Optimization-Based Verification: An Overview

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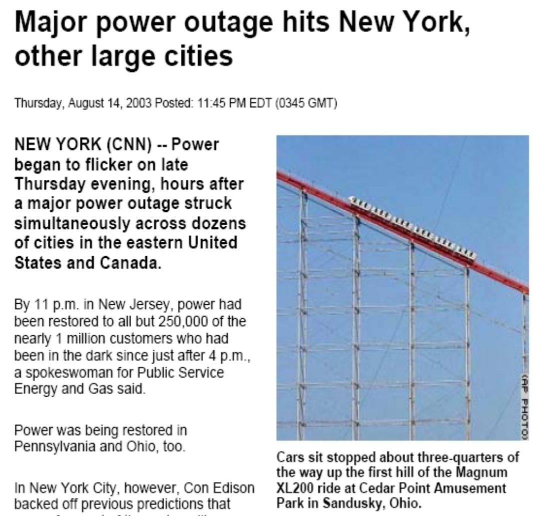
(Joint with A. Jadbabaie, G. J. Pappas, A. Rantzer)

Technological Disasters


Major power outage hits New York, other large cities

Thursday, August 14, 2003 Posted: 11:45 PM EDT (0345 GMT)

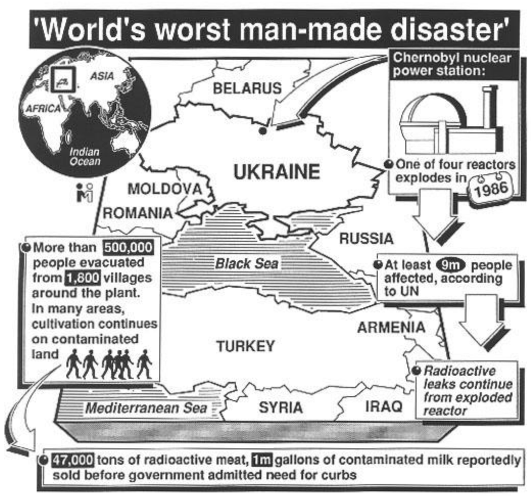
NEW YORK (CNN) -- Power began to flicker on late Thursday evening, hours after a major power outage struck simultaneously across dozens of cities in the eastern United States and Canada.

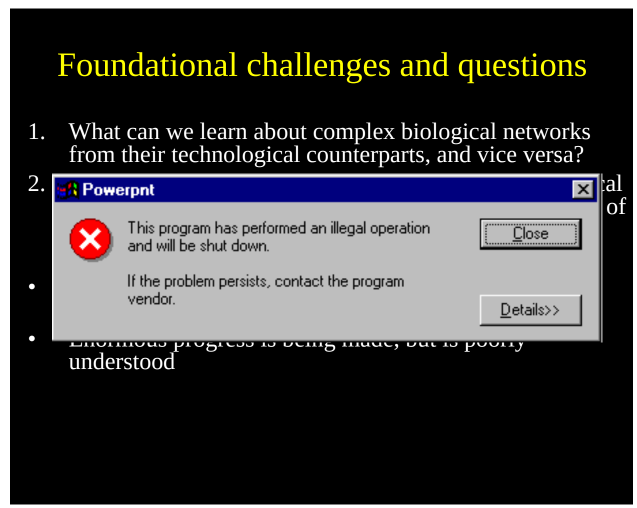
By 11 p.m. in New Jersey, power had been restored to all but 250,000 of the nearly 1 million customers who had been in the dark since just after 4 p.m., a spokeswoman for Public Service Energy and Gas said.

Power was being restored in Pennsylvania and Ohio, too.

In New York City, however, Con Edison backed off previous predictions that

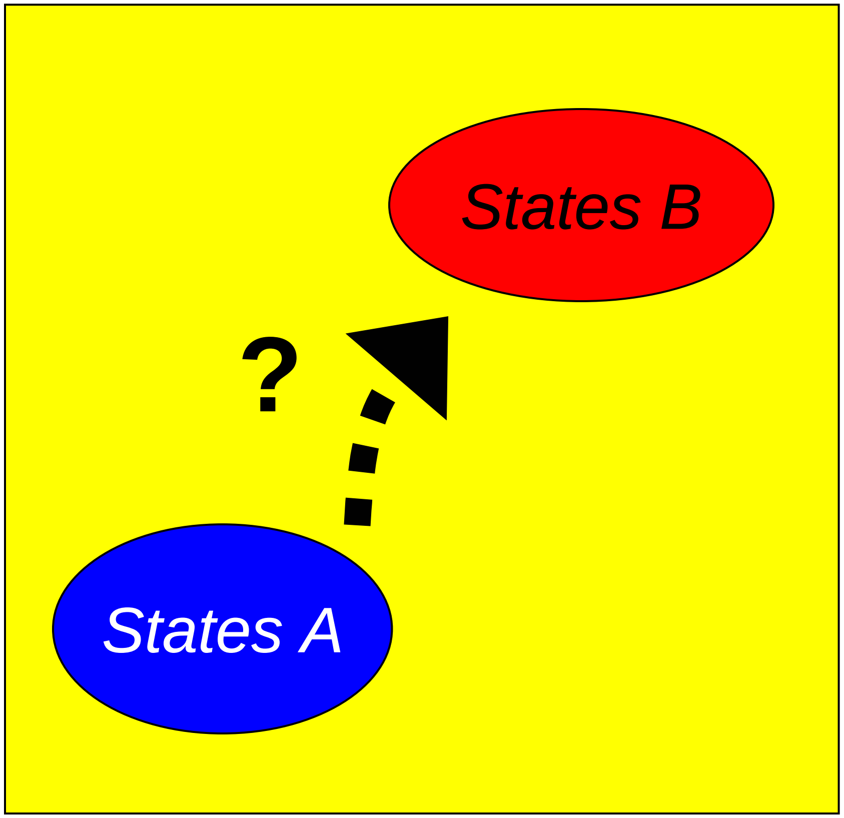
Cars sit stopped about three-quarters of the way up the first hill of the Magnum XL200 ride at Cedar Point Amusement Park in Sandusky, Ohio.





Properties of Interest

Focus today is on properties that are expressed in terms of reachability / temporal specifications.



- Some bad states are never reached (*safety*)
- Some good states will be reached (*reachability/eventuality*)
- Some states will be reached before some other states, etc.

In many cases, more natural than stability and optimality.

Status

- Many results from computer science, for discrete transition systems. Much less for continuous or hybrid systems.

- Standard tools in systems and control (e.g., Lyapunov functions) can handle only some special cases.

- Various methods have been recently proposed:
  - Continuous systems: quantifier elimination, set propagation, ….
  - Hybrid systems: abstraction, ….
  - Stochastic systems: discretization, randomization, …

Some Challenges

- Common framework
- Automated computation
- Easily checkable proofs
- Scalability

Our Approach

- Use algebraic proofs and deductive inference to prove temporal properties.

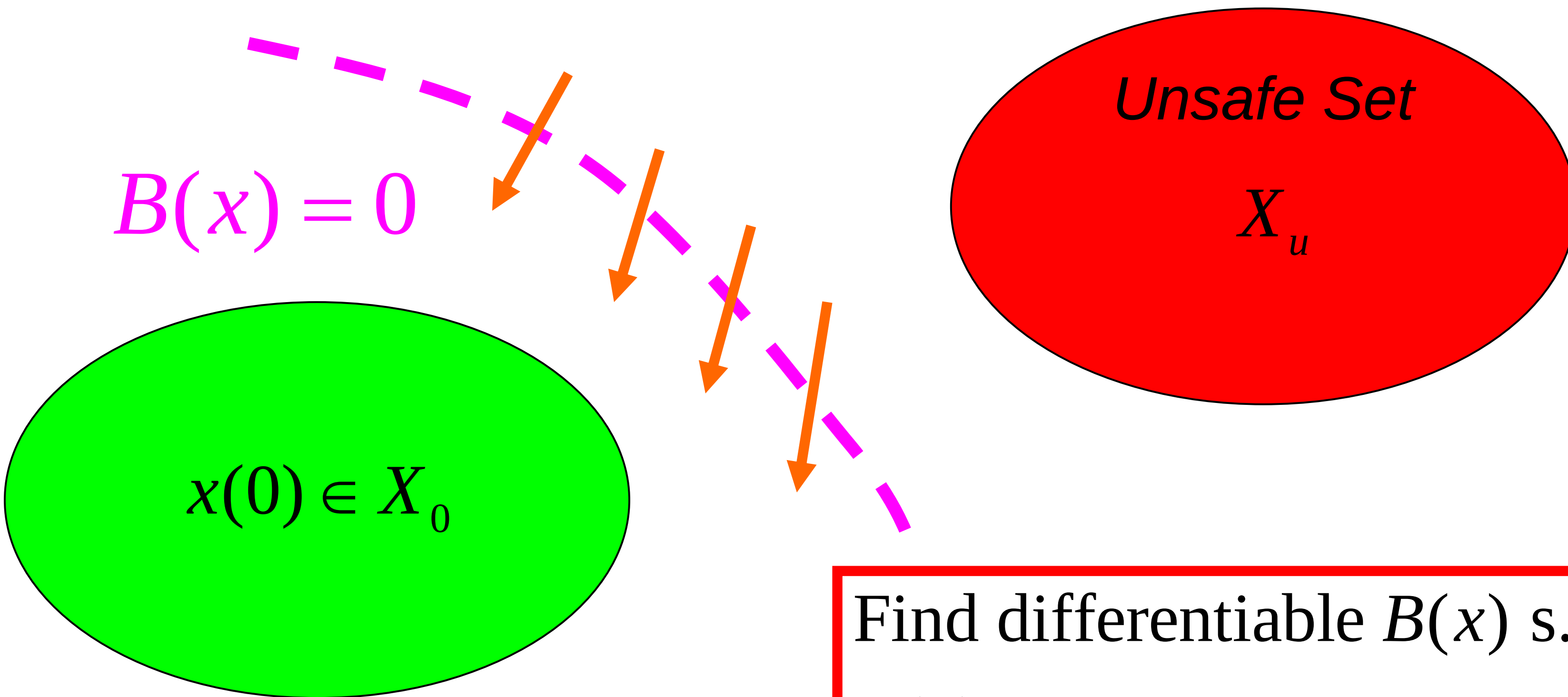
- Framework is applicable to a very large class of systems:
  - Nonlinear
  - Hybrid
  - Uncertain
  - Constrained
  - Stochastic
  - Time-delay, etc.

- Efficient computation of proofs using convex optimization.

Outline of the Talk

- Background
- **Safety Verification**
- Reachability Verification
- Other Results (Hybrid, Stochastic, etc)
- Future Directions

Safety Verification – Barrier Certificate



$$\dot{x} = f(x, d)$$
$$d \in D$$

Find differentiable $B(x)$ s.t.

$$B(x) \le 0 \quad \forall x \in X_0$$
$$B(x) > 0 \quad \forall x \in X_u$$
$$\frac{\partial B}{\partial x} f(x, d) \le 0 \quad \forall (x, d) \in X \times D$$

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(HSCC 2004)

Computation of Barrier Certificate

- For polynomial systems, a polynomial barrier certificate *B(x)*, can be searched using sum of squares (SOS) programming.

- SOS polynomials:

$p(x)$ is an SOS, if $\exists$ polynomials $f_1(x), \ ..., \ f_N(x)$, s.t.

$$p(x) = \sum_{i=1}^{N} f_i^2(x).$$

Equivalently, if $\exists$ monomial vector $Z(x)$ and p.s.d. matrix Q, s.t.

$$p(x) = Z^T(x) Q Z(x).$$

(sufficient condition for non-negativity).

- Software: SOSTOOLS (http://www.cds.caltech.edu/sostools)

Example

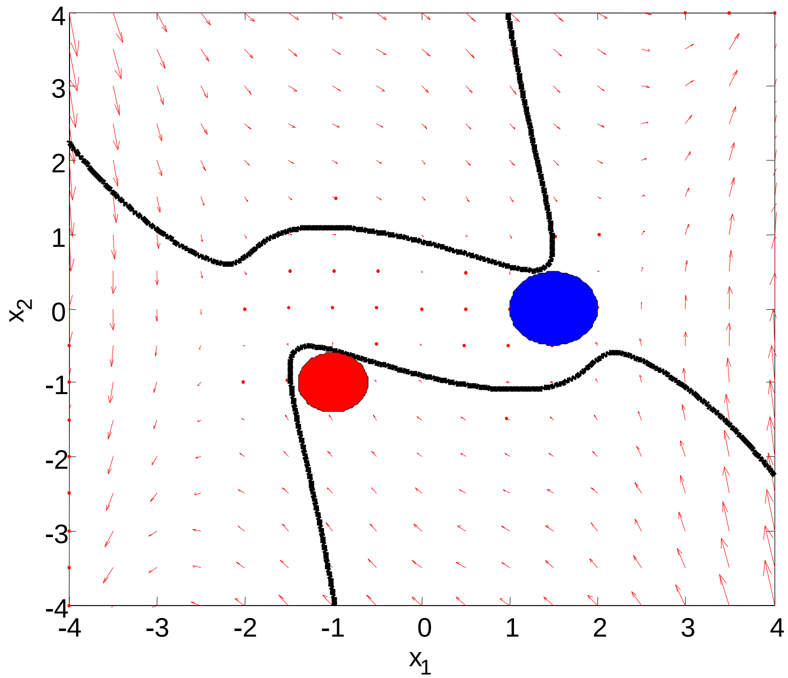
$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -x_1 + \frac{1}{3}x_1^3 - x_2$$

$$X = \mathbb{R}^2$$

$$X_0 = \{(x_1 - 1)^2 + x_2^2 \le 0.5^2\}$$

$$X_u = \{(x_1 + 1)^2 + (x_2 + 1)^2 \le 0.4^2\}$$



Computation using SOS programming yields (after some rounding and re-check):

$$B(x) = -13 + 7x_1^2 + 16x_2^2 - 6x_1^2x_2^2 - \frac{7}{6}x_1^4 - 3x_1x_2^3 + 12x_1x_2 - \frac{12}{3}x_1^3x_2$$

Proof Correctness

For example, to check that $\frac{\partial B}{\partial x} f(x) \le 0$, use the quadratic form

$$-\frac{\partial B}{\partial x} f(x) = 30x_1 x_2 + 3x_1 x_2^3 - 22x_1^3 x_2 + 3x_2^4 + 20x_2^2 - 9x_1^2 x_2^2 + 4x_2 x_1^5 + 3x_1^4 x_2^2 + 12x_1^2 - 8x_1^4 + \frac{4}{3} x_1^6$$

$$= \begin{bmatrix} x_2 \\ x_2^2 \\ x_1 \\ x_1 x_2 \\ x_1^2 x_2 \\ x_1^3 \end{bmatrix}^T \underbrace{\begin{bmatrix} 20 & 0 & 15 & 0 & -7.5 & -5 \\ 0 & 3 & 0 & 1.5 & 0 & 0 \\ 15 & 0 & 12 & 0 & -6 & -4 \\ 0 & 1.5 & 0 & 6 & 0 & 0 \\ -7.5 & 0 & -6 & 0 & 3 & 2 \\ -5 & 0 & -4 & 0 & 2 & 4/3 \end{bmatrix}}_{\ge 0} \begin{bmatrix} x_2 \\ x_2^2 \\ x_1 \\ x_1 x_2 \\ x_1^2 x_2 \\ x_1^3 \end{bmatrix}$$

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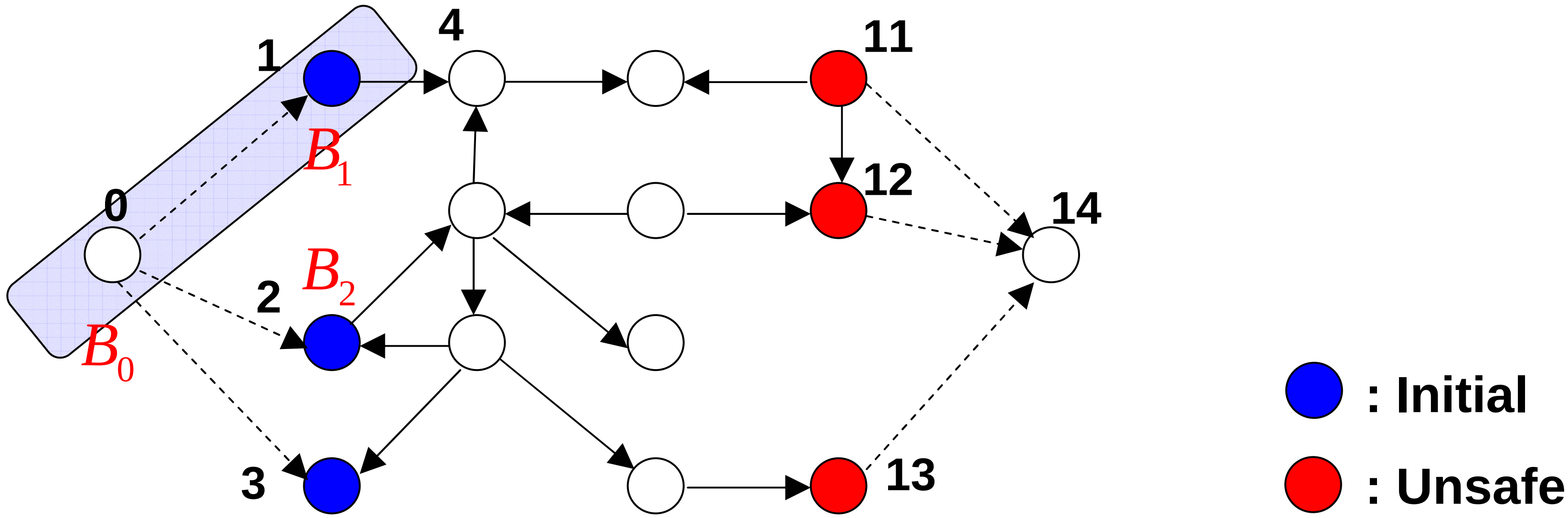
- Safety Verification
- **Reachability Verification**
- Some Other Results
- Future Directions

Lesson from Finite State Systems



Barrier:

Maximize $B_{14} - B_0$, subject to

$$\boxed{B_1 - B_0 \le 0}$$

$$B_2 - B_0 \le 0$$

$$\dots$$

$$B_{14} - B_{13} \le 0$$

Safe if obj > 0.

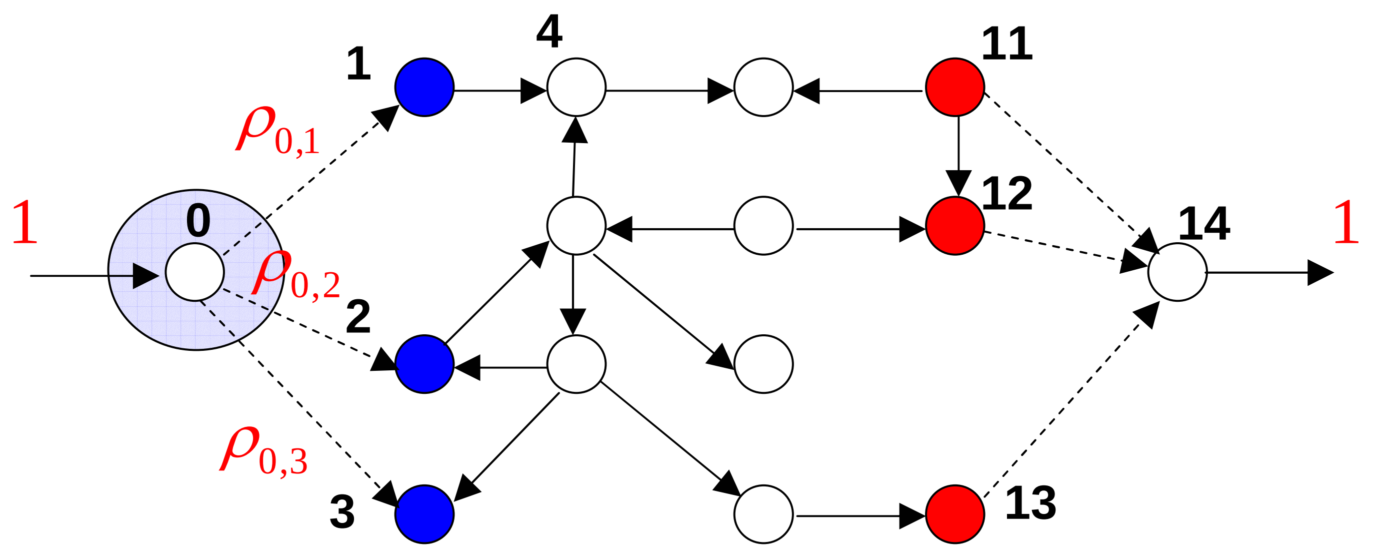
Compare to:

$$B(x) \le 0 \quad \forall x \in X_0$$

$$B(x) > 0 \quad \forall x \in X_u$$

$$\frac{\partial B}{\partial x} f(x,d) \le 0 \quad \forall (x,d) \in X \times D$$

Lesson from Finite State Systems



: Initial

: Unsafe

Barrier:

Maximize $B_{14} - B_0$, subject to

$$B_1 - B_0 \le 0$$

$$B_2 - B_0 \le 0$$

...

$$B_{14} - B_{13} \le 0$$

Safe if obj > 0.

LP duality

$\longleftrightarrow$

Flow:

Minimize 0, subject to

$$\rho_{0,1}, \rho_{0,2}, \ldots, \rho_{13,14} \ge 0$$

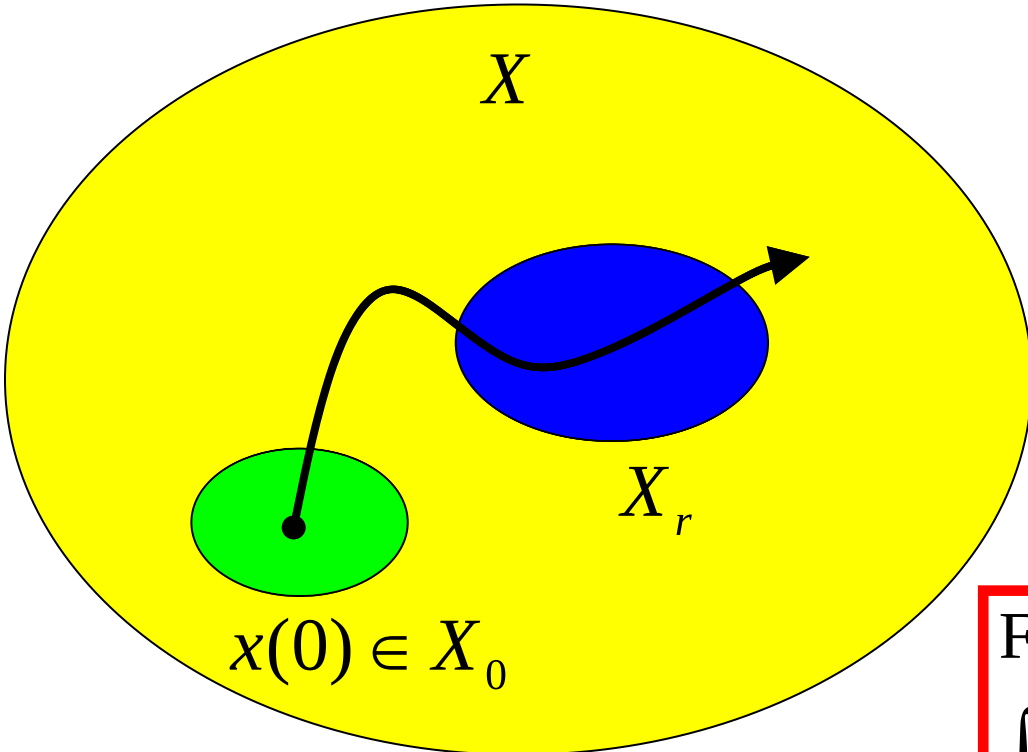
$$1 = \rho_{0,1} + \rho_{0,2} + \rho_{0,3}$$

$$\rho_{11,14} + \rho_{11,12} + \rho_{13,14} = 1$$

$\rho_{0,1} = \rho_{1,4}$, and so on

Reachable if LP is feasible.

Reachability Verification



$$\dot{x} = f(x)$$

Assume X is bounded, and X_0 has non-empty interior.

Let ε be a positive number.

Find cont. diff. $\rho(x)$ s.t.

$$\int_{X_0} \rho(x)dx \geq 0,$$

$$\rho(x) \leq -\varepsilon \quad \forall x \in \partial X \setminus \partial X_r,$$

$$\nabla \cdot (\rho f)(x) \geq \varepsilon \quad \forall x \in X \setminus X_r.$$

Prajna & Rantzer, (HSCC 2005)

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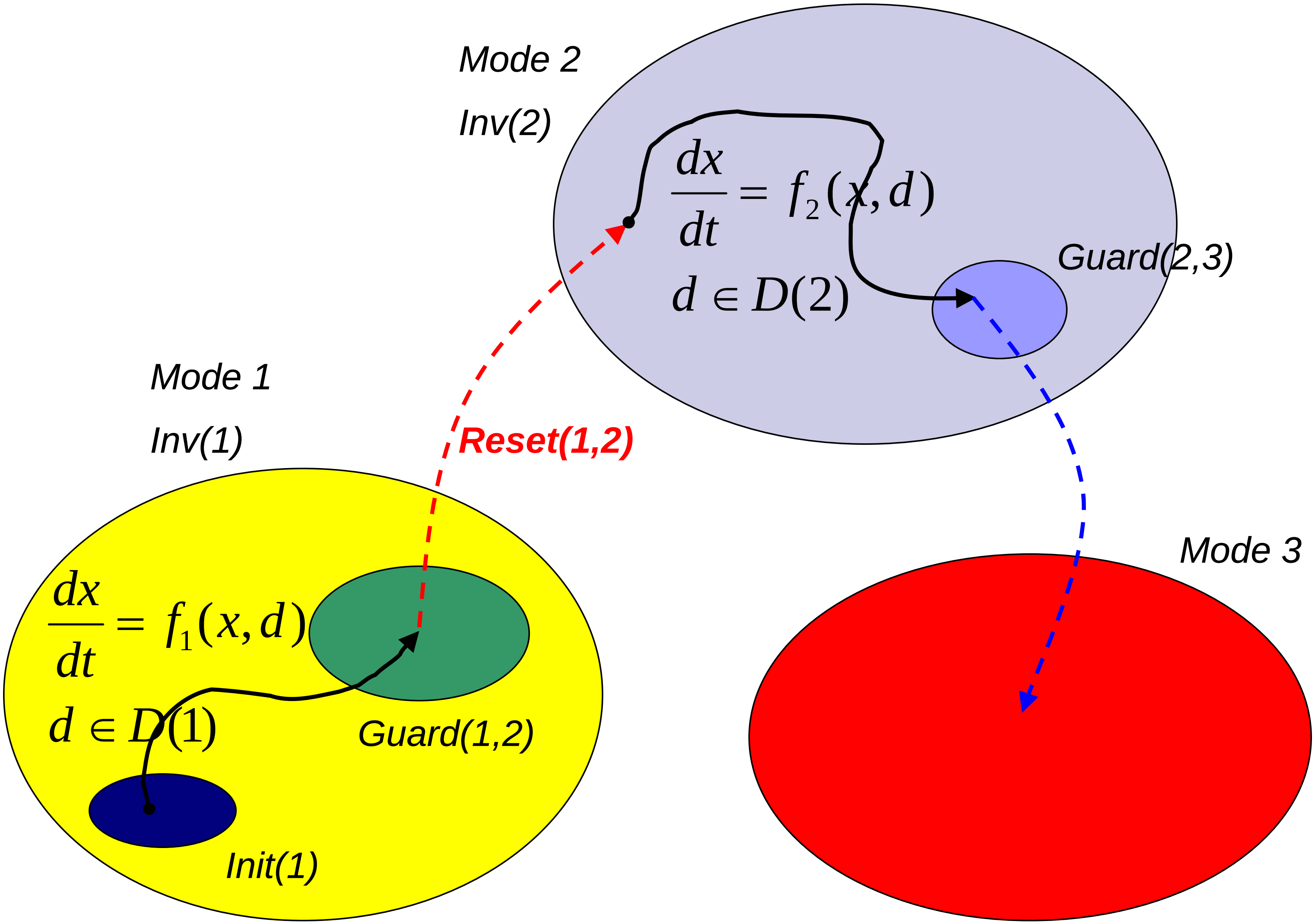
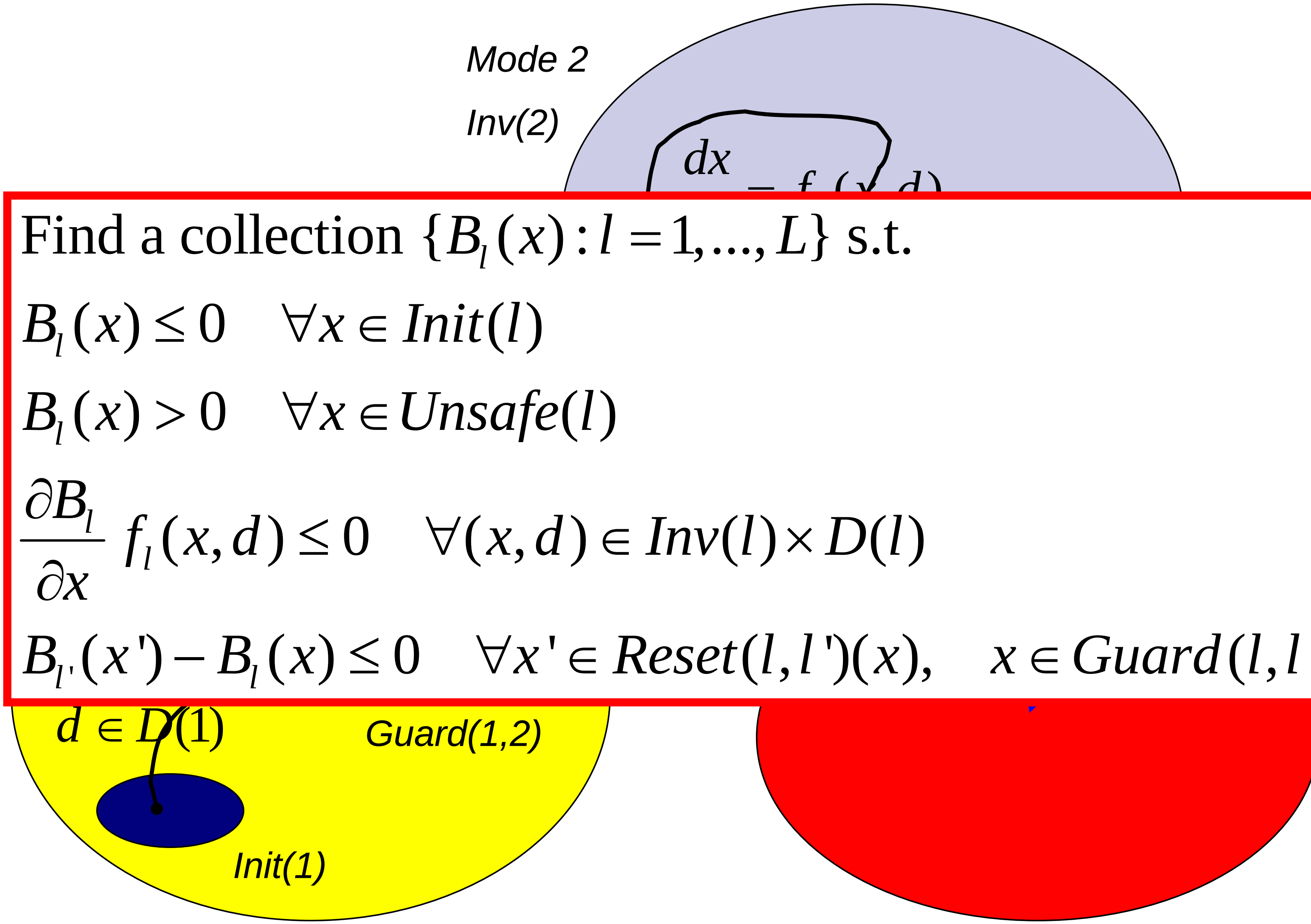
Hybrid Systems



Hybrid Systems



Find a collection $\{B_l(x) : l = 1, \dots, L\}$ s.t.

$$B_l(x) \le 0 \quad \forall x \in Init(l)$$

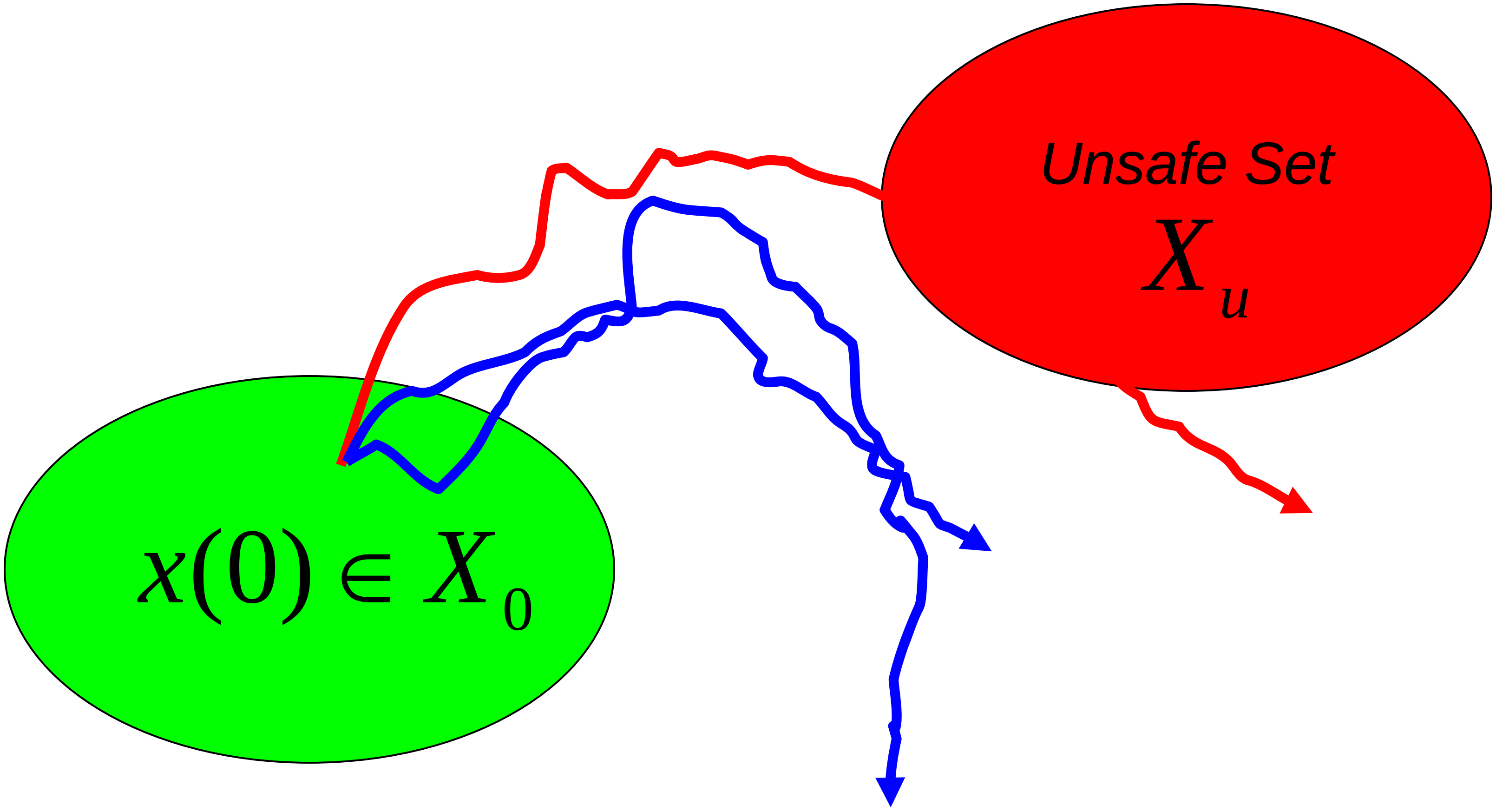
$$B_l(x) > 0 \quad \forall x \in Unsafe(l)$$

$$\frac{\partial B_l}{\partial x} f_l(x,d) \le 0 \quad \forall (x,d) \in Inv(l) \times D(l)$$

$$B_{l'}(x') - B_l(x) \le 0 \quad \forall x' \in Reset(l,l')(x), \quad x \in Guard(l,l')$$

Stochastic Systems

$$dx = f(x)dt + g(x)dw_t$$

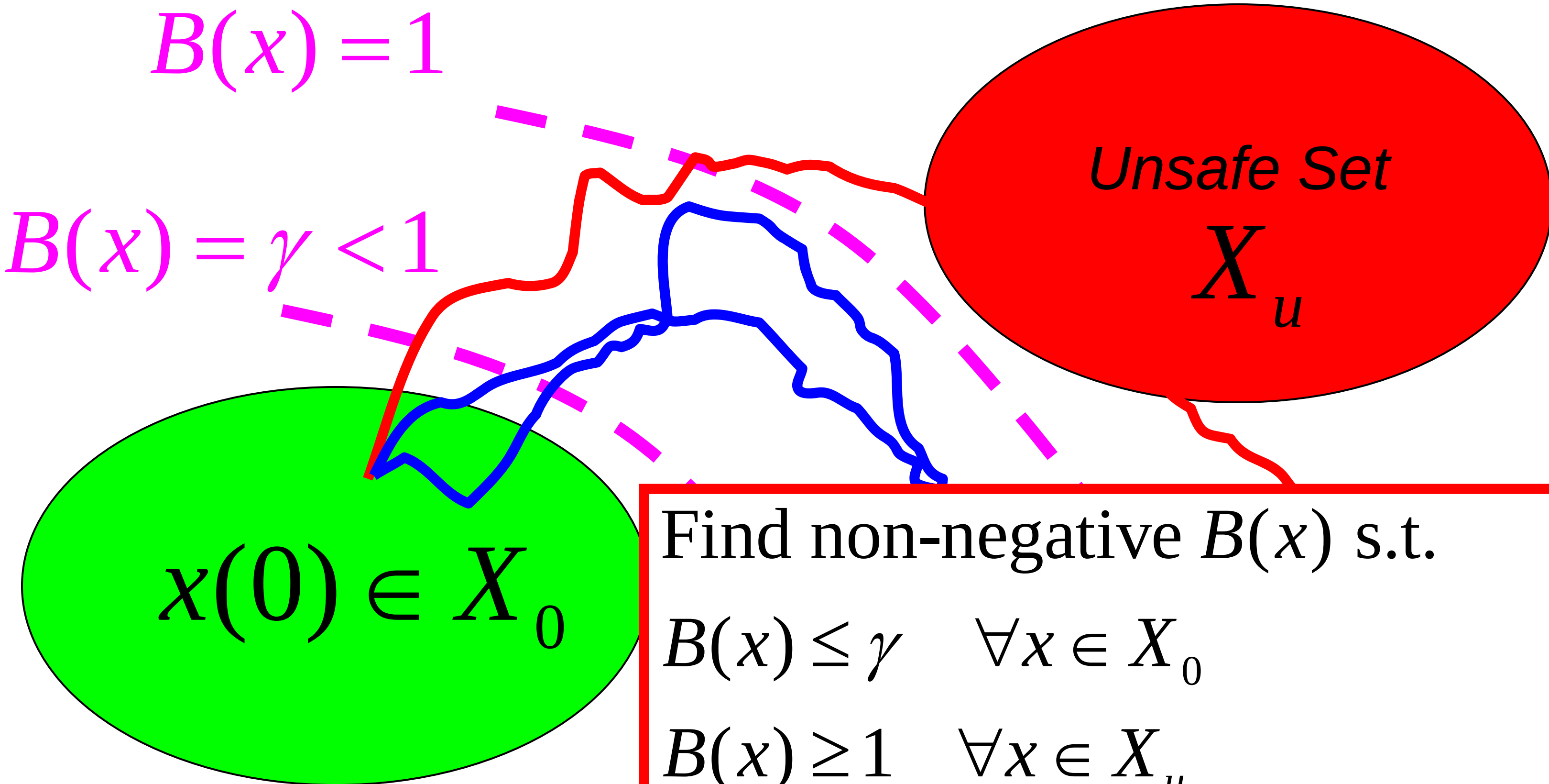


What is the reach probability???

Prajna, Jadbabaie, Pappas (CDC 2004)

Stochastic Systems

$$dx = f(x)dt + g(x)dw_t$$



Find non-negative $B(x)$ s.t.

$$B(x)\le\gamma \quad \forall x\in X_0$$

$$B(x)\ge 1 \quad \forall x\in X_u$$

$$\frac{\partial B}{\partial x} f(x) + g^T(x)\frac{\partial^2 B}{\partial x^2} g(x) \le 0 \quad \forall x\in X$$

$$\Rightarrow \text{Prob(Reach)} \le \gamma$$

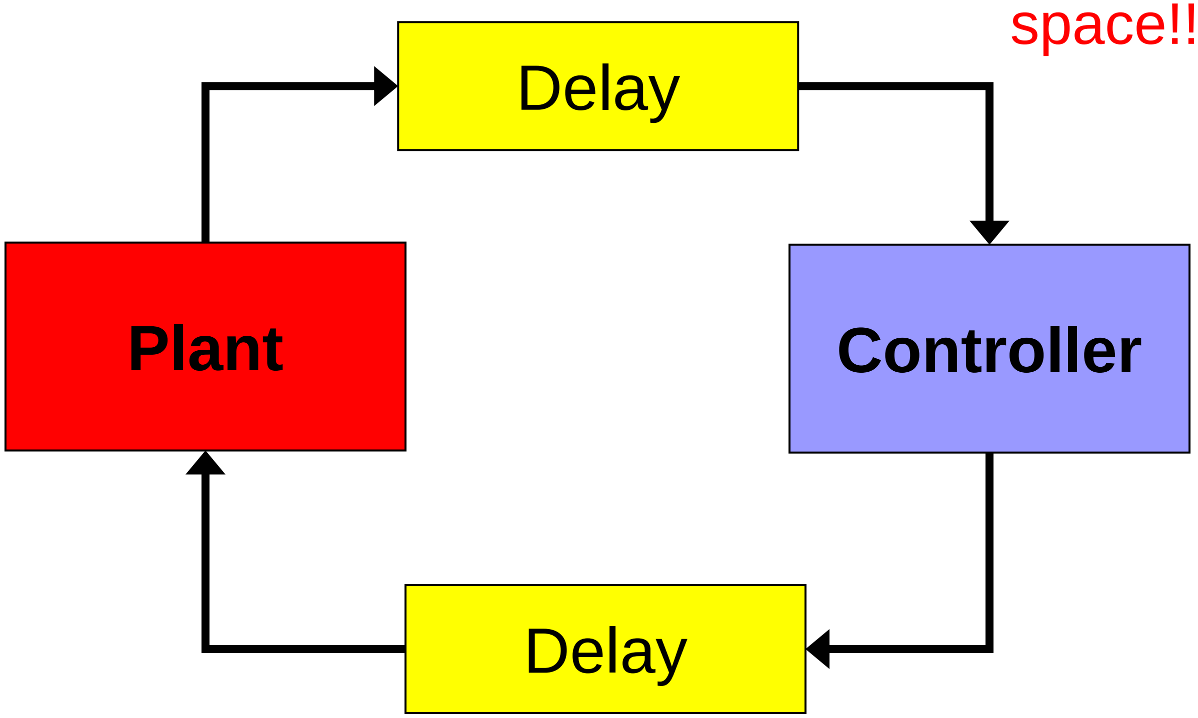
Prajna, Jadbabaie, Pappas (CDC 2004)

Time-Delay Systems

Networked control systems:

$$\dot{x}(t) = f\,(x(t), x(t-r))$$

Infinite dimensional state space!!



Prajna & Jadbabaie,
(CDC 2005)

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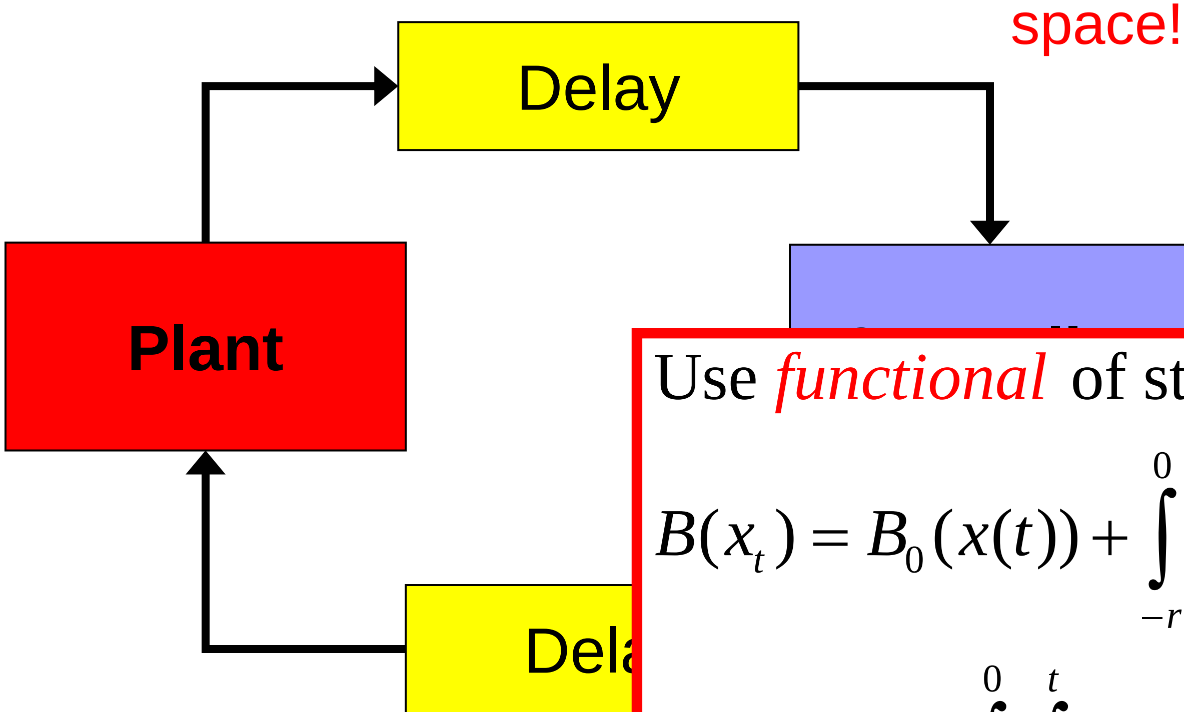

Use *functional* of state:

$$B(x_t) = B_0(x(t)) + \int_{-r}^{0} B_1(\theta, x(t), x(t+\theta))\,d\theta$$

$$+ \int_{-r}^{0}\int_{t+\theta}^{t} B_2(x(\eta))\,d\eta\,d\theta$$

Conditions for safety can be derived.

Prajna & Jadbabaie, (CDC 2005)

Existence (Converse) Theorem

There is $B(x)$ s.t.

$$B(x) \le 0 \quad \forall x \in X_0$$

$$B(x) > 0 \quad \forall x \in X_u$$

$$\frac{\partial B}{\partial x} f(x) \le 0 \quad \forall x \in X$$

$\Rightarrow$

$\Leftarrow$?

Safety property holds for system $\dot{x} = f(x)$, sets X_0, X_u, X

Yes, under some reasonable technical conditions.

Proof: Use strong duality between convex programs for safety and reachability.

Prajna & Rantzer, (IFAC 2005)

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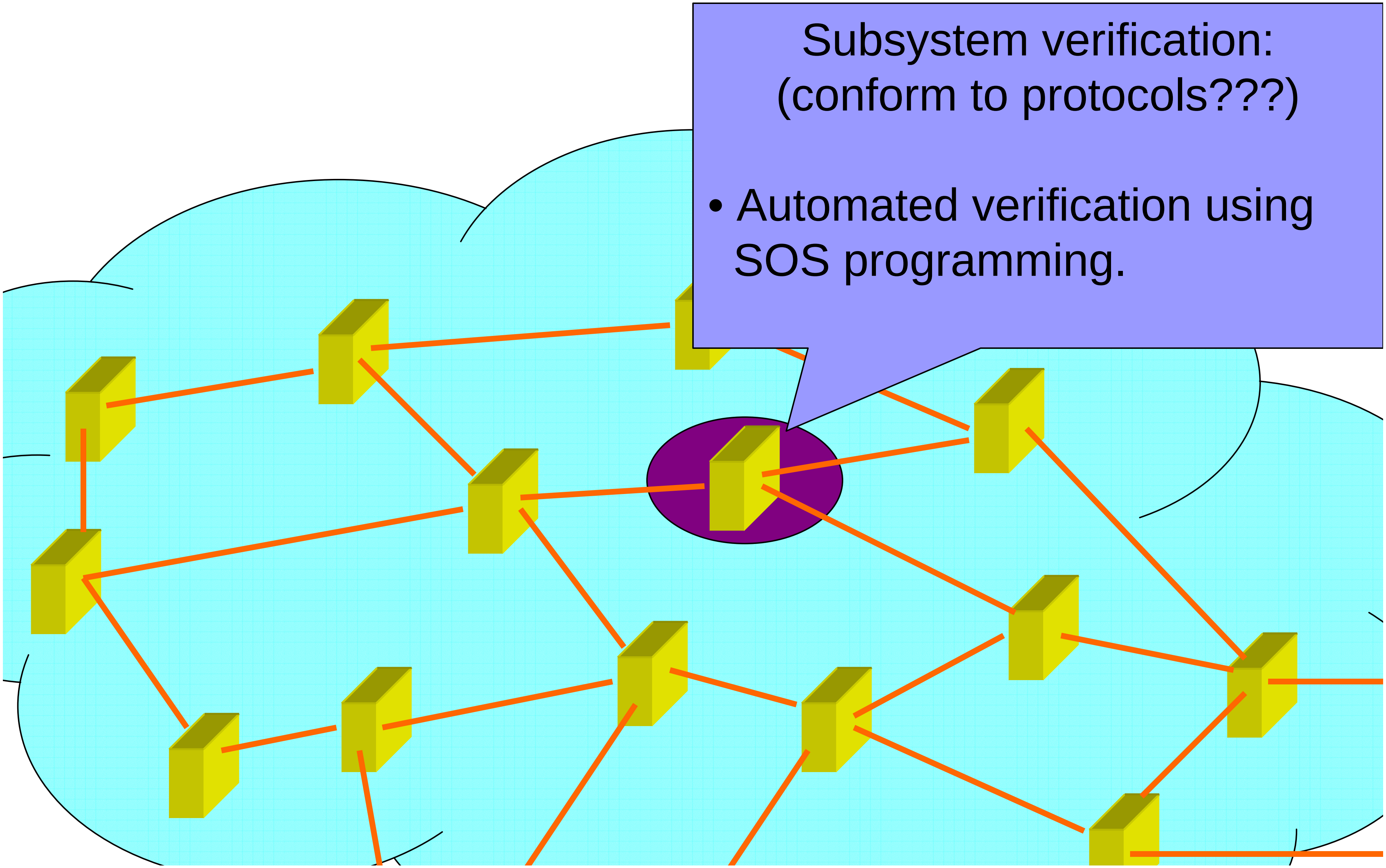
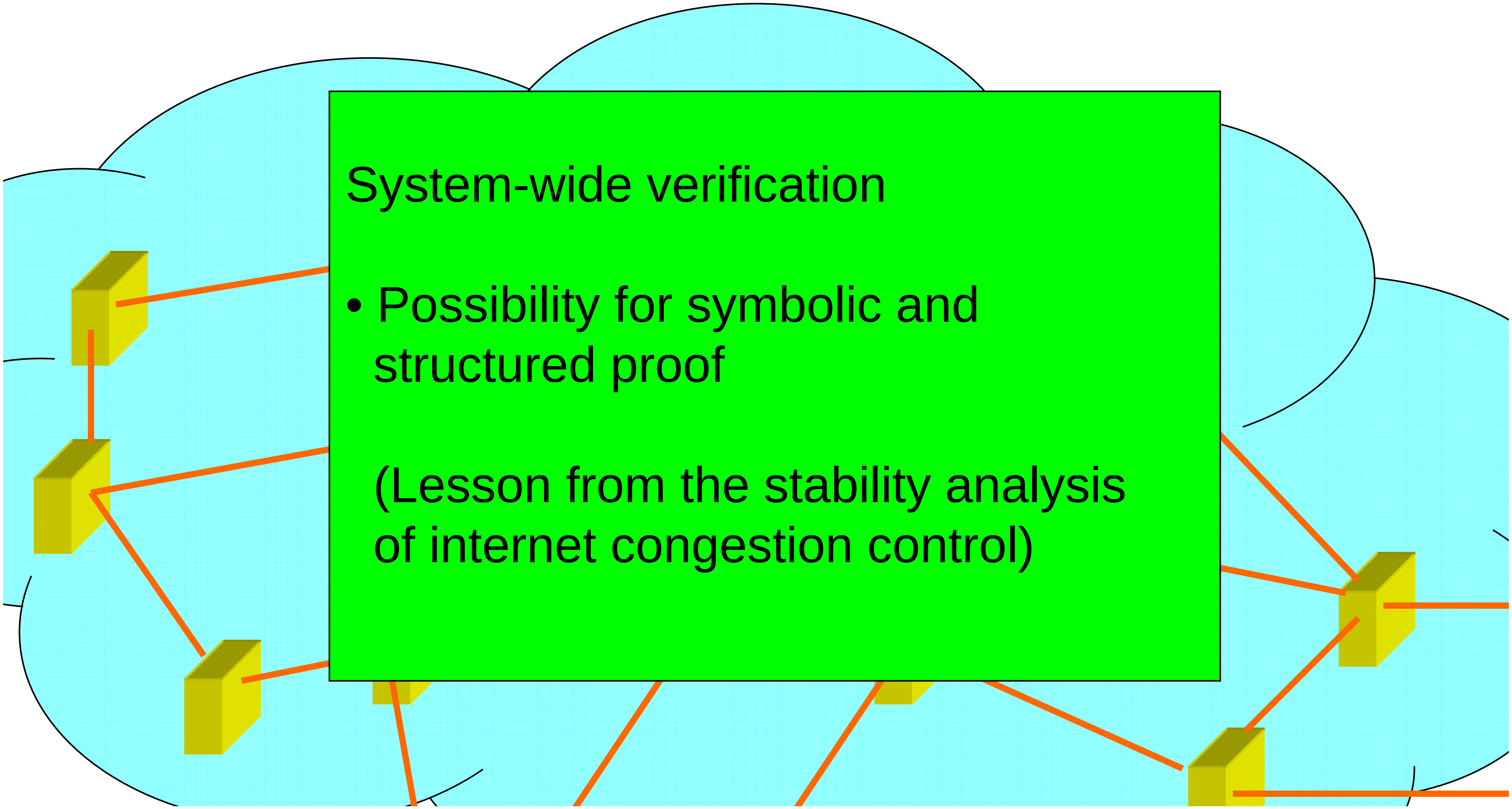
Hierarchical Large-Scale Verification



Hierarchical Large-Scale Verification



System-wide verification

- Possibility for symbolic and structured proof

(Lesson from the stability analysis of internet congestion control)

From Control Algorithm to Implementation

- Sampling
- Quantization
- Code generation

Control Algorithm

Control Implementation

Differential equations, difference equations, logic-based control, etc.

Embedded controllers, C++ code, DSP, etc.

Certificates/Proofs

Should carry over!!

Barrier certificates, density functions, Lyapunov functions, etc.

Barrier certificates, density functions, invariants, ranking functions, etc.

Conclusions

Challenges	Our Methodology
Common framework	Encompasses a very large class of systems.
Automated computation	SOS programming, software tools are available.
Easily checkable proofs	Algebraic proofs, easy to check.
Scalability	Potential both at small – medium scale (numerical) and large scale (symbolic, structured).