Terminology

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Two-player games between Player 0 and 1

An infinite game $\langle G, \phi \rangle$ consists of

- a game graph G and
- a winning condition ϕ.

G defines the "playground", in which the two players compete.

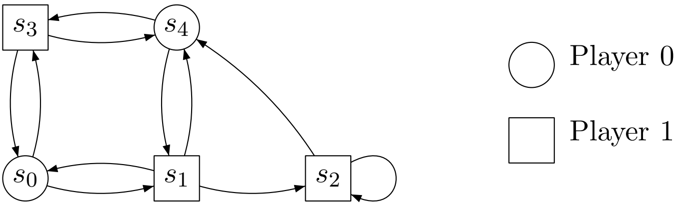
ϕ defines which plays are won by Player 0.

If a play does not satisfy ϕ, then Player 1 wins on this play.

Game Graphs

A game graph is a tuple $G = \langle S, S_0, T \rangle$ where:

- S is a finite set of states,
- $S_0 \subseteq S$ is the set of Player-0 states ($S_1 = S \setminus S_0$ are the Player-1 states),
- $T \subseteq S \times S$ is a transition relation. We assume that each state has at least one successor.



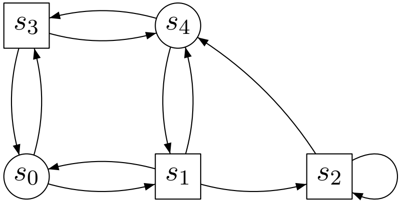
Plays

A play is an infinite sequence of states $\rho = s_0 s_1 s_2 \cdots \in S^\omega$ such that for all $i \geq 0$ $\langle s_i, s_{i+1} \rangle \in T$.

It starts in s_0 and it is built up as follows:

If $s_i \in S_0$, then Player 0 chooses an edge starting in s_i, otherwise Player 1 picks such an edge.

Intuitively, a token is moved from state to state via edges: From S_0-states Player 0 moves the token, from S_1-states Player 1 moves the token.



Winning Condition

The winning condition describes the plays won by Player 0.

A winning condition or winning objective ϕ is a subset of plays, i.e., $\phi \subseteq S^\omega$.

We use logical conditions (e.g., LTL formulas) or automata theoretic acceptance conditions to describe ϕ.

Example:

- always eventually s for some state $s \in S$
- All plays that stay within a safe region $F \subseteq S$ are in ϕ.
- Given a priority function $p : S \to \{0, 1, \ldots, d\}$, all plays in which the smallest priority visited is even.

Games are named after their winning condition, e.g., Safety game, Reachability game, LTL game, Parity game,...

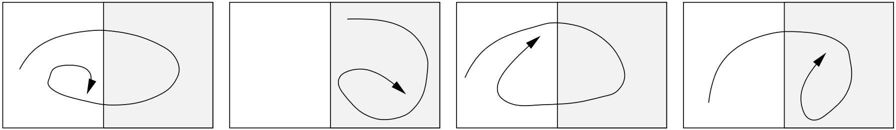
Types of Games

Given a play ρ, we define

- $\mathrm{Occ}(\rho) = \{s \in S \mid \exists i \geq 0 : s_i = s\}$
- $\mathrm{Inf}(\rho) = \{s \in S \mid \forall i \geq 0 \exists j > i : s_j = s\}$

Given a set $F \subseteq S$,

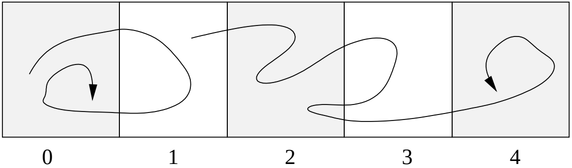
Reachability Game	$\phi = \{\rho \in S^\omega \mid \mathrm{Occ}(\rho) \cap F \neq \emptyset\}$
Safety Game	$\phi = \{\rho \in S^\omega \mid \mathrm{Occ}(\rho) \subseteq F\}$
Büchi Game	$\phi = \{\rho \in S^\omega \mid \mathrm{Inf}(\rho) \cap F \neq \emptyset\}$
Co-Büchi Game	$\phi = \{\rho \in S^\omega \mid \mathrm{Inf}(\rho) \subseteq F\}$



Types of Games

Given a priority function $p : S \to \{0, 1, \ldots, d\}$ or an LTL formula φ

Weak-Parity Game	$\phi = \{\rho \in S^\omega \mid \min_{s \in \mathrm{Occ}(\rho)} p(s) \text{ is even}\}$
Parity Game	$\phi = \{\rho \in S^\omega \mid \min_{s \in \mathrm{Inf}(\rho)} p(s) \text{ is even}\}$
LTL Game	$\phi = \{\rho \in S^\omega \mid \rho \models \varphi\}$



We will refer to the type of a game and give F, p, or φ instead of defining ϕ.

Strategies

A strategy for Player 0 from state s is a (partial) function

$$f : S^* S_0 \to S$$

specifying for any sequence of states $s_0, s_1, \ldots s_k$ with $s_0 = s$ and $s_k \in S_0$ a successor state s_j such that $(s_k, s_j) \in T$.
A play $\rho = s_0 s_1 \ldots$ is compatible with strategy f if for all $s_i \in S_0$ we have that $s_{i+1} = f(s_0 s_1 \ldots s_i)$.
(Definitions for Player 1 are analogous.)

Given strategies f and g from s for Player 0 and 1, respectively. We denote by $G_{f,g}$ the (unique) play that is compatible with f and g.

Winning Strategies and Regions

Given a game (G, ϕ) with $G = (S, S_0, E)$, a strategy f for Player 0 from s is called a winning strategy if for all Player-1 strategies g from s, if $G_{f,g} \in \phi$ holds. Analogously, a Player-1 strategy g is winning if for all Player-0 strategies f, $G_{f,g} \notin \phi$ holds.

Player 0 (resp. 1) wins from s if s/he has a winning strategy from s.

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The winning regions of Player 0 and 1 are the sets

$$W_0 = \{s \in S \mid \text{Player 0 wins from } s\}$$

$$W_1 = \{s \in S \mid \text{Player 1 wins from } s\}$$

Note each state s belongs at most to W_0 or W_1. Otherwise pick winning strategies f and g from s for Player 0 and 1, respectively, then $G_{f,g} \in \phi$ and $G_{f,g} \notin \phi$: Contradiction.

Questions About Games

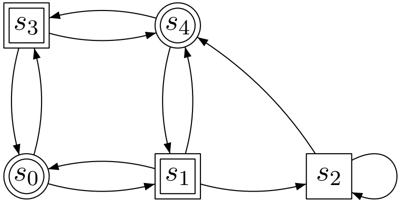
Solve a game (G, ϕ) with $G = (S, S_0, T)$:

1. Decide for each state $s \in S$ if $s \in W_0$.
2. If yes, construct a suitable winning strategy from s.

Further interesting question:

- Optimize construction of winning strategy (e.g., time complexity)
- Optimize parameters of winning strategy (e.g., size of memory)

Example



Safety game (G, F) with $F = \{s_0, s_1, s_3, s_4\}$, i.e., $\mathrm{Occ}(\rho) \subseteq F$

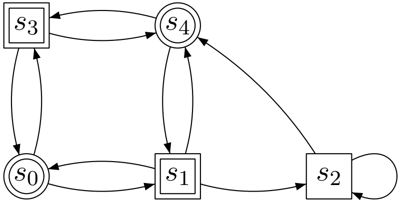
A winning strategy for Player 0 (from state s_0, s_3, and s_4):

- From s_0 choose s_3 and from s_4 choose s_3

A winning strategy for Player 1 (from state s_1 and s_2):

- From s_1 choose s_2, from s_2 choose s_4, and from s_3 choose s_4

Example



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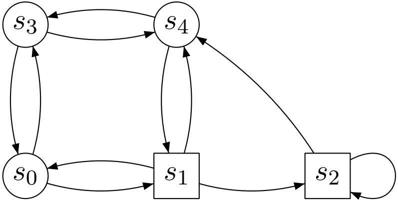
- From s_1 choose s_2, from s_2 choose s_4, and from s_3 choose s_4

$W_0 = \{s_0, s_3, s_4\}$, $W_1 = \{s_1, s_2\}$

Another Example



LTL game (G, φ) with $\varphi = \mathsf{eventually}(s_0) \land \mathsf{eventually}(s_4)$

Winning strategy for Player 0 from s_0:

- From s_0 to s_3, from s_3 to s_4, and from s_4 to s_1.

Note: this strategy is not winning from s_3 or s_4.

Winning strategy for Player 0 from s_3:

- From s_0 to s_3, from s_4 to s_3, and from s_3 to s_0 on first visit, otherwise to s_4.

Determinacy

Recall: the winning regions are disjoint, i.e., $W_0 \cap W_1 = \emptyset$

Question: Is every state winning for some player?

A game (G, ϕ) with $G = (S, S_0, E)$ is called determined if $W_0 \cup W_1 = S$ holds.

Remarks:

1. We will show that all automata theoretic games we consider here are determined.
2. There are games which are not determined (e.g., concurrent games: even/odd sum, paper-rock-scissors)

Strategy Types

In general, a strategy is a function $f : S^+ \to S$.

(Note that sometimes we might define f only partially.)

1. Computable or recursive strategies: f is computable
2. Finite-state strategies: f is computable with a finite-state automaton meaning that f has bounded information about the past (history).
3. Memoryless or positional strategies: f only depends on the current state of the game (no knowledge about history of play)

Positional Strategies

Given a game (G, ϕ) with $G = (S, S_0, E)$, a strategy $f : S^+ \to S$ is called positional or memoryless if for all words $w, w' \in S^+$ with $w = s_0 \dots s_n$ and $w' = s'_0 \dots s'_m$ such that $s_n = s'_m$, $f(w) = f(w')$ holds.

A positional strategy for Player 0 is representable as

1. a function $f : S_0 \to S$
2. a set of edges containing for every Player-0 state s exactly one edge starting in s (and for every Player-1 state s' all edges starting in s')

Finite-state Strategies

A strategy automaton over a game graph $G = (S, S_0, E)$ is a finite-state machine $A = (M, m_0, \delta, \lambda)$ (Mealy machine) with input and output alphabet S, where

- M is a finite set of states (called memory),
- $m_0 \in M$ is an initial state (the initial memory content),
- $\delta : M \times S \to M$ is a transition function (the memory update fct),
- $\lambda : M \times S \to S$ is a labeling function (called the choice function).

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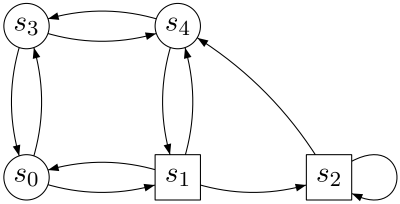
The strategy for Player 0 computed by A is the function

$$f_A(s_0 \dots s_k) := \lambda(\delta(m_0, s_0 \dots s_{k-1}), s_k) \text{ with } s_k \in S_0$$

and the usual extension of δ to words: $\delta(m_0, \epsilon) = m_0$ and $\delta(m_0, s_0 ... s_k) = \delta(\delta(m_0, s_0 ... s_{k-1}), s_k)$. Any strategy f, such that there exists an A with $f_A = f$, is called finite-state strategy.

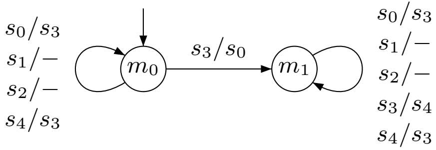
Recall Example



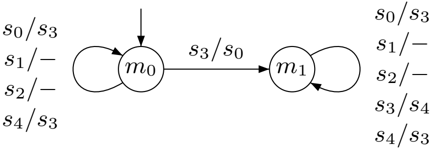
Objective: visit s_0 and s_4, i.e, $\{s_0, s_4\} \subseteq \mathrm{Occ}(\rho)$

Winning strategy for Player 0 from s_0, s_3 and s_4:

- From s_0 to s_3, from s_4 to s_3, and from s_3 to s_0 on first visit, otherwise to s_4.



Extended Game



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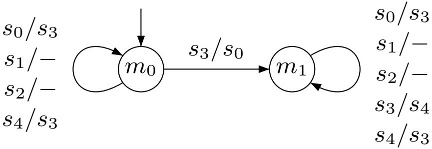

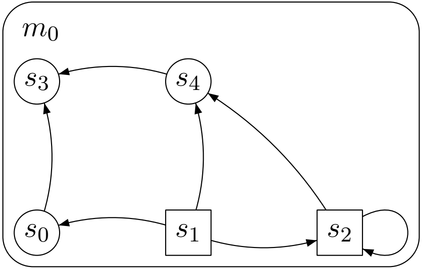

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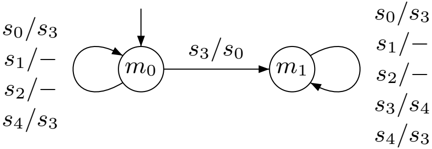

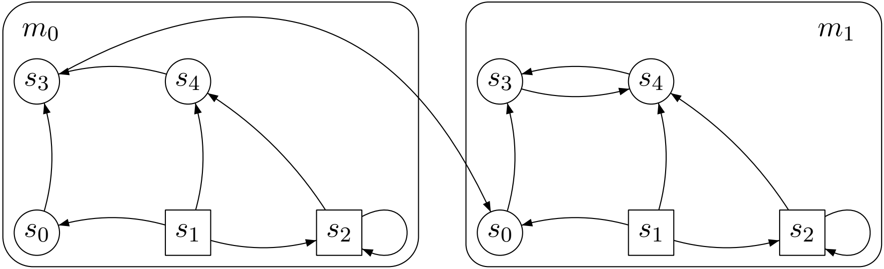

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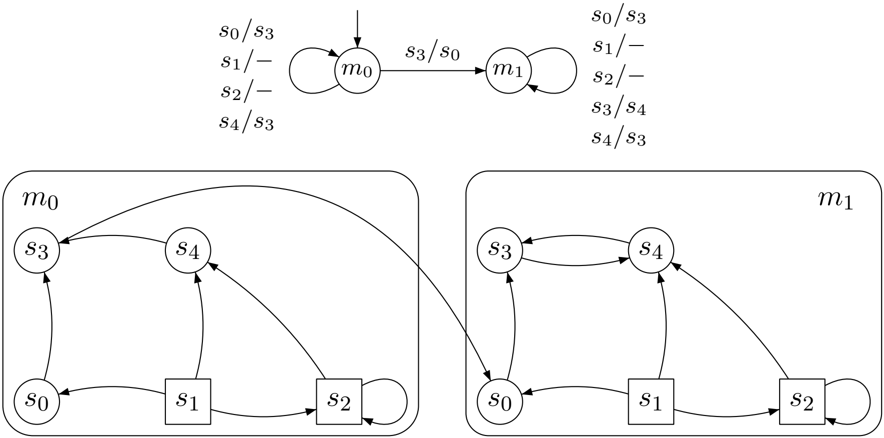

Note: the strategy in the extended grame graph is memoryless.