5th Asian-Pacific Summer School on Formal Methods

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Polymorphic types

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Notes adapted from
Assia Mahboubi
(coq school 2010, Paris) and
Benjamin Pierce (software
foundations course, UPenn)

COMPUTER BUGS
ARE NEVER EXPECTED

Plan

- polymorphic lists
- polymorphic functions
- implicit arguments
- induction on polymorphic lists
- polymorphic trees, products, options
- higher-order functions

一日为师，终生为父

Yī rì wéi shī, zhōngshēng wèi fù

COMPUTER BUGS
ARE NEVER EXPECTED

- Testing
- Static analysis
- Formal methods

Polymorphic lists (1/5)

lists of any type X.

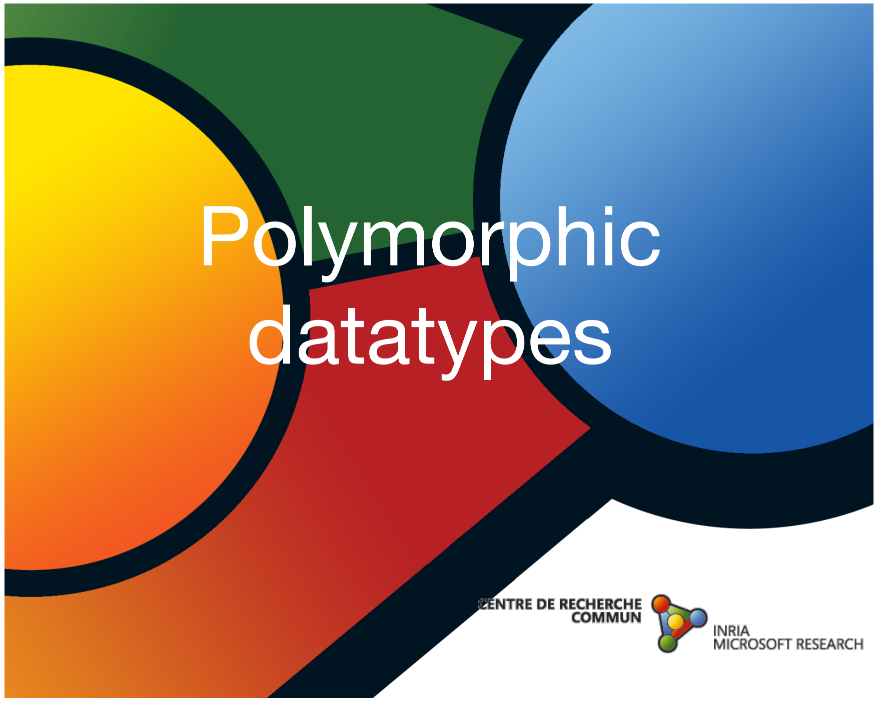
```
Inductive list (X:Type) : Type :=
  | nil : list X
  | cons : X -> list X -> list X.
```

Exercice 14 Check *list*, *nil*, *cons*.
Exercice 15 Check *cons nat 1 (cons nat 2 (nil nat))*.

```
Definition daylist := list (day).
Definition natlist := list (nat).

Check (cons day monday (cons day tuesday (nil day))).
Check (cons nat 2 (cons nat 3 (nil nat))).

Check (cons _ monday (cons _ tuesday (nil _))).
```



Polymorphic lists (2/5)

```
Fixpoint app (X:Type) (l1 l2 : list X) {struct l1}
                : (list X) :=
  match l1 with
  | nil => l2
  | cons h t => cons X h (app X t l2)
  end.
```

Exercice 16 Associativity of append. Etc..

```
Fixpoint rev (X:Type) (l:list X) {struct l} : list X :=
  match l with
  | nil => nil X
  | cons h t => app X (rev X t) (cons X h (nil X))
  end.
```

Synthetizing arguments (1/4)

```
Fixpoint length (X:Type) (l:list X) {struct l} : nat :=
  match l with
  | nil => 0
  | cons h t => S (length _ t)
  end.

Example test_length2 :
    length _ (cons _ 1 (cons _ 2 (nil _))) = 2.
Proof. reflexivity. Qed.
```

Synthetizing arguments (2/4)

```
Arguments nil [X].
Arguments cons [X] _ _.

Check cons 2 nil.
Check cons monday nil.
```

or simply with argument in braces at function definition.

```
Fixpoint length {X:Type} (l:list X) {struct l} : nat :=
  match l with
  | nil => 0
  | cons h t => S (length t)
  end.

Example test_length3 :
    length (cons 1 (cons 2 (nil))) = 2.
Proof. reflexivity. Qed.
```

`@length is notation for function with all arguments.`

Synthetizing arguments (3/4)

```
Notation "x :: l" := (cons x l) (at level 60, right
associativity).

Notation "[ ]" := nil.
Notation "[ x , .. , y ]" := (cons x .. (cons y nil) ..).

Check 3 :: 4 :: nil.
Check monday :: tuesday :: nil.
Check [3, 4, 5].
```

Synthetizing arguments (4/4)

Also decreasing argument is implicit when clear from definition.

```
Fixpoint length {X:Type} (l:list X) : nat :=
  match l with
  | nil => 0
  | cons h t => S (length t)
  end.


Fixpoint app {X : Type} (l1 l2 : list X) : (list X) :=
  match l1 with
  | nil => l2
  | cons h t => cons h (app t l2)
  end.
```

Exercice 17 Write definition of *rev* with implicit arguments.

Polymorphic lists (4/5)

Let iterative reverse be:

```
Fixpoint irev {X: Type} (l1 l2: list X) : list X :=
  match l1 with
  | [ ] => l2
  | v1 :: l1' => irev l1' (v1 :: l2)
  end.
```

Exercice 18 Show for any lists ℓ_1, ℓ_2, ℓ_3:

$\ell_1 ++ (\ell_2 ++ \ell_3) = (\ell_1 ++ \ell_2) ++ \ell_3$
$\text{length}(\ell_1 ++ \ell_2) = (\text{length}\,\ell_1) + (\text{length}\,\ell_2)$
$\text{rev}\ \ell_1 = \text{irev}\ \ell_1\, [\,]$
$\ell ++ [\,] = \ell$
$\text{rev}(\ell_1 ++ \ell_2) = (\text{rev}\,\ell_2) ++ (\text{rev}\,\ell_1)$
$\text{rev}(\text{rev}\,\ell) = \ell$
$\ell = \text{rev}\ \ell' \ \Rightarrow\ \ell' = \text{rev}\ \ell$

Polymorphic binary trees (1/2)

```
Inductive binTree (X : Type) :=
| leaf : X -> binTree X
| node : X -> binTree X -> binTree X -> binTree X.

Fixpoint count_leaves {X: Type} (t : binTree X) :=
  match t with
  | leaf _ => 1
  | node _ t1 t2 => (count_leaves t1) + (count_leaves t2)
  end.
```

Polymorphic binary trees (2/2)

```
Lemma height_le_size : forall (X: Type) (t : binTree X),
  height t <= size t.
Proof.
intros X t. induction t as [| x t1 IHt1 t2 IHt2].
- reflexivity.
- simpl. apply Le.le_n_S.
  apply Max.max_case.
  + apply (Le.le_trans _ (size t1) _).
    apply IHt1. apply Plus.le_plus_l.
  + apply (Le.le_trans _ (size t2) _).
    apply IHt2. apply Plus.le_plus_r.
Qed.
```

Polymorphic Option and Product

A polymorphic non recursive option type:

```
Inductive option (X : Type) : Type :=
  Some : X -> option X | None : option X
```

Use it for default value:

```
Fixpoint last {X : Type} (l : list X) : option X :=
  match l with
    | [ ] => None
    | v :: nil => Some v
    | _ :: l' => last l'
  end.
```

We also define polymorphic product.

```
Inductive prod {X Y : Type} : Type :=
    pair : X -> Y -> prod X Y
```

The notation `X * Y` denotes `(prod X Y)`.
The notation `(x, y)` denotes `(pair x y)` (implicit argument).

Higher order functions

```
Fixpoint map{X Y: Type}(f : X->Y) (l : list X){struct l}: list Y :=
  match l with
  | [ ] => [ ]
  | x :: l' => (f x) :: map f l'
  end.

Example map_negb : map negb [true, false] = [false, true].
Example map_next_weekday :
    map next_weekday [monday, friday] = [tuesday, monday].
```

Exercice 19 Show
map f (rev ℓ) = rev(map f ℓ)
map f (ℓ_1 ++ ℓ_2) = (map f ℓ_1) ++ (map f ℓ_2)