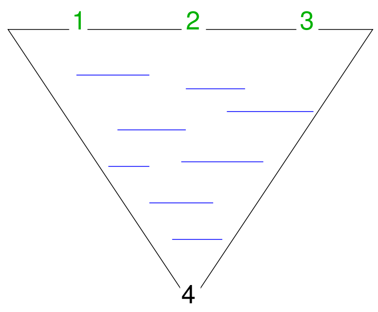
Some rules:

- 1 input comes from exactly 1 output
- an output can be plugged (used) 1, several or 0 times

Alltogether

We get 1 output
from many inputs
using a complex (composed) howto

General shape: trees

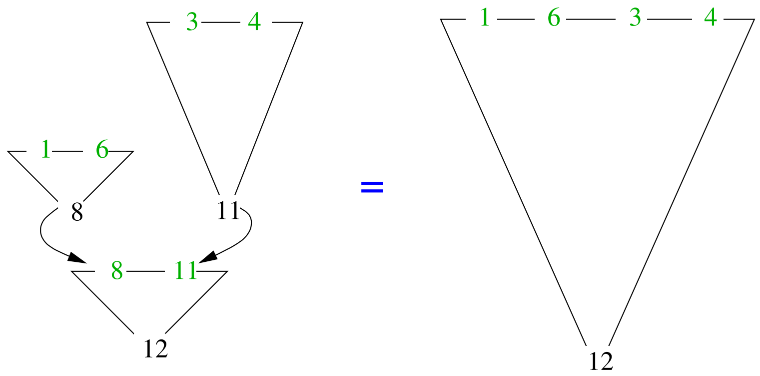


Interpretation

- At positions 1, 2, 3, 4: types
- 1, 2, 3: inputs
- 4: output (or result)

Makes the output from the inputs

Plugging trees



Subtrees can have local additional inputs

Simple examples with 0 input

$\dfrac{}{\texttt{day}}\,\texttt{monday}$ $\dfrac{}{\texttt{day}}\,\texttt{tuesday}$ $\dfrac{}{\texttt{day}}\,\texttt{wednesday}$ $\dfrac{}{\texttt{day}}\,\texttt{thursday}$

$\dfrac{}{\texttt{day}}\,\texttt{friday}$ $\dfrac{}{\texttt{day}}\,\texttt{saturday}$ $\dfrac{}{\texttt{day}}\,\texttt{sunday}$

$\dfrac{}{\texttt{color}}\,\texttt{white}$ $\dfrac{}{\texttt{color}}\,\texttt{black}$

The horizontal bar means: MAKES

Intermezzo: definitions

```
Definition Mo := monday.  Definition Th := thursday.
Definition Tu := tuesday.   Definition Fr := friday.
Definition We := wednesday.  Definition Sa := saturday.
Definition Su := sunday.
```

$\dfrac{}{\texttt{day}}\,\texttt{Mo}$ $\dfrac{}{\texttt{day}}\,\texttt{Tu}$ $\dfrac{}{\texttt{day}}\,\texttt{We}$ $\dfrac{}{\texttt{day}}\,\texttt{Th}$ $\dfrac{}{\texttt{day}}\,\texttt{Fr}$

$\dfrac{}{\texttt{day}}\,\texttt{Sa}$ $\dfrac{}{\texttt{day}}\,\texttt{Su}$

These trees are considered the same as the previous ones (top of previous slide)

Simple example with 4 identical inputs

Making a 4-tuple of days

$$\dfrac{\texttt{day}\quad\texttt{day}\quad\texttt{day}\quad\texttt{day}}{\texttt{tuple4}}\,\texttt{Mk4day}$$

`Mk4day` makes a `tuple4` from

- a `day`
- a `day`
- a `day`
- a `day`

Plugging day into Mk4day

Building blocks

$\dfrac{}{\texttt{day}}\,\texttt{Mo}$ $\dfrac{}{\texttt{day}}\,\texttt{We}$ $\dfrac{}{\texttt{day}}\,\texttt{Fr}$ $\dfrac{}{\texttt{day}}\,\texttt{We}$

$$\dfrac{\texttt{day}\qquad\texttt{day}\qquad\texttt{day}\qquad\texttt{day}}{\texttt{tuple4}}\,\texttt{Mk4day}$$

Connecting them yields the concrete 4-tuple of day

$$\dfrac{\dfrac{}{\texttt{day}}\,\texttt{Mo}\quad\dfrac{}{\texttt{day}}\,\texttt{We}\quad\dfrac{}{\texttt{day}}\,\texttt{Fr}\quad\dfrac{}{\texttt{day}}\,\texttt{We}}{\texttt{tuple4}}\,\texttt{Mk4day}$$

Another view on Mk4day

As a rule

$$\frac{\texttt{day}\qquad\texttt{day}\qquad\texttt{day}\qquad\texttt{day}}{\texttt{tuple4}}\;\texttt{Mk4day}$$

As a tree

$$\frac{\overbrace{\texttt{day}}^{x_1}\qquad\overbrace{\texttt{day}}^{x_2}\qquad\overbrace{\texttt{day}}^{x_3}\qquad\overbrace{\texttt{day}}^{x_4}}{\texttt{tuple4}}\;\texttt{Mk4day}$$

This is called an open tree

Closed and open trees

The meaning (or value) of

$$\frac{\dfrac{}{\texttt{day}}\,\texttt{Mo}\qquad\dfrac{}{\texttt{day}}\,\texttt{We}\qquad\dfrac{}{\texttt{day}}\,\texttt{Mo}\qquad\dfrac{}{\texttt{day}}\,\texttt{Th}}{\texttt{tuple4}}\;\texttt{Mk4day}$$

is completely defined: this is called a closed tree.

In contrast, the meaning of the open tree

$$\frac{\dfrac{}{\texttt{day}}\,\texttt{Mo}\qquad\overbrace{\texttt{day}}^{x_2}\qquad\dfrac{}{\texttt{day}}\,\texttt{We}\qquad\overbrace{\texttt{day}}^{x_4}}{\texttt{tuple4}}\;\texttt{Mk4day}$$

depends on x_2 and x_4.

Environment

The meaning of the open tree

$$\frac{\dfrac{}{\texttt{day}}\,\texttt{Mo}\qquad\overbrace{\texttt{day}}^{x_2}\qquad\dfrac{}{\texttt{day}}\,\texttt{We}\qquad\overbrace{\texttt{day}}^{x_4}}{\texttt{tuple4}}\;\texttt{Mk4day}$$

depends on x_2 and x_4. It has a meaning for all trees plugged into x_2 and x_4.

The variables x_2 : day and x_4 : day make up the environment of this tree

More 4-tuples

$$\frac{\overbrace{\texttt{day}}^{x_1}\quad\overbrace{\texttt{day}}^{x_2}\quad\overbrace{\texttt{day}}^{x_3}\quad\overbrace{\texttt{day}}^{x_4}}{\texttt{tuple4}}\;\texttt{Mk4day}$$

$$\frac{\overbrace{\texttt{color}}^{x_1}\quad\overbrace{\texttt{color}}^{x_2}\quad\overbrace{\texttt{color}}^{x_3}\quad\overbrace{\texttt{color}}^{x_4}}{\texttt{tuple4}}\;\texttt{Mk4co}$$

$$\frac{\overbrace{\texttt{tuple4}}^{x_1}\quad\overbrace{\texttt{tuple4}}^{x_2}\quad\overbrace{\texttt{tuple4}}^{x_3}\quad\overbrace{\texttt{tuple4}}^{x_4}}{\texttt{tuple4}}\;\texttt{Mk4t4}$$

Lists of days

$$\dfrac{}{\texttt{daylist}}\ \texttt{nil}$$

$$\dfrac{\overbrace{\texttt{day}}^{d}\quad \overbrace{\texttt{daylist}}^{u}}{\texttt{daylist}}\ \texttt{cons}$$

Examples with proofs

Consider basic propositions, e.g.:

Proposition	**Intended meaning**
H	hot – the temperature is greater than 35 C
W	the glass contains water
C	the blackboard is clean
HaW	the temperature is greater than 35 C and the glass contains water
HoW	the temperature is greater than 35 C or the glass contains water

Given a proof h of H and a proof w of W,
what is a proof of HaW?
what is a proof of HoW?

Examples with proofs

$$\dfrac{\overbrace{\texttt{H}}^{h}\quad \overbrace{\texttt{W}}^{w}}{\texttt{HaW}}\ \texttt{conj}_{\texttt{HW}}$$

$$\dfrac{\overbrace{\texttt{H}}^{h}}{\texttt{HoW}}\ \texttt{case1}_{\texttt{HoW}} \qquad \dfrac{\overbrace{\texttt{W}}^{w}}{\texttt{HoW}}\ \texttt{case2}_{\texttt{HoW}}$$

To go farther: making propositions

Given 2 propositions P and Q, make a new proposition whose intended meaning is:
P and Q hold together.

This proposition is noted $P \wedge Q$,
Note that it is itself a tree (at the level of propositions, not at the level of proofs)

$$\dfrac{\overbrace{\texttt{Prop}}^{P}\quad \overbrace{\texttt{Prop}}^{Q}}{\texttt{Prop}}\ \texttt{and}$$

Similarly, $P \vee Q$ represents the tree:

$$\dfrac{\overbrace{\texttt{Prop}}^{P}\quad \overbrace{\texttt{Prop}}^{Q}}{\texttt{Prop}}\ \texttt{or}$$

General conjunctions and disjunctions

$$\dfrac{\overbrace{\texttt{P}}^{p} \qquad \overbrace{\texttt{Q}}^{q}}{\texttt{P} \wedge \texttt{Q}}\ \texttt{conj}$$

$$\dfrac{\overbrace{\texttt{P}}^{p}}{\texttt{P} \vee \texttt{Q}}\ \texttt{or\_left} \qquad\qquad \dfrac{\overbrace{\texttt{Q}}^{q}}{\texttt{P} \vee \texttt{Q}}\ \texttt{or\_right}$$

Outline

Decomposing, case analysis

- Given a 4-tuple t, extract its components
- Given a day d, provide a color depending on d
- Given a color c, provide a day depending on c
- Given a proof of $A \wedge B$, provide a proof of A
- Given a proof of $A \vee B$, provide a proof of $B \vee A$
  A proof of $B \vee A$ is needed in each case

Case analysis

Question

Give a `day` for each possible value `c` in `color`

$$\dfrac{}{\texttt{color}}\ \texttt{white} \qquad\qquad \dfrac{}{\texttt{color}}\ \texttt{black}$$

Example

`white` maps to `thursday`, `black` maps to `monday`

$$\dfrac{\begin{array}{c}\vdots\ \texttt{c}\\ \texttt{color}\end{array} \qquad \dfrac{}{\texttt{day}}\ \texttt{thursday} \qquad \dfrac{}{\texttt{day}}\ \texttt{monday}}{\texttt{day}}\ \texttt{destruct}$$

Warning: the order of branches matters

Does it make sense? Subtle point!

Statement of the previous question

Give a `day` for each possible value in `color`

Here assume that all trees with `color` as output are (in this order)

$$\text{either}\qquad \dfrac{}{\texttt{color}}\,\texttt{white}$$

$$\text{or}\qquad \dfrac{}{\texttt{color}}\,\texttt{black}$$

The real story is more subtle

- Claim of exhaustivity: related to inductive types
- However, there are (infinitely) many trees which make a `color`
- However, they eventually reduce to one of the declared cases: related to computations and so-called *strong normalization*

Building block of a case analysis

In this presentation, the order of contructors matters:
`white`, `black`
The `destruct` construct is driven by 2 parameters

- the type of the value to be analyzed
  each enumerated type (e.g., `color`) comes automatically with its `destruct` construct, which should actually be written, e.g. $\texttt{destruct}_{\texttt{color}}$
- the type of the result (e.g., `day`)

$$\dfrac{\overbrace{\texttt{Set}}^{A}\qquad \overbrace{\texttt{color}}^{c}\qquad \overbrace{\texttt{A}}^{x_2}\qquad \overbrace{\texttt{A}}^{x_3}}{\texttt{A}}\,\texttt{destruct}$$

Correct version of the previous example

$$\dfrac{\dfrac{}{\texttt{Set}}\,\texttt{day}\qquad \overbrace{\texttt{color}}^{c}\qquad \dfrac{}{\texttt{day}}\,\texttt{thursday}\qquad \dfrac{}{\texttt{day}}\,\texttt{monday}}{\texttt{day}}\,\texttt{destruct}$$

Decomposing

Original question

Given a proof tree p of $\texttt{A} \wedge \texttt{B}$, provide a proof of `A`
We know that the only possible shape of p is

$$\dfrac{\begin{matrix}\vdots\ \texttt{a}\\ \texttt{A}\end{matrix}\qquad \begin{matrix}\vdots\ \texttt{b}\\ \texttt{B}\end{matrix}}{\texttt{A} \wedge \texttt{B}}\,\texttt{conj}$$

where $\boxed{\begin{matrix}\vdots\ \texttt{a}\\ \texttt{A}\end{matrix}}$ is an arbitrarily large proof tree concluding to `A`

and similarly for $\boxed{\begin{matrix}\vdots\ \texttt{b}\\ \texttt{B}\end{matrix}}$

So the answer is `a`, but how to get it?

Decomposing

More general question

Given a proof tree $\texttt{p}$ of $\texttt{A} \wedge \texttt{B}$,
prove some proposition $\texttt{C}$ using the two components building p.

Rule

$$\dfrac{\dfrac{}{\texttt{Prop}}\,\texttt{C} \qquad \begin{matrix}\vdots\,\texttt{p}\\ \texttt{A} \wedge \texttt{B}\end{matrix} \qquad \begin{matrix}\overbrace{\texttt{A}}^{a}\ \overbrace{\texttt{B}}^{b}\\ \vdots\,\texttt{f}\\ \texttt{C}\end{matrix}}{\texttt{C}}\ \texttt{destruct [a b]}$$

Reading

Let us prove $\texttt{C}$ assuming a, a proof of $\texttt{A}$ and b, a proof of $\texttt{B}$; as we have a proof of $\texttt{A} \wedge \texttt{B}$, we get a proof of $\texttt{C}$.

Warning

$$\dfrac{\dfrac{}{\texttt{Prop}}\,\texttt{C} \qquad \begin{matrix}\vdots\,\texttt{p}\\ \texttt{A} \wedge \texttt{B}\end{matrix} \qquad \begin{matrix}\overbrace{\texttt{A}}^{a}\ \overbrace{\texttt{B}}^{b}\\ \vdots\,\texttt{f}\\ \texttt{C}\end{matrix}}{\texttt{C}}\ \texttt{destruct [a b]}$$

The subtree $\boxed{\begin{matrix}\overbrace{\texttt{A}}^{a}\ \overbrace{\texttt{B}}^{b}\\ \vdots\,\texttt{f}\\ \texttt{C}\end{matrix}}$ contains additional local inputs a and b

a and b are not available outside $\texttt{f}$

Example: a proof of $\texttt{B} \wedge \texttt{A}$ from $\texttt{A} \wedge \texttt{B}$

$$\dfrac{\dfrac{}{\texttt{Prop}}\,\texttt{B} \wedge \texttt{A} \qquad \overbrace{\texttt{A} \wedge \texttt{B}}^{ab} \qquad \dfrac{\overbrace{\texttt{B}}^{b} \quad \overbrace{\texttt{A}}^{a}}{\texttt{B} \wedge \texttt{A}}\ \texttt{conj}}{\texttt{B} \wedge \texttt{A}}\ \texttt{destruct [a b]}$$

Shape of this proof tree

$$\begin{matrix}\overbrace{\texttt{A} \wedge \texttt{B}}^{ab}\\ \vdots\\ \texttt{B} \wedge \texttt{A}\end{matrix}$$

Case analysis and decomposition

For a disjunction $\texttt{A} \vee \texttt{B}$, we have 2 cases, and each one has 1 input

$$\dfrac{\dfrac{}{\texttt{Prop}}\,\texttt{C} \qquad \begin{matrix}\vdots\,\texttt{p}\\ \texttt{A} \vee \texttt{B}\end{matrix} \qquad \begin{matrix}\overbrace{\texttt{A}}^{a}\\ \vdots\,\texttt{f}\\ \texttt{C}\end{matrix} \qquad \begin{matrix}\overbrace{\texttt{B}}^{b}\\ \vdots\,\texttt{g}\\ \texttt{C}\end{matrix}}{\texttt{C}}\ \texttt{destruct [a | b]}$$

The hypothesis a is available only inside $\texttt{f}$
The hypothesis b is available only inside $\texttt{g}$

Exercises

- a proof of $\mathtt{B} \lor \mathtt{A}$ from $\mathtt{A} \lor \mathtt{B}$
- a proof of $\mathtt{A}$ from $\mathtt{A} \land \mathtt{B}$
- a proof of $\mathtt{B} \land \mathtt{B}$ from $\mathtt{A} \land \mathtt{B}$
- draw the case analysis and decomposition rules for 4-tuples

Outline

Implication

What is a function from day to color?

- open a new scope where d represents an arbitrary tree with day as output
- make a tree with output color from d
- in this subtree, d is available; but not outside

What is a proof that P implies Q?

- open a new scope with p an arbitrary proof of P (an arbitrary proof tree with P as the conclusion)
- make a proof tree with conclusion Q from p
- in this subtree, p is available; but not outside

It is just a function from P to Q

Again: some subtrees have additional inputs

Introduction rules for implications/functions

$$\cfrac{\begin{array}{c}\overbrace{\mathtt{A}}^{a}\\ \vdots\ \mathtt{u}\\ \mathtt{B}\end{array}}{\mathtt{A} \rightarrow \mathtt{B}}\ \mathtt{intro}\ \mathtt{a}$$

Warning: this a is available only in $\mathtt{u}$

Example

$$
\dfrac{
\dfrac{
\dfrac{}{\texttt{Prop}}\,\texttt{B} \land \texttt{A}
\qquad
\overbrace{\texttt{A} \land \texttt{B}}^{ab}
\qquad
\dfrac{\overbrace{\texttt{B}}^{b} \quad \overbrace{\texttt{A}}^{a}}{\texttt{B} \land \texttt{A}}\,\texttt{conj}
}{\texttt{B} \land \texttt{A}}\,\texttt{destruct}\,[\texttt{a b}]
}{\texttt{A} \land \texttt{B} \rightarrow \texttt{B} \land \texttt{A}}\,\texttt{intro}\ \texttt{ab}
$$

Applying a function / modus ponens

$$
\dfrac{\begin{array}{c}\vdots\ \texttt{f} \\ \texttt{A} \rightarrow \texttt{B}\end{array} \qquad \begin{array}{c}\vdots\ \texttt{t} \\ \texttt{A}\end{array}}{\texttt{B}}\,\texttt{apply}
$$