Solutions Exercise 2: CTL Modelchecking

Note that I use ‘!’ for ‘not’, ‘∨’ for ‘or’, and ‘∧’ for ‘and’ for simplicity.

1) (a) \(AG(AF \ p \rightarrow AF \ q) = !EF !\ (AF \ p \rightarrow AF \ q) = !EF !(AF \ p \lor AF \ q) = \)
\(!EF(\ AF \ p \land !AF \ q) = !EF (!EG!p \land EG!q) = !E \ true \ U (!EG!p \land EG!q) =!

(b) \(AF(p \lor AG \ q) = !EG!(p \lor AG \ q) = !EG(!p \land !AG \ q) = !EG(!p \land EF!q) = \)
\(!!(EG(!p \land E \ true \ U !q))

2) \(!E \ true \ U (!E \ true \ U p))

3) (a) Proof: \(EF(p \lor q) = EF \ p \lor EF \ q \) (and \(AG \ p \land q = AG \ p \land AG \ q\))
Forward:
Assume \(K \models EF(p \lor q)\), then \(\exists\) a path \(s_0, s_1, \ldots\) such that \(\exists \ i \geq 0\) with \(s_i \models p \lor q\). It follows that either \(s_i \models p\) or \(s_i \models q\) (or both). So \(\exists\) a path \(t_0, t_1, \ldots\) such that \(t_i \models p\) or \(\exists\) a path \(t_0, t_1, \ldots\) such that \(t_i \models q\) and it follows that \(K \models EF(p) \lor EF(q)\)

(b), (c), and (d) are not true (see the counterexamples below).
4) Proof: \( \text{EG } p = \forall Z. p \land \text{EX } Z \)

\( \text{EG } p \) is a fix-point because
\( \text{EG } p = p \land \text{EX } \text{EG } p \) is true.

\( \text{EG } p \) is the greatest fix-point, if forall \( Z \) s.t. \( Z = p \land \text{EX } Z \), we have that \( Z \leq \text{EG } p \)

From \( Z = p \land \text{EX } Z \) is follows that \( Z \leq p \) and \( Z \leq \text{EX } Z \).
So for all \( s \in Z \), \( s \models p \) and \( s \) has a successor in \( Z \), so \( s \) has an infinite path along which \( p \) holds, and so 
\( s \models \text{EG } p \) (and therefore \( s \in \text{EG } p \) and \( Z \leq \text{EG } p \), since \( s \) was chosen arbitrarily).

Furthermore, from the exists of a greatest fix-point and since \( Z \leq \text{EG } p \) for any arbitrary fix-point, it follows that \( \text{EG } p \) is the greatest fix-point.

5) To proof \( \text{EG } p = \text{EG } \text{EG } p \).
Note that from fix-point theory, we know that if \( f \) is a monotonic function over a finite lattice \( L \) and \( x \) is an element of \( L \), then \( x \geq \forall Z. f = \forall Z. x \land f \) holds. (1)

We note that \( \text{EG } p \leq p \), and so \( \text{EG } p = p \land \text{EG } p \).
Furthermore, due to the previous exercise and (1) we know that
\( \text{EG } \text{EG } p = \forall Z. (\text{EG } p) \land \text{EX } Z = \forall Z. p \land (\text{EG } p) \land \text{EX } Z = \forall Z. p \land \text{EX } Z = \text{EG } p \).

6) \ E p U q
\((Z0={})
\((Z1={c} \lor \{b,d\} \land \text{EX}{} = \{c\})
\((Z2={c} \lor \{b,d\} \land \text{EX}\{c\} = \{c\} \lor \{b,d\} \land \{b\} = \{b,c\})
\((Z3={c} \lor \{b,d\} \land \text{EX}\{b,c\} =\{c\} \lor \{b,d\} \land \{a,b\} = \{b,c\} \text{ fix-point}

Bonus)
(a) \( Z_i = S \) trivial
(b) \( Z_i = p \lor q \), all states not in \( p \lor q \) cannot be in \( E p U q \) since either \( p \) or \( q \) have to be satisfied along a path satisfying \( E p U q \)
(c) \( Z_i \geq p \), we know that \( Z_i \leq p \lor q \) and since \( Z_1={q} \) it follows that if \( Z_i\geq p \), then \( Z_i = p \lor q \).