Synthesis for Regular Specifications over Unbounded Domains

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***Abstract*—Synthesis from specifications is a promising method of obtaining systems that are correct by construction. Previous work includes synthesis of reactive finite-state systems from linear temporal logic and its fragments. Further recent work focuses on a different application area by doing functional synthesis over unbounded domains, using a modified Presburger arithmetic quantifier elimination algorithm. We present new algorithms for functional synthesis over unbounded domains based on automata-theoretic methods, with advantages in the expressive power and in the efficiency of synthesized code.**

Our approach synthesizes functions that meet given regular specifications defined over unbounded sequences of input and output bits. Thanks to the translation from weak monadic second-order logic to automata, this approach supports full Presburger arithmetic as well as bitwise operations on arbitrary length integers. The presence of quantifiers enables finding solutions that optimize a given criterion. Unlike synthesis of reactive systems, our notion of realizability allows functions that require examining the entire input to compute the output. Regardless of the complexity of the specification, our algorithm synthesizes linear-time functions that read the input and directly produce the output. We also describe a technique to synthesize functions with bounded lookahead when possible, which is appropriate for streaming implementations. We implemented our synthesis algorithm and show that it synthesizes efficient functions on a number of examples.

I. Introduction

Automated synthesis of systems from specifications is a promising method to increase development productivity. Automata-based methods have been the core technique for reactive synthesis of finite-state systems [1], [2], [3]. In this paper, we show that automata-based techniques can also be used to perform functional synthesis over unbounded data domains. In functional synthesis, we are interested in synthesizing functions that accept a tuple of input values (ranging over possibly unbounded domains), and generate a tuple of output values that satisfy a given specification. Recently, researchers have proposed [4] a technique for functional synthesis based on quantifier elimination of Presburger arithmetic.

In the previous approach, the functions generated by quantifier elimination can be inefficient if the input contains inequalities, possibly performing search over a very large space of integer tuples. Furthermore, this approach handles disjunctions by a transformation into disjunctive normal form. In general, the efficiency of the function generated by quantifier elimination depends not only on the semantics of the specification, but also on its syntactic form. Finally, the specification language accepts integer arithmetic but not bitwise constructs on integers.

In this paper we present a synthesis procedure that is guaranteed to produce an efficient function, which finds a solution of a given constraint on unbounded integers in time linear in the combined length of input and the shortest output, represented in binary. Moreover, our specification language supports Presburger arithmetic operations, but also bitwise operations and quantifiers. We achieve this expressive power by representing integers as sets in weak monadic second-order logic of one successor (WS1S) which is known to be more expressive than pure Presburger arithmetic [5], [6]. We use an off-the-shelf procedure, MONA, to obtain a deterministic automaton that represents a given WS1S specification.

As our central result, we show how to convert an automaton recognizing the input/output relation into a function that reads the input sequence, and produces an output sequence that satisfies the input/output relation. Consequently, we obtain functions that are guaranteed to run in linear-time on arbitrarily large integers (represented as bit sequences). What is important is that the running time of the synthesized functions is independent of the automaton size. These properties are a consequence of our algorithm, and we have also experimentally verified them on a number of examples. Finally, we describe a technique to identify a prefix of the output before reading the entire input, which enables the use of our method to synthesize function implementations with bounded memory. We therefore believe we have identified a promising technique to generate implementations with predictable run times from specifications in an expressive logic.

II. Examples

This section presents several examples to illustrate the expressive power of our synthesis implementation, as well as the efficiency of the synthesized systems.

A. Parity Bit Computation

The goal of our first example is to illustrate the form of the functions produced by our synthesizer. For a non-negative integer x, let $x[k]$ denote the k-th least significant bit in the binary representation of x. (We write the binary digits starting with the least significant one on the left, so $\overline{11001}_2$ is a binary representation of 19.) Our first specification states that the first output bit, $y[0]$ indicates the parity of the number of one-bits in the input (Figure 2): $y[0] = |\{k \mid x[k] = 1\}| \% 2$.

Consequently, the synthesized function must examine the entire input before emitting the first bit of the output.

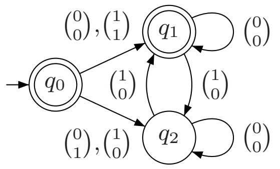

Fig. 1. Automaton A for parity specification between x and y

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x: 0 1 1 0 1
y: 1 0 0 0 0
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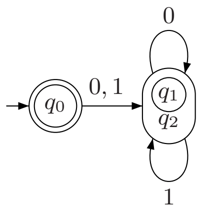
Fig. 2. Input x and output y satisfying parity specification



Transition	State	τ
$\{q_0\} \xrightarrow{0} \{q_1, q_2\}$	q_1	$(q_0, 0)$
$\{q_0\} \xrightarrow{0} \{q_1, q_2\}$	q_2	$(q_0, 1)$
$\{q_0\} \xrightarrow{1} \{q_1, q_2\}$	q_1	$(q_0, 1)$
$\{q_0\} \xrightarrow{1} \{q_1, q_2\}$	q_2	$(q_0, 0)$
$\{q_1, q_2\} \xrightarrow{0} \{q_1, q_2\}$	q_1	$(q_1, 0)$
$\{q_1, q_2\} \xrightarrow{0} \{q_1, q_2\}$	q_2	$(q_2, 0)$
$\{q_1, q_2\} \xrightarrow{1} \{q_1, q_2\}$	q_1	$(q_2, 0)$
$\{q_1, q_2\} \xrightarrow{1} \{q_1, q_2\}$	q_2	$(q_1, 0)$

Fig. 3. Automaton A' for computing parity y of input x

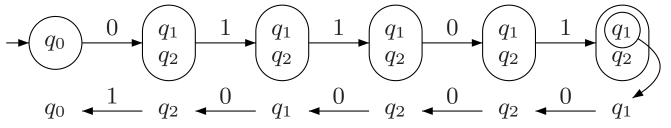


Fig. 4. Running synthesized function on input shown in Fig. 2

One way to specify this computation is as follows. Let n_{max} have the property $\forall k > n_{max}.x[k] = 0$. Then introduce an auxiliary sequence of bits z such that

$$z[n] = \ |\{k \leq n \mid x[k] = 1\}|\%2$$

for all $n \leq n_{max}$ (by defining $z[k+1]$ as xor of $z[k]$ and $x[k+1]$), and let $y[0]$ to be $z[n_{max}]$.

Figure 1 shows the generated automaton A for this specification, accepting the words $\binom{x[0]}{y[0]}\binom{x[1]}{y[1]}\ldots\binom{x[n]}{y[n]}$ which satisfy the given relation between x and y. After applying our construction to compute a function from x to y, we obtain the input-determinstic automaton A' shown on the left of Figure 3, augmented with two labeling functions τ and ϕ. The automaton is the result of first projecting out the part of A' labels corresponding to the output, then applying the subset construction. Therefore, the labels in A' correspond to input bits, and the states are sets of states of the automaton A. Function τ tells us how to move backwards within a run of A' to construct an accepting run of the underlying automaton A; it thus recovers information lost in applying the projection to A. Finally, function ϕ tells us for every accepting state in A' at which state of A to start the backward reconstruction. The table on the right of Figure 3 shows τ for A': it maps every transition $S \xrightarrow{\sigma_i} S'$ of A' and every state $q' \in S'$ into a predecessor state $q \in S$, and a matching output value σ_o, such that $(q, (\sigma_i \cup \sigma_o), q')$ is a transition in the automaton A. We indicate function ϕ in A' by additional circles around individual states, e.g., $\phi(\{q_1, q_2\}) = q_1$. Figure 4 shows the run of A' on the input 01101. The synthesized function first runs the deterministic automaton A' (the upper part of Figure 4, ending in state $\{q_1, q_2\}$). The synthesized function then picks a state q according to ϕ (the state q_1 in case of our example), and runs backwards according to τ while computing the output bits. The lower part of Figure 4 shows the backward computation following τ defined in Figure 3; the backward run generates the bits 10000 of the output.

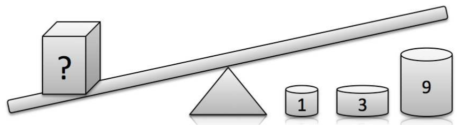


Fig. 5. Beam balance with three weights

B. Synthesizing Specialized Constraint Solvers

Our next examples illustrate a range of problems to which our synthesis technique applies. Consider first the beam balance (scale) depicted in Figure 5. We are interested in a function that tells us, for any object on the left-side of the beam, how to arrange the weights to balance the beam.[1] We are given three weights, with 1, 3, and 9kg, respectively. We use the variable w for the weight of the unknown object. For each available weight i, we use two variables l_i to indicate whether the weight is placed on the left side and r_i to indicate it is placed on the right side of the beam. We obtain the constraint:

$$w + l_1 + 3l_3 + 9l_9 = r_1 + 3r_3 + 9r_9. \tag{1}$$

Because each weight can only be use at most once, we require that the solution also respects the following three constraints

$$l_1 + r_1 \leq 1,\ l_3 + r_3 \leq 1,\ l_9 + r_9 \leq 1. \tag{2}$$

When we give these four constraints to our tool, it compiles them into a function. The function accepts arbitrary input values and returns corresponding output values, performing computation in time linear in the number of bits in the input. E.g., if the object weights 11kg, then the program tells us that we should use Weight 1 on the left and Weight 3 and 9 on the right side to balance the beam. It is easy to verify that this response is correct by insertion into Equation 1 leading to $11 + 1 \cdot 1 = 3 \cdot 1 + 9 \cdot 1$. When asked for $w = 15$, the program correctly responds with "There is no output for your input."

C. Modifying Example to Minimize Output

Next, we consider a modified version of the balance example to show that neither inputs nor outputs need to be bounded. It also shows how to specify a function that minimizes the output. In the previous example, we could only balance objects up to 13kg because only one copy of each weight was available. Assume we want to balance arbitrary heavy objects with the minimal number of balance weights of 1, 3, and 9kg. We keep the constraint from Eqn. 1 and replace the constraints in Eqn. 2 by a constraint that asks for a minimal solution:

$$\forall l'_1, l'_3, l'_9, r'_1, r'_3, r'_9.\quad \mathsf{balance}(w, l'_1, l'_3, l'_9, r'_1, r'_3, r'_9) \rightarrow$$
$$\mathsf{sum}(l_1, l_3, l_9, r_1, r_3, r_9) \leq \mathsf{sum}(l'_1, l'_3, l'_9, r'_1, r'_3, r'_9)$$

where $\mathsf{balance}(w, l'_1, l'_3, l'_9, r'_1, r'_3, r'_9)$ is the constraint obtained from Eqn. 1 by replacing l_i and r_i by l'_i and r'_i, respectively, and sum refers to the sum of the listed variables. This

[1] A similar problem appears among the examples of the Comfusy synthesis tool [4] distribution at http://lara.epfl.ch/dokuwiki/comfusy.

constraint requires that every other solution that would also balance the scale for the given object has to use more weights than the solution returned.

The newly synthesized program gives correct answers for arbitrary large natural numbers. E.g., let us assume the object weighs 12345123451234512345123456789kg, then the program tells us to take 1371680383470501371680384088 times Weight 9 on the right side and once Weight 3 on the left side.

D. Finding Approximate Solutions

Consider the constraint $6x+9y = z$, where z is the input and x, y are inputs. The solution exists only when z is a multiple of 3, so we may wish to find x, y that minimizes $|6x + 9y - z|$, using a similar encoding with quantifiers as in the previous example. The support for disjunctions allows us to encode the absolute value operator that is useful for finding approximate solutions. The tool synthesizes a function that given a value of z, computes x, y to be as close to z as possible. For example, given the input 104, the tool outputs $x = 13$ and $y = 3$.

E. Folding and Inverting Computations

Consider the Syracuse algorithm function, whose one step is given by $f(x) = \text{ if } (2 \mid x) \text{ then } x/2 \text{ else } 3x + 1$. Consider a relation on integers corresponding to iterating f six times: $r(x, y) \leftrightarrow f^6(x) = y$. (We could use such function to speed-up experimental verification of the famous $3n + 1$ conjecture that states $\forall x > 0.\exists n. f^n(x) = 1$.) When we use $r(x, y)$ as the specification and indicate x as input and y as output, our synthesizer generates a function that accepts a sequence of bits of x and outputs in linear time a sequence of bits of y that is given by 6-fold iteration of f. For example, given 9834092834092384039284092384 the function produces 921946203196161003682883662 and given an input decremented by one, it produces 33190063315061796132583811795. Moreover, using the same description, the synthesizer allows us to specify y as the input and x as the output and also produces a linear-time function that maps y to one of the possible values of x. For example, given the value 1000, it gives 1777 (which happens not to be the trivial solution $2^6 y$). Note that, if the synthesis for $y = f^n(x)$ succeeds, the computation of the result is independent of n, so our approach can effectively folds a number of iterations of f into one linear-time function on the binary representations of inputs and outputs.

F. Processing Sequences of Bits

We next illustrate the use of specification of unbounded numbers in simple signal processing task. Suppose we have an input signal X with discrete values in the range $\{0, 1, 2, \ldots, 15\}$ and we wish to compute a smoothed output signal Y by averaging signal values with its neighbors, using the formula $Y_i = (X_{i-1} + 2X_i + X_{i+1}) \operatorname{div} 4$. We specify this function in WS1S as a relation between unbounded integers x and y, where we reserve 4 bits for value of the signal at each time point (see Figure 6). For constants a, b, let $x[k+a, k+b]$

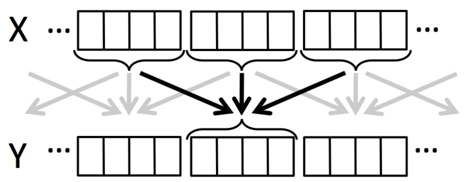


Fig. 6. Averaging signal values

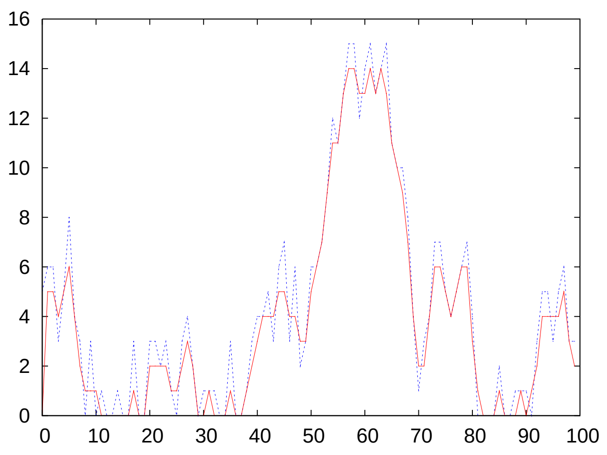


Fig. 7. The result of applying the synthesized function that computes a smoothed version of a signal. The function on an arbitrarily long signal was specified in WS1S.

denote the number represented by the subrange of digits of x between $k + a$ and $k + b$:

$$x[k+a, k+b] = x[k+a] + 2x[k+a+1] + \ldots + 2^{b-a}x[k+b]$$

We define the smoothing relation between numbers x and y by:

$$\forall i. \quad (4|i) \rightarrow y[i{+}4..i{+}7] = \\ (x[i..i{+}3] + 2x[i{+}4..i{+}7] + x[i{+}8..i{+}11]) \operatorname{div} 4$$

Our synthesizer generates a function that, given the sequence of bits x, produces a sequence of bits y. Figure 7 shows an input signal (dotted line) and the resulting smoothed signal (full line) that results after we applied the linear-time function synthesized by our tool to the input.

III. Preliminaries

A. Words and Automata

Given a finite set of variables V, we use Σ_V to denote the alphabet $\Sigma_V = 2^V$. We omit V in Σ_V if it is clear from the context. When used as a letter, we denote $\emptyset \in \Sigma_V$ by $\mathbf{0}$. Given a finite word $w \in \Sigma_V^*$, we use $|w|$ to denote the length of w, and w_i to denote the letter on the i-th position of w. By ϵ we denote the empty word, of length zero. Given a partitioning of V into the sets I and O and a letter $\sigma \in \Sigma_V$, we use $\sigma|_I$ to denote the projection of σ to I, i.e., $\sigma|_I = \sigma \cap I$. We extend projection in usual sense to words and languages.

A *finite automaton* A over a finite set of variables V is a tuple $(\Sigma, Q, \text{init}, F, T)$, where $\Sigma = 2^V$ is the *alphabet*, Q is a finite-state of *states*, init $\in Q$ is the *initial state*, $T \subseteq Q \times \Sigma \times Q$ is the *transition relation*, and $F \subseteq Q$ is a set of *final state*. Automaton A is *deterministic*, if for all transitions

$$\begin{array}{rcl} F & ::= & F \wedge F \mid F \vee F \mid \neg F \mid t_N < t_N \mid t_N = t_N \\ & \mid & t_N[t_P] \mid t_P < t_P \mid t_P = t_P \mid (C \,\|\, t_N) \mid t_N \underset{t_P}{\sim} t_N \\ & \mid & \forall_{\mathsf{pos}} k.F \mid \exists_{\mathsf{pos}} k.F \mid \forall x.F \mid \exists x.F \\ t_N & ::= & x \mid C \mid t_N + t_N \mid C \cdot t_N \mid t_N \,\mathsf{div}\, C \mid t_N \,\%\, C \\ & \mid & (t_N \veebar t_N) \mid (t_N \barwedge t_N) \mid t_N \ll C \mid t_N \gg C \\ & \mid & 2^{t_P} \mid t_N[t_P..^{+}C] \mid t_N[0..t_P] \\ t_P & ::= & k \mid C \mid k + C \mid k \dot{-} C \mid \mathsf{maxBit}(t_N) \\ C & ::= & \text{non-negative integer constant} \end{array}$$

Fig. 8. Syntax of WS1S where sets denote natural numbers (T_N) and elements denote positions (T_P) in binary representations of numbers

$(q_1, \sigma_1, q_1'), (q_2, \sigma_2, q_2') \in T$, $q_1 = q_2$ and $\sigma_1 = \sigma_2$ implies $q_1' = q_2'$ holds. A is *complete*, if for all states $q \in Q$ and letters $\sigma \in \Sigma$, there exists a state $q' \in Q$ such that $(q, \sigma, q') \in T$. Note that if A is deterministic and complete T describes a total function from Q and Σ to Q.

Given an automaton $A = (\Sigma, Q, \mathrm{init}, F, T)$ and a state $q \in Q$, we use A_q to refer to the automaton (Σ, Q, q, F, T) that has the same structure as A but starts at q.

A *run* ρ of A on a word $w \in \Sigma^*$ is a sequence of states $q_1 \dots q_{|w|+1}$ such that (i) $q_1 = \mathrm{init}$ and (ii) for all $1 \le i \le |w| : (q_i, w_i, q_{i+1}) \in T$. A run is *accepting* if $q_{|w|+1} \in F$. We say w *is accepted by* A if there exists a run of A on w that is accepting. We denote by $\mathcal{L}(A) \subseteq \Sigma^*$ the set of words accepted by A.

The *exhaustive run* ρ of A on a word $w \in \Sigma^*$ is a sequence of sets of states $S_1 \dots S_{|w|+1}$ such that (i) $S_1 = \{\mathrm{init}\}$ and (ii) for all $1 \le i \le |w|$, $S_{i+1} = \{q' \in Q \mid \exists q, (q, w_i, q') \in T\}$. An exhaustive run is *accepting* if $S_{|w|+1} \cap F \neq \emptyset$. Note that if A is deterministic, then the run of A on a word w is unique and the elements in the exhaustive run of A on w are singletons.

Lemma 1: For an automaton A with a set of states Q, computing an exhaustive run of A for a word $w \in \Sigma^*$ can be done in time $O(|Q| \cdot |w|)$ for a non-deterministic A, and can be done in time $O(|w|)$ for a deterministic A.

Given an automaton $A = (\Sigma_V, Q, \mathrm{init}, F, T)$ over variables V and a set $I \subset V$, the *projection of A to I*, denoted by $A|_I$, is the automaton $(\Sigma_I, Q, \mathrm{init}, F, T_I)$ with $T_I = \{(q, \sigma_I, q') \in Q \times \Sigma_I \mid \exists \sigma \in \Sigma_V, (q, \sigma, q') \in T \wedge \sigma|_I = \sigma\}$.

In the remainder, we fix I to be the set of input and O to be the set of output variables.

B. WS1S as extension of Presburger Arithmetic

Figure 8 shows the syntax of weak monadic second-order logic of one successor, which we use as our specification language for unbounded non-negative integers. The logic contains all integer linear arithmetic operations and quantifiers, thus subsuming Presburger arithmetic. Furthermore, it contains the expression $x[k]$ to extract the k-th least significant bit of the number x. It is also possible to find a c-successor of position k, denoted $k+c$. Together with quantification over positions, this allows the specification of arbitrary uniform bitwise relations on integer variables. To illustrate the expressive power of WS1S, we introduce shorthands for some of the constraints that can be defined in this way: bitwise operations ($\barwedge$, $\veebar$), left and right shifting ($\ll$, $\gg$), a sub-word of length c at position k of a given integer x (denoted $x[k..^{+}c]$), congruence modulo 2^p (denoted $x \underset{p}{\sim} y$), the initial prefix of an integer $x[0..k]$, the integer 2^p for a position p, and the smallest p such that $x < 2^p$, denoted $\mathsf{maxBit}(x)$.

C. Amortized Cost of Synthesis

We describe the cost of synthesis and synthesized program in a unified framework, by considering the entire amortized cost of applying a given specification a on a series of inputs $b_1, \dots, b_n$. Let f be a function with two arguments, so that $f(a, b) = c$ if the input-output pair (b, c) satisfies the specification a. We implement function f using a function of the form $g(a, b, s) = (f(a, b), s')$ that computes f and updates its local state from s to s'. We assume a fixed initial state s_0. The presence of local state can make the computation more efficient on a series of inputs. This framework accounts for simple cases such as memoization and caching, as well as the more general case of on-the-fly specialization.

Given the specification a and the inputs $b_1, \dots, b_n$ we define $s_i = g(a, b_i, s_{i-1})$ for $i \in \{1, \dots, n\}$. Let $g'(a, b, s)$ denote the time to compute $g(a, b, s)$. Let $|x|$ denote the length of value x. We define the amortized cost of g on inputs $a; b_1, \dots, b_n$ by $\frac{1}{n} \sum_{i=1}^{n} g'(a, b_i, s_{i-1})$. Our main complexity measure is then $c(s_a, s_b, n)$, which we define as the maximum amortized cost over all $a; b_1, \dots, b_n$ for which $|a| \le s_a$ and $|b_i| \le s_b$ for all i.

Observe that $c(s_a, s_b, 1)$ is simply the complexity of running function f once on inputs of size s_a and s_b, respectively. Another useful measure, of particular interest in synthesis, is $c_\infty(s_a, s_b) = \lim_{n \to \infty} c(s_a, s_b, n)$, which amortizes any pre-computation that happens in finitely many steps. We next present several examples to illustrate the cost function $c_\infty(s_a, s_b)$ for implementations of several problems.

Example 1 (Finding an enclosing interval): Consider the problem of computing the smallest interval enclosing a given number. More precisely, the goal is to compute $f([x_1, \dots, x_m], y) = (L, U)$ where $L = \max\{x_i \mid x_i \le y\}$ and $U = \min\{x_j \mid y \le x_j\}$ given an unordered list of numbers $x_1, \dots, x_m$ (with the result arbitrary if the $\max$ or $\min$ expressions above are not defined). In this example, we assume that each number takes constant space to represent, so $|[x_1, \dots, x_m]| = m$ and $|y| = 1$. An algorithm for one invocation can simply make a single pass through the list, computing the current $\max$ of lower bounds of y and the current $\min$ of the upper bounds up to a given position in the list. This gives the worst-case complexity m of the algorithm. If we use this algorithm as the implementation g (without making use of state), we obtain $c_\infty(m, 1)$ of $O(m)$.

Consider next an alternative implementation, given by $g'([x_1, \dots, x_m], y, s)$, which behaves as follows: on the first invocation, $g([x_1, \dots, x_m], y, s_0)$, builds a balanced binary search tree storing the set of numbers $x_1, \dots, x_m$ in time $O(m \log m)$, and returns this tree in the resulting state s'. On subsequent invocations, g uses this tree to find the enclosing interval (L, U), which can be done in time $O(\log m)$ by doing a lookup in the tree. Therefore, we obtain that n invocations require $O(m \log m + n \log m)$, which gives $c(m, 1, n) \in O(\frac{1}{n}(m \log m) + \log m)$ and $c_\infty(m, 1) = O(\log m)$. Thus, we have seen that precomputation improves the amortized time $c_\infty(m, 1)$ from $O(m)$ to $O(\log m)$. □

IV. Synthesis Algorithm

A. Constructing Specification Automaton

The input to our algorithm is a WS1S formula F whose free variables $z_1, \ldots, z_r$ denote unbounded integers. We assume a partitioning of the index set $\{1, \ldots, r\}$ into inputs I and the outputs O. In the first step, our algorithm constructs a deterministic specification automaton A accepting words in the alphabet $\Sigma_{I\cup O}$. We use a standard automaton construction [7] and obtain an automaton A characterizing the satisfying assignments of F, i.e. whose language $L(A)$ contains precisely the words $\sigma_0\sigma_1 \ldots \sigma_n \in \Sigma^*_{I\cup O}$ for which F holds in the variable assignment $(z_1, \ldots, z_r)$ in which the k-th least significant bit of z_i is one iff $0 \leq k \leq n$ and $i \in \sigma_k$. From this correctness property it follows that $w \in L(A)$ implies $w\mathbf{0}^p \in L(A)$ for every $p \geq 0$.

B. Overview

All subsequent steps of our algorithm work with the specification automaton A and do not depend on how this automaton was obtained. Given A, our goal is to construct a function that computes, for a given sequence of inputs bits a corresponding sequence of output bits such that the combined word is accepted by the deterministic automaton. [2] We show our construction in several steps. First, we assume that we are only interested in outputs whose length does not exceed the length of inputs. For this case we start by describing a less time-efficient implementation (Subsection IV-C) that depends on the size of A, then describe an efficient version, showing that we can avoid the dependence on the size of A (Subsection IV-D). Finally, we show how to lift the assumption that the outputs are no longer than the inputs (Subsection IV-E).

C. Input-Bounded Synthesis of Unspecialized Implementations

In the first version of our solution we assume that, given an input bit sequence, we seek an output sequence of the *same length* such that the input and output pair are accepted by the specification automaton A.

Our unspecialized implementation P_{unspec} simulates the given automaton $A = (\Sigma_{I\cup O}, Q, \text{init}, F, T)$ on the input word $w \in \Sigma^*_I$ and tries to find an accepting run. P_{unspec} first constructs the exhaustive run $\rho = S_1 \ldots S_{|w|+1}$ of the projected automaton $A|_I$ on w (see preliminaries for the definition of automaton projection and exhaustive run). If ρ is not accepting, then there is no matching output word and P_{unspec} terminates. Otherwise, P_{unspec} picks a state $q_{|w|+1}$ in $S_{|w|+1} \cap F$ and constructs an accepting run $q_1 \ldots q_{|w|+1}$ of A and the output word v by proceeds backwards over i, from $i = |w|$ to $i = 1$, as follows: it picks $v_i \in \Sigma_O$ and $q_i \in S_i$ such that $(q_i, w_i \cup v_i, q_{i+1}) \in T$. When it reaches one of the initial states in S_1, the result is an accepting run of the automaton A; the desired output is the sequence $v_1 \ldots v_{|w|}$ of the output components of the labels in the reconstructed run.

The P_{unspec} implementation repeats the above construction for each input word w. From Lemma 1 we obtain the amortized cost of P_{unspec}.

Lemma 2: If s_A denotes the size of the input automaton A and s_w denotes the size of the input word, then the unspecialized implementation P_{unspec} solves the synthesis for input-bounded specifications in amortized time $c(s_A, s_w, n)$ of $O(s_A \cdot s_w)$ (consequently, $c_\infty(s_A, s_w)$ is also $O(s_A \cdot s_w)$).

D. Input-Bounded Synthesis of Specialized Implementations

We next present our main construction (illustrated in the Example II-A), which avoids the dependence of the running time of computation of on the (potentially large) number of states of the automaton A. To obtain an implementation with optimal runtime, we transform the given automaton A into an input-deterministic automaton A' using the subset construction on the projection $A|_I$. The challenge is to extend the subset construction with the additional labeling functions that allow us to efficiently reconstruct an accepting run of A from an accepting run of A'. Given such additional information, our specialized implementation P_{spec} runs A' on the input w and uses the labeling to construct the output v.

Our construction introduces two labeling functions, ϕ and τ. The function ϕ maps each accepting state S of A' into one state $q \in S$ that is accepting in A. The τ function indicates how to move backwards through the accepting run; it maps each transition (S, σ_i, S') of A' and a state $q' \in S'$ into a pair $(q, \sigma_o) \in S \times \Sigma_o$ of new a state and an output letter, such that $(q, \sigma_i \cup \sigma_o), q')$ is a transition of the original automaton A.

Definition of synthesized data structure A', ϕ, τ. Given an automaton $A = (\Sigma_{I\cup O}, Q, \text{init}, F, T)$, we construct an automaton $A' = (\Sigma_I, Q', \text{init}', F', T')$ and two labeling functions $\phi : F' \to Q$ and $\tau : (T' \times Q) \to (Q \times \Sigma_O)$ such that (i) A' is deterministic, (ii) $\mathcal{L}(A)|_I = \mathcal{L}(A')$, and (iii) for every word $u \in \mathcal{L}(A')$ with an accepting run $S_1 \ldots S_{n+1}$ of A', there exists a word $w \in \mathcal{L}(A)$ with $w|_I = u$ and an accepting run $q_1 \ldots q_{n+1}$ of A such that $\phi(S_{n+1}) = q_{n+1}$ and for all $1 \leq i \leq n$, $(q_i, w_i|_O) \in \tau((S_i, u_i, S_{i+1}), q_{i+1})$. We define A' as follows:

$$\begin{array}{lcl}
Q' & = & 2^Q \\
\text{init}' & = & \{\text{init}\} \\
F' & = & \{S \in Q' \mid S \cap F \neq \emptyset\} \\
T' & = & \{(S, i, S') \in Q' \times \Sigma_I \times Q' \mid \\
 & & \quad S' = \{q' \mid \exists q, \sigma.(q, \sigma, q') \in T \wedge q \in S \wedge \sigma|_I = i\}\}
\end{array}$$

We define $\phi : F' \to Q$ such that if $S \in F'$ then $\phi(S) \in S \cap F$; such value exists by definition of F'.

We define $\tau : (T' \times Q) \to (Q \times \Sigma_O)$ for $(S, i, S') \in T'$ and $q' \in S'$ as follows. By definition of T', there exists a transition $(q, \sigma, q') \in T$ of the original automaton such that $\sigma|_I = i$. We

[2]Note that we seek an implementation that works uniformly for arbitrarily long sequences of bits, which means that it is not possible to pre-compute all possible input/output pairs.

pick an arbitrary such transition and define $\tau((S, i, S'), q') = (q, \sigma|_O)$.

Computing A' and τ through automata transformations. In our implementation, we represent both A' and τ in one automaton, which we compute using the following sequence of automata transformations. Because τ refers to sets of transitions, we first turn each transition of A into a state, i.e, given $A = (\Sigma_{I\cup O}, Q, \text{init}, F, T)$, we construct an automaton $B = (\Sigma_{I\cup O}, Q_B, \text{init}_B, F_B, T_B)$ such that

$$\begin{aligned}
\text{init}_B &= (q, \sigma, \text{init}_A) \quad \text{for arbitrarily chosen } q, \sigma \\
Q_B &= \{\text{init}_B\} \cup T \\
F_B &= \{(q, \sigma, q') \in Q_B \mid q' \in F\} \\
T_B &= \{(t, \sigma, t') \in Q_B \times \Sigma_{I\cup O} \times Q_B \mid \\
&\qquad \exists q, q', q'' \in Q.\ \exists \sigma' \in \Sigma_{I\cup O}. \\
&\qquad t = (q, \sigma', q') \text{ and } t' = (q, \sigma, q'')\}.
\end{aligned}$$

Next, we project B to I, i.e., we replace every transition (q, σ, q') in B by (q, σ_I, q'). Finally, we obtain automaton C by determinizing $B|_I$ using the classical subset construction. Now, every reachable state in C (other than init_B) corresponds to a transition in A'. Assume we are given a state q_c in C, then q_c has the form $\{(q_1, \sigma_1, q'_1), \ldots, (q_k, \sigma_k, q'_k)\}$ with $\forall i, j, \sigma_i|_I = \sigma_j|_I = \sigma_I$ and corresponds to the transition (t, σ_I, t') in A', where $t = \{q_i \mid 1 \leq i \leq k\}$ and $t' = \{q'_i \mid 1 \leq i \leq k\}$. So, every state q_c in C defines a labeling function for the corresponding transition (t, σ_I, t') that maps every state $q'_i \in t'$ to a set of available pairs $(q_i, \sigma_i|_O)$. Our final labeling function τ picks for each state q_i one of the available pairs.

Specialized implementation and its complexity. The specialized implementation P_{spec} runs A' on the input word w and constructs a run $\rho = S_1 \ldots S_{|w|+1}$. If ρ is not accepting, then there is no matching output word and the function terminates. Otherwise, it computes an accepting run $q_1 \ldots q_{|w|+1}$ of A and the output word v as follows: $\phi(S_{|w|+1}) = q_{|w|+1}$ and, for all $1 \leq i \leq |w|$, $(q_i, v_i) = \tau((S_i, w_i, S_{i+1}), q_{i+1})$.

The following theorem states the correctness of P_{spec} and follows by construction.

Theorem 1: Consider an automaton A and an input $w_1 \ldots w_n$. Then if there exists an output $v_1 \ldots v_n$ such that $(w_1 \cup v_1) \ldots (w_n \cup v_n)$ is accepted by A, then P_{spec} computes one such output $v_1 \ldots v_n$. If there is no corresponding output then P_{spec} indicates that there is no output.

The following theorem states that our construction achieves the desired linear-time behavior and independence from the size of the initial automaton. The construction of A', ϕ, τ takes time singly exponential in the size of the automaton, but is done only once, so it is amortized for each invocation of the automaton. Extracting the output for a given input takes time independent of the number of states in A' because A' and τ have deterministic transitions.

Theorem 2: If s_A denotes the size of the specification automaton A and s_w denotes the size of the input word, then P_{spec} solves the synthesis for input-bounded specifications in amortized time $c(s_A, s_w, n)$ of $O(\frac{1}{n} 2^{s_A} + s_w)$. Consequently, the amortized time $c_\infty(s_A, s_w)$ as the number of queries approaches infinity is $O(s_w)$.

E. Extending Synthesis to Arbitrary Regular Specifications

In this section we extend the result of the previous section to allow computing an output that satisfies the specification even if the output has a larger number of bits than the input. Consider the simple specification $x < y$, where x is the input and y is the output. Given the input $\overline{111}_2$ of length three (representing the number 7), every value of output satisfying the specification has the length at least four.

To adapt the solution in the previous section to the full synthesis problem we generalize the notion of acceptance to take into account any number of zeros that could be appended to the input without changing the meaning of the input. Therefore, if the automaton A' finishes reading the input word and none of the states reached in the last step are accepting, it checks whether one of the states can reach an accepting state while reading only the input letter $\mathbf{0}$. The closure with the input $\mathbf{0}$ can be computed in polynomial time by computing the states that are backward-reachable from an accepting state using only edges with input label $\mathbf{0}$.

To be able to emit the appropriate segment of the output, the backward-reachability computation keeps, for every state, an output word that leads to an accepting state. We use the function $\psi : Q \to \Sigma_O^* \cup \{\perp\}$ to store these words, where Q are the states of the specification automaton A. We write $\psi(q) = \perp$ to denote that there is no input word $w \in \mathbf{0}^*$ that is accepted starting from q. Formally, given the automaton $A = (\Sigma_{I\cup O}, Q, \text{init}, F, T)$, we set $\psi = \psi^{|Q|}$ and define ψ^i inductively: for all $q \in Q$:

(i) $\psi^0(q) = \begin{cases} \epsilon & \text{if } q \in F \\ \perp & \text{otherwise} \end{cases}$

(ii) let R^i be the set of states q for which $\psi^i(q) \neq \perp$,

$$\psi^{i+1}(q) = \begin{cases} \psi^i(q) & \text{if } q \in R^i \\ \sigma|_O \psi^i(q') & \text{elsif } \exists (q, \sigma, q') \in T : \sigma|_I = \mathbf{0} \wedge q' \in R^i, \\ \perp & \text{otherwise.} \end{cases}$$

Observe that if $\psi(q) \neq \perp$ then $\psi(q)$ is a word of length bounded by the number of states of the specification automaton A. Therefore, the maximal amount by which the output is longer than the input is bounded by the size of the specification automaton.

To recognize leading zeros, we adapt the final states F' of A' (computed as for P_{spec} in the previous section) and extend the labeling function ϕ as follows. Let $\mathsf{fin}(S) = \{q \in S \mid \psi(q) \neq \perp\}$ be the states in S that can reach input on zeros.

$$\begin{aligned}
F' &= \{S \in Q' \mid \mathsf{fin}(S) \neq \emptyset\} \\
\phi(S) &= q \in \mathsf{fin}(S) \text{ s.t. } |\psi(q)| = \min\{|\psi(q') \mid q' \in \mathsf{fin}(S)|\}
\end{aligned}$$

Note that the function $\phi(S)$ chooses one of the states that lead to an accepting state with an output word of minimal length.

The implementation and its complexity. Given an input word $w_1 \ldots w_n$, the implementation P_{gspec} generates, as P_{spec} in the previous Subsection (IV-D), a run $S_1, \ldots, S_{n+1}$. If $S_{n+1} \notin F'$, then there is no corresponding output; the implementation indicates this and stops. Otherwise, suppose $S_{n+1} \in F'$ and $q_{n+1} = \phi(S_{n+1})$. The implementation generates the backward run from q_{n+1} as in Subsection (IV-D), producing the output bits $v_1 \ldots v_n$. The final output is then the word $v_1 \ldots v_n \psi(q_{n+1})$.

Theorem 3: Let s_A denote the size of the specification automaton A, s_w the size of the input word, and s_v the size of the shortest output word that satisfies the input word. Then $s_v \leq s_w + s_A$. The P_{gspec} implementation solves the synthesis for arbitrary regular specifications in amortized time $c(s_A, s_v, s_w, n)$ of $O(\frac{1}{n}2^{s_A} + s_w + s_v)$. Consequently, the amortized time $c_\infty(s_A, s_w)$ is $O(s_w + s_v)$.

Together with the correctness of the construction of specification automata from WS1S, we obtain the following soundness and completeness theorem.

Theorem 4: Consider a WS1S formula F with input variables $(z_k)_{k\in I}$ and output variables $(z_k)_{k\in O}$. Consider a binary representation of input variables $(z_k)_{k\in I}$. If there exist values of output variables $(z_k)_{k\in O}$ such that F holds for $(z_k)_{k\in I\cup O}$, then P_{gspec} outputs a sequence of bits for one such $(z_k)_{k\in O}$. If there is no corresponding output then P_{gspec} indicates that there is no output.

V. Lookahead-Causal Specifications

The algorithms presented so far first read the entire input and then generate a corresponding output. In some cases (e.g., in streaming applications), one might prefer an implementation that starts outputting before reading the entire input. In general, there are specifications in WS1S that require reading the complete input to decide on the output, such as the bit counting example (Subsection II-A). Other specifications, such as the signal processing example require reading a bounded number of bits ahead (three, in this case) to compute an output bit.

For notational simplicity we consider specifications $\mathsf{spec}(x, y)$ containing a single input-output pair x and y. Furthermore, we assume that the specifications are total, that is, $\forall x.\exists y.\mathsf{spec}(x, y)$. If a specification is not total, we can transform it into a total specification $\mathsf{spec}'(x, y, e)$ given by

$$(\mathsf{spec}(x, y) \wedge e = 0) \vee ((\neg\exists y.\mathsf{spec}(x, y)) \wedge y = 0 \wedge e = 1)$$

Definition of k-causality. We next define lookahead-k-causality, or k-causality for short. Recall that $z' \underset{p}{\sim} z$ means that z' and z have identical the initial p bits. We say that an input output pair x, y is k-causal for spec, written $\mathsf{causal}_k(x, y)$ iff

$$\forall p.\ \forall x' \underset{p+k}{\sim} x.\ \exists y' \underset{p}{\sim} y.\ \mathsf{spec}(x', y')$$

We say that spec is k-causal iff it implies $\mathsf{causal}_k(x, y)$ for all x, y.

Note that, if spec is not k-causal but some inputs have multiple possible outputs, a general strategy to turn spec it into a causal specification is to simply conjoin it with $\mathsf{causal}_k(x, y)$ and check whether the resulting specification is still total, that is, whether $\forall x.\exists y.\mathsf{spec}(x, y) \wedge \mathsf{causal}_k(x, y)$. We next show how to construct, given a k-causal specification, an implementations that emits the input after reading k steps after the output. The basic idea is to apply our current construction to go backward k steps. The key intuition behind the correctness is to consider a sequence of zeros as a possible continuation of the current input sequence, and apply the definition of k-causality.

Synthesized system for a k-causal specifications. Let $\mathsf{spec}(x, y)$ be a k-causal and total specification. We construct the specification automaton A and apply the construction described in Section IV-E. We obtain the automaton A' and the labeling functions τ, ϕ, and ψ. We extend A', τ, ϕ, and ψ so that they include, for all states q of A, the determinized version A'_q of A_q, where A_q is the automaton that differs from A only in that its initial state is changed to q.

The synthesized program P_{caus} for k-causal specification has a fill parameter $\mu > 0$. It uses a buffer of length at least $(1 + \mu)k$ and alternates operations Read and Flush. The Read operation reads one more input bit into the buffer and advances the state S of A' accordingly, as for P_{gspec}. The Flush operation is invoked when the input buffer contains at least j input bits for $j \geq \lceil (1+\mu)k \rceil$. It runs backwards k steps from the current state $q = \phi(S)$ following τ and reaches state q'. It then treats q' as a final state of the entire input, emits $j - k$ outputs going backwards and reaching state q''. It then empties the corresponding buffer elements and moves forward from q' using $A_{q'}$ until the current position of the input. This gives a streaming implementation that traverses the input bits only $1/\mu$ more times compared to P_{gspec}, regardless of k.

VI. Experimental Results

We implemented our algorithm in Scala [8] using MONA [9] as backend to construct the specification automaton. Our tool accepts specifications written in WS1S using MONA's syntax. It is called using two arguments: (i) the name of the MONA file containing the specification, and (ii) the list of input variables. Our tool first invokes MONA to construct the specification automaton, then applies the construction described in Section IV-C and computes additional information necessary to produce output values with larger bit-length as their corresponding input values (see Section IV-E). Then, the synthesized function can be invoked at a later point in the program any number of times. Currently, we have deployed this in interactive shell, where the user can query the function.

We have tested our tool on several specifications including simple addition and the examples described in Section II and the appendix. The specifications were written in WS1S. In Table I, we summarize the results. In the second column, we give a short description that relates the results to the description of the example in Section II. In the column labeled *MONA* we show the time (in ms) that MONA needs to create the specification automaton A. The number of states of A is shown in Column labeled $|A|$. Column *Synthesis* and $|A'|$ show the time (in ms) necessary to create the synthesized function and the number of states of the augmented automaton A', respectively. The last 4 columns show the time (in ms) to run these functions on 1000 random inputs of different bit length (512, 1024, 2048, and 4096 bits). The tests were performed

TABLE I
SYNTHESIS TIMES, SIZE OF THE GENERATED AUTOMATA, AND AVERAGE RUNNING TIMES OF SYNTHESIZE FUNCTIONS

No	*Example*	*MONA (ms)*	*Synthesis (ms)*	$\|A\|$	$\|A'\|$	512b	1024b	2048b	4096b
1	addition	318	132	4	9	509	995	1967	3978
2	approx	719	670	27	35	470	932	1821	3641
3	company	8’291	1’306	58	177	608	1312	2391	4930
4	parity	346	108	4	5	336	670	1310	2572
5	mod-6-test	341	242	23	27	460	917	1765	3567
6	3-weights-min	26’963	640	22	13	438	875	1688	3391
7	4-weights	2’707	1’537	55	19	458	903	1781	3605
8	smooth-4bits	51’578	1’950	1781	955	637	1271	2505	4942
9	smooth-f-2bits	569	331	73	67	531	989	1990	3905
10	smooth-b-2bits	569	1’241	73	342	169	347	628	1304
11	forward-6-3n+1	834	1’007	233	79	556	953	1882	4022

on an Intel Core 2 Duo P7450 (2,13 GHz) CPU with 4096 MB DDR2 (667 MHz) RAM.

In most cases (9 out of 11) creating the augmented automaton (column *Synthesis*) is faster than creating the initial automaton (column *MONA*). In some cases (e.g., Line 6 and 8) the synthesis time is is only a fraction of the overall time. The running time of the synthesized function is (as expected) linear in the number of bits of the input.

We are not aware of any other tool that can handle the examples that we present in Table I. The closest tool that we are aware of is Comfusy [4], which does not support quantifiers and large integers for implementation reasons, and does not support bitwise operations due to a fundamental restriction of the underlying quantifier elimination algorithm. Conversely, there are examples supported by Comfusy (multiplication with symbolic constants and set constraints) that our approach does not support, so a combination of the two approaches would be fruitful.

VII. RELATED WORK

Our work is related to synthesizing combinational circuits from relations (e.g.,[10]). We also synthesize a function implementing the given relation but our implementation works for arbitrarily long input sequences.

Techniques [1], [2], [11] to synthesize reactive systems that implement a given S1S specification can handle arbitrarily long input sequences. However, they assume that the specification can be implemented by a (usually finite-state) system that produces the output immediately while reading the input, i.e., the system cannot look ahead. Such techniques have applications to control the behavior of hardware and embedded systems or concurrent programs [12]. These techniques usually take specifications in a fragment of temporal logic [13] and have resulted in tools that can synthesize useful hardware components [14], [3]. Recent work [15] establishes theoretical results (without implementation) regarding the problem of deciding when an S1S specification can be implemented using a system with lookahead. The (k-bounded) causality checks in our problem could be performed using this decision procedure based on infinite game theory. Our specification language uses finite instead of infinite words, which allows us to eliminate the non-causal behaviors and thus simplify the synthesis process. Moreover, our technique is not restricted to k-bounded specifications.

The work on graph types [16] proposes to synthesize fields given by definitions in monadic second-order logic and also uses the MONA tool. However, it focuses on computing assignments to update fields of linked data structures as opposed to numerical and bit constraints.

VIII. CONCLUSION

We presented the first algorithm to synthesize systems from general WS1S specifications. We have shown that all total relations can be implemented using a function that maps input into output and takes time proportional to the sum of the lengths of input and the output. Our implementation suggests that our approach is promising, even when used with general-purpose off-the-shelf tools that represent automata states explicitly.

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APPENDIX

A. *Specialized Solvers for Optimization Problems*

Assume we have a company producing Product A and Product B using two different types of resources: R_1 and R_2. In order to produce Product A, we need 4 units of resource R_1 and 1 units of resource R_2. Similarly, for Product B we need 3 units from each resource. The resources are delivered once a month but the exact amounts vary. The company sells Product A for 9 CHF and Product B for 6 CHF. Now, we construct a program that given the current amount of available resources, computes how many instances of Product A and of Product B should be produced in order to maximize the profit. To specify the problem, we use variables x_A and x_B to refer to the number of Product A and B should be producted and r_1 and r_2 for the available resources. We require that the sum of the resource units use for Product A and B do not exceed the available resources and obtain the following two constraints.

$$4 \cdot x_A + x_B \leq r_1 \tag{3}$$

$$3 \cdot x_A + 3 \cdot x_B \leq r_2 \tag{4}$$

The profit of the company is given by $9 \cdot x_A + 6 \cdot x_B$. In order to optimize the profit, we add the following constraint:

$$\forall x'_A, x'_B . ((4 \cdot x'_A + x'_B \leq r_1) \wedge (3 \cdot x'_A + 3 \cdot x'_B \leq r_2)) \rightarrow (9 \cdot x'_A + 6 \cdot x'_B) \leq (9 \cdot x_A + 6 \cdot x_B)$$

This constraint states that for any values of x'_A and x'_B that satisfies the constraints given in Eqn. 3 and 4, the profit with respect to x_A and x_B will be at least as large as the profit for x'_A and x'_B.

The synthesized program then tells us, e.g., that if we have 3000 units of R_1 and 7000 units of R_2, we optimize the profit by producing 222 instance of Product A and 2111 of Product B. While with 5000 units each, we obtain the maximal profit with 1111 and 555 instances of A and B, respectively.