Semialgebraic Relaxations using Moment-SOS Hierarchies

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Personal Background

- 2008 – 2010: Master at Tokyo University
  **Hierarchical Domain Decomposition Methods**
  (S. Yoshimura)

- 2010 – 2013: PhD at Inria Saclay LIX/CMAP
  **Formal Proofs for Nonlinear Optimization**
  (S. Gaubert and B. Werner)

- 2014–now: Postdoc at LAAS-CNRS
  **Moment-SOS applications**
  (J.B. Lasserre and D. Henrion)
Mathematicians want to eliminate all the uncertainties on their results. Why?

M. Lecat, Erreurs des Mathématiciens des origines à nos jours, 1935.

130 pages of errors! (Euler, Fermat, Sylvester, …)
Possible workaround: proof assistants

COQ (Coquand, Huet 1984)

HOL-LIGHT (Harrison, Gordon 1980)

Built in top of OCAML
Complex Proofs

- Complex mathematical proofs / mandatory computation

K. Appel and W. Haken, Every Planar Map is Four-Colorable, 1989.

From Oranges Stack...

Kepler Conjecture (1611):

The maximal density of sphere packings in 3D-space is \( \frac{\pi}{\sqrt{18}} \)

- Face-centered cubic Packing
- Hexagonal Compact Packing

Computation: check thousands of nonlinear inequalities.

Robert MacPherson, editor of The Annals of Mathematics: “[...] the mathematical community will have to get used to this state of affairs.”

Flyspeck [Hales 06]: Formal Proof of Kepler Conjecture
The proof of T. Hales (1998) contains mathematical and computational parts

Computation: check thousands of nonlinear inequalities

Robert MacPherson, editor of The Annals of Mathematics: “[...] the mathematical community will have to get used to this state of affairs.”

**Flyspeck** [Hales 06]: Formal Proof of Kepler Conjecture

Project Completion on 10 August by the Flyspeck team!!
A “Simple” Example

In the computational part:

- Multivariate Polynomials:

\[
\Delta x := x_1 x_4 (-x_1 + x_2 + x_3 - x_4 + x_5 + x_6) + x_2 x_5 (x_1 - x_2 + x_3 + x_4 - x_5 + x_6) + x_3 x_6 (x_1 + x_2 - x_3 + x_4 + x_5 - x_6) - x_2 (x_3 x_4 + x_1 x_6) - x_5 (x_1 x_3 + x_4 x_6)
\]
A “Simple” Example

In the computational part:

- **Semialgebraic** functions: composition of polynomials with $|\cdot|, \sqrt{}, +, -, \times, /, \text{sup}, \text{inf}, \ldots$

\[
p(x) := \partial_4 \Delta x \quad q(x) := 4x_1 \Delta x \\
 r(x) := p(x) / \sqrt{q(x)}
\]

\[
l(x) := -\frac{\pi}{2} + 1.6294 - 0.2213 (\sqrt{x_2} + \sqrt{x_3} + \sqrt{x_5} + \sqrt{x_6} - 8.0) + 0.913 (\sqrt{x_4} - 2.52) + 0.728 (\sqrt{x_1} - 2.0)
\]
A “Simple” Example

In the computational part:

- **Transcendental** functions $\mathcal{T}$: composition of semialgebraic functions with arctan, exp, sin, $+, -, \times, \ldots$
A “Simple” Example

In the computational part:

- Feasible set \( K := [4, 6.3504]^3 \times [6.3504, 8] \times [4, 6.3504]^2 \)

Lemma_9922699028 from Flyspeck:

\[
\forall x \in K, \arctan\left( \frac{p(x)}{\sqrt{q(x)}} \right) + l(x) \geq 0
\]
Existing Formal Frameworks

Formal proofs for Global Optimization:

- Bernstein polynomial methods [Zumkeller 08] restricted to polynomials

- Taylor + Interval arithmetic [Melquiond 12, Solovyev 13] robust but subject to the CURSE OF DIMENSIONALITY
Lemma from Flyspeck:

\[
\forall x \in K, \arctan\left(\frac{\partial_4 \Delta x}{\sqrt{4x_1 \Delta x}}\right) + l(x) \geq 0
\]

- Dependency issue using Interval Calculus:
  - One can bound \(\frac{\partial_4 \Delta x}{\sqrt{4x_1 \Delta x}}\) and \(l(x)\) separately

- Too coarse lower bound: \(-0.87\)

- Subdivide \(K\) to prove the inequality
Introduction

**Moment-SOS relaxations**

Another look at Nonnegativity

New Applications of Moment-SOS Hierarchies

Conclusion
Polynomial Optimization Problems

- Semialgebraic set $K := \{ x \in \mathbb{R}^n : g_1(x) \geq 0, \ldots, g_m(x) \geq 0 \}$

- $p^* := \min_{x \in K} p(x)$: NP hard

- Sums of squares $\Sigma[x]$
  e.g. $x_1^2 - 2x_1x_2 + x_2^2 = (x_1 - x_2)^2$

- $Q(K) := \left\{ \sigma_0(x) + \sum_{j=1}^m \sigma_j(x)g_j(x), \text{ with } \sigma_j \in \Sigma[x]\right\}$
Archimedean module
The set $K$ is compact and the polynomial $N - \|x\|^2_2$ belongs to $Q(K)$ for some $N > 0$.

- Assume that $K$ is a box: product of closed intervals
- Normalize the feasibility set to get $K' := [-1, 1]^n$
  
  $K' := \{ x \in \mathbb{R}^n : g_1 := 1 - x_1^2 \geq 0, \cdots, g_n := 1 - x_n^2 \geq 0 \}$

- $n - \|x\|^2_2$ belongs to $Q(K')$
Convexification and the K Moment Problem

- Borel $\sigma$-algebra $\mathcal{B}$ (generated by the open sets of $\mathbb{R}^n$)

- $\mathcal{M}_+(K)$: set of probability measures supported on $K$. If $\mu \in \mathcal{M}_+(K)$ then
  1. $\mu : \mathcal{B} \to [0, 1]$, $\mu(\emptyset) = 0$, $\mu(\mathbb{R}^n) < \infty$
  2. $\mu(\bigcup_i B_i) = \sum_i \mu(B_i)$, for any countable $(B_i) \subset \mathcal{B}$
  3. $\int_K \mu(dx) = 1$

- $\text{supp}(\mu)$ is the smallest set $K$ such that $\mu(\mathbb{R}^n \setminus K) = 0$
Convexification and the K Moment Problem

\[
p^* = \inf_{x \in K} p(x) = \inf_{\mu \in \mathcal{M}_+(K)} \int_K p \, d\mu
\]
Convexification and the K Moment Problem

- Let \((x^\alpha)_{\alpha \in \mathbb{N}^n}\) be the monomial basis

**Definition**

A sequence \(y\) has a representing measure on \(K\) if there exists a finite measure \(\mu\) supported on \(K\) such that

\[
y_\alpha = \int_K x^\alpha \mu(dx), \quad \forall \alpha \in \mathbb{N}^n.
\]
Convexification and the K Moment Problem

\[ L_y(q) : q \in \mathbb{R}[x] \mapsto \sum_\alpha q_\alpha y_\alpha \]

**Theorem [Putinar 93]**

Let \( K \) be compact and \( Q(K) \) be Archimedean. Then \( y \) has a representing measure on \( K \) iff

\[ L_y(\sigma) \geq 0, \quad L_y(g_j \sigma) \geq 0, \quad \forall \sigma \in \Sigma[x]. \]
Lasserre’s Hierarchy of SDP relaxations

- Moment matrix
  \[ M(y)_{u,v} := L_y(u \cdot v), \text{ } u, v \text{ monomials} \]

- Localizing matrix \( M(g_j y) \) associated with \( g_j \)
  \[ M(g_j y)_{u,v} := L_y(u \cdot v \cdot g_j), \text{ } u, v \text{ monomials} \]
Lasserre’s Hierarchy of SDP relaxations

- $M_k(y)$ contains $\binom{n+2k}{n}$ variables, has size $\binom{n+k}{n}$

- Truncated matrix of order $k = 2$ with variables $x_1, x_2$:

\[
M_2(y) = \begin{pmatrix}
  1 & x_1 & x_2 & x_1^2 & x_1x_2 & x_2^2 \\
  1 & y_{1,0} & y_{0,1} & y_{2,0} & y_{1,1} & y_{0,2} \\
  x_1 & y_{1,0} & y_{2,0} & y_{1,1} & y_{3,0} & y_{2,1} & y_{1,2} \\
  x_2 & y_{0,1} & y_{1,1} & y_{0,2} & y_{2,1} & y_{1,2} & y_{0,3} \\
  x_1^2 & y_{2,0} & y_{3,0} & y_{2,1} & y_{4,0} & y_{3,1} & y_{2,2} \\
  x_1x_2 & y_{1,1} & y_{2,1} & y_{1,2} & y_{3,1} & y_{2,2} & y_{1,3} \\
  x_2^2 & y_{0,2} & y_{1,2} & y_{0,3} & y_{2,2} & y_{1,3} & y_{0,4}
\end{pmatrix}
\]
Consider $g_1(x) := 2 - x_1^2 - x_2^2$. Then $v_1 = \lceil \deg g_1 / 2 \rceil = 1$.

\[
M_1(g_1 y) = x_1 \begin{pmatrix} 1 \\ 2 - y_{2,0} - y_{0,2} \\ 2y_{1,0} - y_{3,0} - y_{1,2} \\ 2y_{0,1} - y_{2,1} - y_{0,3} \end{pmatrix} x_1 + x_2 \begin{pmatrix} 2y_{1,0} - y_{3,0} - y_{1,2} \\ 2y_{2,0} - y_{4,0} - y_{2,2} \\ 2y_{1,1} - y_{3,1} - y_{1,3} \end{pmatrix} x_2
\]

\[
M_1(g_1 y)(3, 3) = L(g_1(x) \cdot x_2 \cdot x_2) = L(2x_2^2 - x_1^2x_2^2 - x_2^4)
\]

\[
= 2y_{0,2} - y_{2,2} - y_{0,4}
\]
Lasserre’s Hierarchy of SDP relaxations

- Truncation with moments of order at most $2k$

- $v_j := \lceil \deg g_j / 2 \rceil$

- Hierarchy of semidefinite relaxations:

\[
\begin{align*}
\inf_y L_y(p) &= \sum_{\alpha} \int_K p_\alpha x^\alpha \mu(dx) = \sum_{\alpha} p_\alpha y_\alpha \\
M_k(y) &\succeq 0, \\
M_{k-v_j}(g_j y) &\succeq 0, \quad 1 \leq j \leq m, \\
 y_1 &= 1.
\end{align*}
\]
- $F_0, F_\alpha$ symmetric real matrices, cost vector $c$

**Primal-dual pair of semidefinite programs:**

$\begin{align*}
\mathcal{P} : \quad & \inf_y \quad \sum_\alpha c_\alpha y_\alpha \\
& \text{s.t.} \quad \sum_\alpha F_\alpha y_\alpha - F_0 \succeq 0
\end{align*}$

$\begin{align*}
\mathcal{D} : \quad & \sup_Y \quad \text{Trace} \left( F_0 Y \right) \\
& \text{s.t.} \quad \text{Trace} \left( F_\alpha Y \right) = c_\alpha , \quad Y \succeq 0 .
\end{align*}$

- Freely available SDP solvers (CSDP, SDPA, SEDUMI)
Primal-dual Moment-SOS

- $\mathcal{M}_+(K)$: space of probability measures supported on $K$

 Polynomial Optimization Problems (POP)

(Primal) \[ \inf \int_K p \, d\mu = \sup \lambda \]

s.t. \[ \mu \in \mathcal{M}_+(K) \]

(Dual) \[ \text{s.t.} \quad \lambda \in \mathbb{R}, \quad p - \lambda \in Q(K) \]
Primal-dual Moment-SOS

- Truncated quadratic module $\mathcal{Q}_k(K) := \mathcal{Q}(K) \cap \mathbb{R}_{2k}[x]$

- For large enough $k$, zero duality gap [Lasserre 01]:

Polynomial Optimization Problems (POP)

(Moment) \[ \inf_{\alpha} \sum p_\alpha y_\alpha \]

(SOS) \[ = \sup \lambda \]

s.t. \[ M_{k-v}(g_j y) \succeq 0, \ 0 \leq j \leq m, \]
\[ y_1 = 1 \]

s.t. \[ \lambda \in \mathbb{R}, \]
\[ p - \lambda \in \mathcal{Q}_k(K) \]
Hierarchy of SOS relaxations:
\[ \lambda_k := \sup_{\lambda} \left\{ \lambda : p - \lambda \in Q_k(K) \right\} \]

Convergence guarantees \( \lambda_k \uparrow p^* \) [Lasserre 01]

If \( p - p^* \in Q_k(K) \) for some \( k \) then:

\[ y^* := (1, x_1^*, x_2^*, (x_1^*)^2, x_1^*x_2^*, \ldots, (x_1^*)^{2k}, \ldots, (x_n^*)^{2k}) \]

is a global minimizer of the primal SDP [Lasserre 01].
Practical Computation

- **Caprasse Problem**

\[
\forall x \in [-0.5, 0.5]^4, -x_1 x_3^3 + 4x_2 x_3^2 x_4 + 4x_1 x_3 x_4^2 + 2x_2 x_4^3 + 4x_1 x_3 + 4x_3^2 - 10x_2 x_4 - 10x_4^2 + 5.1801 \geq 0.
\]

- Scale on \([0, 1]^4\)

- SOS of degree at most 4

- Redundant constraints \(x_1^2 \leq 1, \ldots, x_4^2 \leq 1\)
The “No Free Lunch” Rule

- Exponential dependency in
  1. Relaxation order $k$ (SOS degree)
  2. number of variables $n$

- Computing $\lambda_k$ involves $\binom{n+2k}{n}$ variables

- At fixed $k$, $O(n^{2k})$ variables
Introduction

Moment-SOS relaxations

Another look at Nonnegativity

New Applications of Moment-SOS Hierarchies

Conclusion
Another look at Nonnegativity

- Knowledge of $K$ through $\mu \in \mathcal{M}_+(K)$

- **Independent** of the representation of $K$

- Typical from **Inverse Problems**
  
  “reconstruct” $K$ from measuring moments of $\mu \in \mathcal{M}_+(K)$
Another look at Nonnegativity

Borel $\sigma$-algebra $\mathcal{B}$ (generated by the open sets of $\mathbb{R}^n$)

**Lemma**

A continuous function $p : \mathbb{R}^n \to \mathbb{R}$ is nonnegative on $K$ iff

the set function $\nu : B \in \mathcal{B} \mapsto \int_{K \cap B} p(x) \, d\mu(x)$ belongs to $\mathcal{M}_+$. 
Another look at Nonnegativity

\[ \nu : B \in \mathcal{B} \mapsto \int_{K \cap B} p(x) \, d\mu(x) \]

Proof

1. “Only if” part is straightforward

2. “If part”
   - If \( \nu \in \mathcal{M}_+ \) then \( p(x) \geq 0 \) for \( \mu \)-almost all \( x \in K \), i.e. there exists \( G \in \mathcal{B} \) such that \( \mu(G) = 0 \) and \( p(x) \geq 0 \) on \( K \setminus G \).
   - \( K = \overline{K \setminus G} \) (from the support definition).
   - Let \( x \in K \). There is a sequence \( (x_l) \subset K \setminus G \) such that \( x_l \to x \), as \( l \to \infty \). By continuity of \( p \) and \( p(x_l) \geq 0 \), one has \( p(x) \geq 0 \).
The K Moment Problem

Definition

A sequence $y$ has a representing measure on $K$ if there exists a finite measure $\mu$ supported on $K$ such that

$$y_\alpha = \int_K x^\alpha \mu(dx), \quad \forall \alpha \in \mathbb{N}^n.$$
The K Moment Problem

\[ L_y(q) : q \in \mathbb{R}[x] \mapsto \sum_{\alpha} q_{\alpha} y_{\alpha} \]

**Theorem [Putinar 93]**

Let \( K \) be compact and \( \mathcal{Q}(K) \) be Archimedean. Then \( y \) has a representing measure on \( K \)

iff

\[ L_y(\sigma) \geq 0, \quad L_y(g_j \sigma) \geq 0, \quad \forall \sigma \in \Sigma[x]. \]
The K Moment Problem

Theorem [Lasserre 11]

Let $K$ be compact and $\mu \in \mathcal{M}_+$ be arbitrary fixed with moments

$$y_\alpha = \int_K x^{\alpha} \mu(dx), \quad \forall \alpha \in \mathbb{N}^n.$$ 

Then a polynomial $p$ is nonnegative on $K$ iff

$$\mathbf{M}_k(p^y) \succeq 0, \quad \forall k \in \mathbb{N}.$$
Hierarchy of Outer Approximations

Cone of nonnegative polynomials $\mathcal{C}(K)_d \subset \mathbb{R}_d[x]$

- Each entry of the matrix $M_k(p \ y)$ is linear in the coefficients of $p$

- The set $\Delta_k := \{p \in \mathbb{R}_d[x] : M_k(p \ y) \succeq 0\}$ is closed and convex, called spectrahedron

- Nested outer approximations:
  $\Delta_0 \supset \Delta_1 \cdots \supset \Delta_k \cdots \supset \mathcal{C}(K)_d$
Hierarchy of Outer Approximations

Hierarchy of upper bounds for \( p^* := \inf_{x \in K} p(x) \)

**Theorem [Lasserre 11]**

Let \( K \) be closed and \( \mu \in \mathcal{M}_+(K) \) be arbitrary fixed with moments \((y_\alpha), \alpha \in \mathbb{N}^n\). Consider the hierarchy of SDP:

\[
u_k := \max\{\lambda : M_k((p - \lambda) y) \succeq 0\} = \max\{\lambda : \lambda M_k(y) \preceq M_k(p y)\}.
\]

Then \( \nu_k \downarrow p^* \), as \( k \to \infty \).
Hierarchy of Outer Approximations

Hierarchy of upper bounds for \( p^* := \inf_{x \in K} p(x) \)

- Computing \( u_k \): **Generalized Eigenvalue Problem** for the pair \([M_k(p y), M_k(y)]\).

- Index the matrices in the basis of orthonormal polynomials w.r.t. \( \mu \):

\[
u_k = \max\{ \lambda : \lambda I \preceq M_k(p y) \} = \lambda_{\min}(M_k(p y)),
\]

a standard **Minimal Eigenvalue Problem**.
Hierarchy of Outer Approximations

Primal-Dual Moment-SOS

(Primal) \[ u'_k := \min \int_K p(x) \sigma(x) d\mu(x) \geq u_k := \max \lambda \]

s.t. \[ \int_K \sigma(x) d\mu(x) = 1, \]
\[ \sigma \in \Sigma_k[x] \]

(Dual) \[ \lambda \mathbf{M}_k(y) \preceq \mathbf{M}_k(p\ y) . \]

Then \( u'_k, u_k \to p^* \), as \( k \to \infty \).
Interpretation of the Primal Problem

\[ u_k^* := \min_{\nu} \left\{ \int_K p \sigma \, d\mu : \nu(K) = 1, \nu(\mathbb{R}^n \setminus K) = 0, \sigma \in \Sigma_k[x] \right\} \]

The measure \( \nu \) approximates the \textsc{Dirac} measure \( \delta_{x=x^*} \) at a global minimizer \( x^* \in K \) and

1. \( \nu \) is absolutely continuous w.r.t. \( \mu \)

2. the density of \( \nu \) is \( \sigma \in \Sigma_k[x] \)
Example with uniform probability measure

Polynomial $p := 0.375 - 5x + 21x^2 - 32x^3 + 16x^4$ on $K := [0, 1]$
Example with uniform probability measure

Probability Density $\sigma \in \Sigma_{10}[x]$
Extension to the non-compact case

In this case, the measure $\mu$

- needs to satisfy CARLEMAN-type sufficient condition to limit the growth of its moments ($y_{\alpha}$):

$$\sum_{t=1}^{\infty} L_y (x_i^{2t})^{-1/(2t)} = +\infty, \quad i = 1, \ldots, n.$$ 

- e.g. $d\mu(x) := \exp(-\|x\|_2^2/2) d\mu_0$, with $\mu_0 \in M_+(K)$ finite
Extension to the non-compact case

\[ d\mu(x) := \exp(-\|x\|_2^2/2) \, dx, \text{ for } K = \mathbb{R}^n \]

\[ d\mu(x) := \exp(-\sum_{i=1}^n x_i) \, dx, \text{ for } K = \mathbb{R}_+^n \]

\[ d\mu(x) := dx, \text{ when } K \text{ is a box, simplex} \]
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New Applications of Moment-SOS Hierarchies

Semialgebraic Maxplus Optimization
Formal Nonlinear Optimization
Pareto Curves
Polynomial Images of Semialgebraic Sets
Program Analysis with Polynomial Templates

Conclusion
Given $K$ a compact set and $f$ a transcendental function, bound $f^* = \inf_{x \in K} f(x)$ and prove $f^* \geq 0$

- $f$ is underestimated by a semialgebraic function $f_{sa}$

- Reduce the problem $f_{sa}^* := \inf_{x \in K} f_{sa}(x)$ to a polynomial optimization problem (POP)
Approximation theory: Chebyshev/Taylor models

- mandatory for non-polynomial problems

- hard to combine with Sum-of-Squares techniques (degree of approximation)
Maxplus Approximation

Initially introduced to solve Optimal Control Problems [Fleming-McEneaney 00]

Curse of dimensionality reduction [McEaneney Kluberg, Gaubert-McEneaney-Qu 11, Qu 13]. Allowed to solve instances of dim up to 15 (inaccessible by grid methods)

In our context: approximate transcendental functions
Maxplus Approximation

Definition

Let $\gamma \geq 0$. A function $\phi : \mathbb{R}^n \to \mathbb{R}$ is said to be $\gamma$-semiconvex if the function $x \mapsto \phi(x) + \frac{\gamma}{2} \|x\|_2^2$ is convex.
Nonlinear Function Representation

Exact parsimonious maxplus representations

$y$

$a$
Nonlinear Function Representation

Exact parsimonious maxplus representations

$y$ vs. $a$
Nonlinear Function Representation

Abstract syntax tree representations of multivariate transcendental functions:

- leaves are semialgebraic functions of $A$
- nodes are univariate functions of $D$ or binary operations
Nonlinear Function Representation

For the “Simple” Example from Flyspeck:
Maxplus Optimization Algorithm

First iteration:

1 control point \( \{a_1\} \): \( m_1 = -4.7 \times 10^{-3} < 0 \)
Maxplus Optimization Algorithm

Second iteration:

2 control points \( \{a_1, a_2\} \): \( m_2 = -6.1 \times 10^{-5} < 0 \)
Maxplus Optimization Algorithm

Third iteration:

3 control points \( \{a_1, a_2, a_3\} \): \( m_3 = 4.1 \times 10^{-6} > 0 \)

OK!
Contributions

Published:


In revision:

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Another look at Nonnegativity

**New Applications of Moment-SOS Hierarchies**
- Semialgebraic Maxplus Optimization
- Formal Nonlinear Optimization
- Pareto Curves
- Polynomial Images of Semialgebraic Sets
- Program Analysis with Polynomial Templates

Conclusion
The General “Formal Framework”

- We check the correctness of SOS certificates for POP
- We build certificates to prove interval bounds for semialgebraic functions
- We bound formally transcendental functions with semialgebraic approximations
Formal SOS bounds

When \( q \in \mathbb{Q}(K) \), \( \sigma_0, \ldots, \sigma_m \) is a positivity certificate for \( q \)

Check symbolic polynomial equalities \( q = q' \) in Coq

- Existing tactic **ring** [Grégoire-Mahboubi 05]

- Polynomials coefficients: arbitrary-size rationals \( \text{bigQ} \) [Grégoire-Théry 06]

- Much simpler to verify certificates using *sceptical approach*

- Extends also to semialgebraic functions
Formal Nonlinear Optimization

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- Comparable with Taylor interval methods in HOL-LIGHT [Hales-Solovyev 13]

- Bottleneck of informal optimizer is SOS solver

- 22 times slower! \(\Rightarrow\) Current bottleneck is to check polynomial equalities
Contribution

For more details on the formal side:

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New Applications of Moment-SOS Hierarchies
  Semialgebraic Maxplus Optimization
  Formal Nonlinear Optimization
  Pareto Curves
  Polynomial Images of Semialgebraic Sets
  Program Analysis with Polynomial Templates

Conclusion
Let $f_1, f_2 \in \mathbb{R}_d[x]$ two conflicting criteria

Let $S := \{x \in \mathbb{R}^n : g_1(x) \geq 0, \ldots, g_m(x) \geq 0\}$ a semialgebraic set

$$\begin{align*}
(P) \quad \left\{ \min_{x \in S} (f_1(x) f_2(x))^\top \right\}
\end{align*}$$

Assumption

The image space $\mathbb{R}^2$ is partially ordered in a natural way ($\mathbb{R}_+^2$ is the ordering cone).
Bicriteria Optimization Problems

\( g_1 := -(x_1 - 2)^3/2 - x_2 + 2.5 \),
\( g_2 := -x_1 - x_2 + 8(-x_1 + x_2 + 0.65)^2 + 3.85 \),
\( S := \{ x \in \mathbb{R}^2 : g_1(x) \geq 0, g_2(x) \geq 0 \} \).

\( f_1 := (x_1 + x_2 - 7.5)^2/4 + (-x_1 + x_2 + 3)^2 \),
\( f_2 := (x_1 - 1)^2/4 + (x_2 - 4)^2/4 \).
Parametric sublevel set approximation

- Inspired by previous research on multiobjective linear optimization [Gorissen-den Hertog 12]

- Workaround: reduce $P$ to a parametric POP

\[
(P_{\lambda}) : \quad f^*(\lambda) := \min_{x \in S} \left\{ f_2(x) : f_1(x) \leq \lambda \right\},
\]
A Hierarchy of Polynomial underestimators

Moment-SOS approach [Lasserre 10]:

\[
(D_d) \left\{ \begin{array}{l}
\max_{q \in \mathbb{R}_{2d}[\lambda]} \sum_{k=0}^{2d} \frac{q_k}{1+k} \\
\text{s.t. } f_2(x) - q(\lambda) \in Q_{2d}(K).
\end{array} \right.
\]

- The hierarchy \((D_d)\) provides a sequence \((q_d)\) of polynomial underestimators of \(f^*(\lambda)\).

- \(\lim_{d \to \infty} \int_0^1 (f^*(\lambda) - q_d(\lambda))d\lambda = 0\)
A Hierarchy of Polynomial underestimators

Degree 4
A Hierarchy of Polynomial underestimators

Degree 6
A Hierarchy of Polynomial underestimators

Degree 8
Contributions

- Numerical schemes that avoid computing finitely many points.

- Pareto curve approximation with polynomials, convergence guarantees in $L_1$-norm

Introduction

Moment-SOS relaxations

Another look at Nonnegativity

New Applications of Moment-SOS Hierarchies
  Semialgebraic Maxplus Optimization
  Formal Nonlinear Optimization
  Pareto Curves
  Polynomial Images of Semialgebraic Sets
  Program Analysis with Polynomial Templates

Conclusion
Approximation of sets defined with “∃”

Let $\mathcal{B} \subset \mathbb{R}^2$ be the unit ball and assume that $f(S) \subset \mathcal{B}$.

- Another point of view:

$$f(S) = \{ y \in \mathcal{B} : \exists x \in S \text{ s.t. } h(x, y) \leq 0 \} ,$$

with

$$h(x, y) := \| y - f(x) \|_2^2 = (y_1 - f_1(x))^2 + (y_2 - f_2(x))^2 .$$

- Approximate $f(S)$ as closely as desired by a sequence of sets of the form :

$$\Theta_d := \{ y \in \mathcal{B} : q_d(y) \leq 0 \} ,$$

for some polynomials $q_d \in \mathbb{R}_{2d}[y]$. 
A Hierarchy of Outer approximations for $f(S)$

$$f(x) := \frac{(x_1 + x_1 x_2, x_2 - x_1^3)}{2}$$

Degree 2
A Hierarchy of Outer approximations for $f(S)$

$$f(x) := (x_1 + x_1x_2, x_2 - x_1^3)/2$$

Degree 4
A Hierarchy of Outer approximations for $f(S)$

$$f(x) := (x_1 + x_1 x_2, x_2 - x_1^3)/2$$

Degree 6
A Hierarchy of Outer approximations for $f(S)$

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Degree 8
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One-loop with Conditional Branching

- $r, s, T^i, T^e \in \mathbb{R}[x]$

- $x_0 \in X_0$, with $X_0$ semialgebraic set

```plaintext
x = x_0;
while (r(x) \leq 0) {
    if (s(x) \leq 0) {
        x = T^i(x);
    }
    else {
        x = T^e(x);
    }
}
```
Bounding Template using SOS

Sufficient condition to get bounding inductive invariant:

\[ \alpha := \min_{q \in \mathbb{R}[x]} \sup_{x \in X_0} q(x) \]

s.t. \[ q - q \circ T^i \geq 0 , \]
\[ q - q \circ T^e \geq 0 , \]
\[ q - \| \cdot \|_2^2 \geq 0 . \]

- Nontrivial correlations via polynomial templates \( q(x) \)
- \( \{ x : q(x) \leq \alpha \} \supset \bigcup_{k \in \mathbb{N}} X_k \)
Bounds for $\bigcup_{k \in \mathbb{N}} X_k$

$X_0 := [0.9, 1.1] \times [0, 0.2]$ \quad $r(x) := 1$ \quad $s(x) := 1 - x_1^2 - x_2^2$

$T^i(x) := (x_1^2 + x_2^3, x_1^3 + x_2^2)$ \quad $T^e(x) := \left( \frac{1}{2}x_1^2 + \frac{2}{5}x_2^3, -\frac{3}{5}x_3^3 + \frac{3}{10}x_2^2 \right)$
Bounds for $\bigcup_{k \in \mathbb{N}} X_k$

$$X_0 := [0.9, 1.1] \times [0, 0.2] \quad r(x) := 1 \quad s(x) := 1 - x_1^2 - x_2^2$$

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Degree 10
Contribution

For more details:

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Conclusion
Conclusion

With **MOMENT-SOS HIERARCHIES**, you can

- Optimize nonlinear (transcendental) functions
- Approximate Pareto Curves, images and projections of semialgebraic sets
- Analyze programs
Conclusion

Further research:

- Alternative polynomial bounds using geometric programming (T. de Wolff, S. Iliman)

- Mixed LP/SOS certificates (trade-off CPU/precision)
Thank you for your attention!

http://homepages.laas.fr/vmagron/