The McNaughton Theorem
**McNaughton Theorem**

**Theorem 1** Let $\Sigma$ be an alphabet. Any recognizable subset of $\Sigma^\omega$ can be recognized by a Rabin automaton.

Determinisation algorithm by S. Safra (1989) uses a special subset construction to obtain a Rabin automaton equivalent to a given Büchi automaton. The Safra algorithm is optimal $2^{O(n \log n)}$.

This proves that recognizable $\omega$-languages are closed under complement (Büchi Theorem).
Oriented Trees

Let $\Sigma$ be an alphabet of labels.

An oriented tree is a pair of partial functions $t = \langle l, s \rangle$:

- $l : \mathbb{N} \mapsto \Sigma$ denotes the labels of the nodes
- $s : \mathbb{N} \mapsto \mathbb{N}^*$ gives the ordered list of children of each node

\[
\text{dom}(l) = \text{dom}(s) \overset{\text{def}}{=} \text{dom}(t)
\]

$\leq$ denotes the successor, and $\preceq$ the lexicographical ordering on tree positions.
Safra Trees

Let $A = \langle S, I, T, F \rangle$ be a Büchi automaton.

A **Safra tree** is a pair $\langle t, m \rangle$, where $t$ is a finite oriented tree labeled with non-empty subsets of $S$, and $m \subseteq \text{dom}(t)$ is the set of **marked positions**, such that:

- each marked position is a leaf
- for each $p \in \text{dom}(t)$, the union of labels of its children is a strict subset of $t(p)$
- for each $p, q \in \text{dom}(t)$, if $p \not\leq q$ and $q \not\leq p$ then $t(p) \cap t(q) = \emptyset$

**Proposition 1** A Safra tree has at most $\|S\|$ nodes.

$$r(p) = t(p) \setminus \bigcup_{q < p} t(q)$$

$$\|\text{dom}(t)\| = \sum_{p \in \text{dom}(t)} 1 \leq \sum_{p \in \text{dom}(t)} \|r(p)\| \leq \|S\|$$
**Initial State**

We build a Rabin automaton $B = \langle S_B, i_B, T_B, \Omega_B \rangle$, where:

- $S_B$ is the set of all Safra trees $\langle t, m \rangle$ labeled with subsets of $S$
- $i_B = \langle t, m \rangle$ is the Safra tree defined as either:
  - $\text{dom}(t) = \{\epsilon\}, t(\epsilon) = I$ and $m = \emptyset$ if $I \cap F = \emptyset$
  - $\text{dom}(t) = \{\epsilon\}, t(\epsilon) = I$ and $m = \{\epsilon\}$ if $I \subseteq F$
  - $\text{dom}(t) = \{\epsilon, 0\}, t(\epsilon) = I, t(0) = I \cap F$ and $m = \{0\}$ if $I \cap F \neq \emptyset$
Classical Subset Move

[Step 1] \( \langle t_1, m_1 \rangle \) is the tree with \( \text{dom}(t_1) = \text{dom}(t) \), \( m_1 = \emptyset \), and
\( t_1(p) = \{ s' \mid s \xrightarrow{\alpha} s', s \in t(p) \} \), for all \( p \in \text{dom}(t) \)
Spawn New Children

[Step 2] $\langle t_2, m_2 \rangle$ is the tree such that, for each $p \in \text{dom}(t_1)$, if $t_1(p) \cap F \neq \emptyset$ we add a new child to the right, identified by the first available id, and labeled $t_1(p) \cap F$, and $m_2$ is the set of all such children.
[Step 3] \( \langle t_3, m_3 \rangle \) is the tree with \( \text{dom}(t_3) = \text{dom}(t_2), \ m_3 = m_2 \), such that, for all \( p \in \text{dom}(t_3) \), \( t_3(p) = t_2(p) \setminus \bigcup_{q \prec p} t_2(q) \)
Delete Empty Nodes

[Step 4] \( \langle t_4, m_4 \rangle \) is the tree such that \( \text{dom}(t_4) = \text{dom}(t_3) \setminus \{ p \mid t_3(p) = \emptyset \} \) and \( m_4 = m_3 \setminus \{ p \mid t_3(p) = \emptyset \} \)
[Step 5] \( \langle t_5, m_5 \rangle \) is \( m_5 = m_4 \cup V \), \( \text{dom}(t_5) = \text{dom}(t_4) \setminus \{ q \mid p \in V, \ q < p \} \), \( V = \{ p \in \text{dom}(t_4) \mid t_4(p) = \bigcup_{p<q} t_4(q) \} \)
Accepting Condition

The Rabin accepting condition is defined as
\[ \Omega_B = \{(N_q, P_q) \mid q \in \bigcup_{(t,m) \in S_B} \text{dom}(t)\}, \]
where:

- \( N_q = \{ \langle t, m \rangle \in S_B \mid q \not\in \text{dom}(t) \} \)
- \( P_q = \{ \langle t, m \rangle \in S_B \mid q \in m \} \)

\[ \Omega_B = \{ (\{R_1\}, \{R_2\}), (\{R_2\}, \{R_1\}) \} \]
Example
Correctness of Safra Construction

Lemma 1 For $0 \leq i \leq n - 1$, $S_{i+1} \subseteq T(S_i, \alpha_{i+1})$. Moreover, for every $q \in S_n$, there is a path in $A$ starting in some $q_0 \in S_0$, ending in $q$ and visiting at least one final state after its origin.

An infinite accepting path in $B$ corresponds to an infinite accepting path in $A$ (König’s Lemma)
Correctness of Safra Construction

Conversely, an infinite accepting path of $A$ over $u = \alpha_0\alpha_1\alpha_2\ldots$

$$\pi : q_0 \xrightarrow{\alpha_0} q_1 \xrightarrow{\alpha_1} q_2 \ldots$$

corresponds to a unique infinite path of $B$:

$$i_B = R_0 \xrightarrow{\alpha_0} R_1 \xrightarrow{\alpha_1} R_2 \ldots$$

where each $q_i$ belongs to the root of $R_i$

If the root is marked infinitely often, then $u$ is accepted. Otherwise, let $n_0$ be the largest number such that the root is marked in $R_{n_0}$. Let $m > n_0$ be the smallest number such that $q_m$ is repeated infinitely often in $\pi$.

Since $q_m \in F$ it appears in a child of the root. If it appears always on the same position $p_m$, then the path is accepting. Otherwise it appears to the left of $p_m$ from some $n_1$ on (step 3). This left switch can only occur a finite number of times.
Complexity of the Safra Construction

Given a Büchi automaton with $n$ states, how many states we need for an equivalent Rabin automaton?

- The upper bound is $2^{O(n \log n)}$ states
- The lower bound is of at least $n!$ states
Maximum Number of Safra Trees

Each Safra tree has at most $n$ nodes.

A Safra tree $\langle t, m \rangle$ can be uniquely described by the functions:

- $S \rightarrow \{0, \ldots, n\}$ gives for each $s \in S$ the characteristic position $p \in \text{dom}(t)$ such that $s \in t(p)$, and $s$ does not appear below $p$
- $\{1, \ldots, n\} \rightarrow \{0, 1\}$ is the marking function
- $\{1, \ldots, n\} \rightarrow \{0, \ldots, n\}$ is the parent function
- $\{1, \ldots, n\} \rightarrow \{0, \ldots, n\}$ is the older brother function

Altogether we have at most $(n + 1)^n \cdot 2^n \cdot (n + 1)^n \cdot (n + 1)^n \leq (n + 1)^{4n}$ Safra trees, hence the upper bound is $2^{O(n \log n)}$. 
The Language $L_n$

$\alpha \in L_n$ if there exist $i_1, \ldots, i_n \in \{1, \ldots, n\}$ such that

- $\alpha_k = i_1$ is the first occurrence of $i_1$ in $\alpha$ and $q_0 \xrightarrow{\alpha_0 \ldots \alpha_k} q_{i_1}$

- the pairs $i_1i_2, i_2i_3, \ldots, i_n i_1$ appear infinitely often in $\alpha$.

**Example 1**

$(3\#32\#21\#1)^\omega \in L_3$

$(312\#)^\omega \not\in L_3$
The Language $L_n$

Lemma 2 (Permutation) For each permutation $i_1, i_2, \ldots, i_n$ of $1, 2, \ldots, n$, the infinite word $(i_1i_2\ldots i_n#)^\omega \notin L_n$.

Lemma 3 (Union) Let $A = (S, i, T, \Omega)$ be a Rabin automaton with $\Omega = \{\langle N_1, P_1 \rangle, \ldots, \langle N_k, P_k \rangle\}$ and $\rho_1, \rho_2, \rho$ be runs of $A$ such that

$$\inf(\rho_1) \cup \inf(\rho_2) = \inf(\rho)$$

If $\rho_1$ and $\rho_2$ are not successful, then $\rho$ is not successful either.
Proving the $n!$ Lower Bound

Suppose that $A$ recognizes $L_n$. We need to show that $A$ has $\geq n!$ states.

Let $\alpha = i_1, i_2, \ldots, i_n$ and $\beta = j_1, j_2, \ldots, j_n$ be two permutations of $1, 2, \ldots, n$. Then the words $(i_1i_2\ldots i_n\#)^\omega$ and $(j_1j_2\ldots j_n\#)^\omega$ are not accepted.

Let $\rho_\alpha$, $\rho_\beta$ be the non-accepting runs of $A$ over $\alpha$ and $\beta$, respectively.

Claim 1 $\inf(\rho_\alpha) \cap \inf(\rho_\beta) = \emptyset$

Then $A$ must have $\geq n!$ states, since there are $n!$ permutations.
Proving the $n!$ Lower Bound

By contradiction, assume $q \in \inf(\rho_\alpha) \cap \inf(\rho_\beta)$. Then we can build a run $\rho$ such that $\inf(\rho) = \inf(\rho_1) \cup \inf(\rho_2)$ and $\alpha, \beta$ appear infinitely often. By the union lemma, $\rho$ is not accepting.

\[
i_1 \ldots i_{k-1} \ i_k \ i_{k+1} \ldots i_{l-1} \ i_l \ldots i_n \\
= = \neq \\
\hat{j}_1 \ldots \hat{j}_{k-1} \ \hat{j}_k \ \hat{j}_{k+1} \ldots \hat{j}_{r-1} \ \hat{j}_r \ldots \hat{j}_n
\]

\[
i_k \ i_{k+1}, \ldots i_l = \hat{j}_k \ \hat{j}_{k+1}, \ldots \hat{j}_{r-1} \ \hat{j}_r = i_k
\]

The new word is accepted since the pairs $i_k i_{k+1}, \ldots, \hat{j}_k \hat{j}_{k+1}, \ldots, \hat{j}_{r-1} i_k$ occur infinitely often. Contradiction with the fact that $\rho$ is not accepting.
Büchi Complementation Theorem
Büchi Complementation Theorem

**Theorem 2** For every Büchi automaton $A$ there exists a Büchi automaton $B$ such that $\mathcal{L}(A) = \mathcal{L}(B)$.

Already a consequence of McNaughton Theorem, since from $A$ we can build a Rabin automaton $R$, complement it to $\overline{R}$, and build $B$ from $\overline{R}$.

Next we present a direct proof.
**Congruences**

**Definition 1** An equivalence relation $R \subseteq \Sigma^* \times \Sigma^*$ is said to be a left-congruence iff for all $u, v, w \in \Sigma^*$ we have $u \equiv v \Rightarrow wu \equiv vw$.

**Definition 2** An equivalence relation $R \subseteq \Sigma^* \times \Sigma^*$ is said to be a right-congruence iff for all $u, v, w \in \Sigma^*$ we have $u R v \Rightarrow uw R vw$.

**Definition 3** An equivalence relation $R \subseteq \Sigma^* \times \Sigma^*$ is said to be a congruence iff it is both a left- and a right-congruence.

Ex: the Myhill-Nerode equivalence $\sim_L$ is a right-congruence.
Congruences

Let $A = \langle S, I, T, F \rangle$ be a Büchi automaton and $s, s' \in S$.

$$W_{s,s'} = \{ w \in \Sigma^* \mid s \xrightarrow{w} s' \}$$

For $s, s' \in S$ and $w \in \Sigma^*$, we denote $s \xrightarrow{w} F s'$ iff $s \xrightarrow{w} s'$ visiting a state from $F$.

$$W_{s,s'}^F = \{ w \in \Sigma^* \mid s \xrightarrow{w} F s' \}$$

For any two words $u, v \in \Sigma^*$ we have $u \equiv v$ iff for all $s, s' \in S$ we have:

- $s \xrightarrow{u} s' \iff s \xrightarrow{v} s'$, and
- $s \xrightarrow{F} s' \iff s \xrightarrow{F} s'$.

The relation $\equiv$ is a congruence of finite index on $\Sigma^*$.
**Congruences**

Let \([w] \sim\) denote the equivalence class of \(w \in \Sigma^*\) w.r.t. \(\sim\).

**Lemma 4** For any \(w \in \Sigma^*\), \([w] \sim\) is the intersection of all sets of the form \(W_{s,s'}, W_{s,s'}^F, \overline{W_{s,s'}}, \overline{W_{s,s'}^F}\), containing \(w\).

\[
T_w = \bigcap_{w \in W_{s,s'}} W_{s,s'} \cap \bigcap_{w \in W_{s,s'}^F} W_{s,s'}^F \cap \bigcap_{w \in \overline{W_{s,s'}}} \overline{W_{s,s'}} \cap \bigcap_{w \in \overline{W_{s,s'}^F}} \overline{W_{s,s'}^F}
\]

We show that \([w] \sim = T_w\).

“\(\subseteq\)” If \(u \approx w\) then clearly \(u \in T_w\).
Congruences

“⊇” Let $u \in T_w$

- if $s \xrightarrow{w} s'$, then $w \in W_{s,s'}$, hence $u \in W_{s,s'}$, then $s \xrightarrow{u} s'$ as well.

- if $s \xleftarrow{w} s'$, then $w \in \overline{W_{s,s'}}$, hence $u \in \overline{W_{s,s'}}$, then $s \xleftarrow{u} s'$.

Also,

- if $s \xrightarrow{F_w} s'$, then $w \in W_{s,s'}^F$, hence $u \in W_{s,s'}^F$, then $s \xrightarrow{F_u} s'$ as well.

- if $s \xleftarrow{F_w} s'$, then $w \in \overline{W_{s,s'}^F}$, hence $u \in \overline{W_{s,s'}^F}$, then $s \xleftarrow{F_u} s'$.

Then $u \cong w$.

This lemma gives us a way to compute the $\cong$-equivalence classes.
Outline of the proof

We prove that:

$$\mathcal{L}(A) = \bigcup_{VW^\omega \cap \mathcal{L}(A) \neq \emptyset} VW^\omega$$

where $V, W$ are $\cong$-equivalence classes

Then we have

$$\Sigma^\omega \setminus \mathcal{L}(A) = \bigcup_{VW^\omega \cap \mathcal{L}(A) = \emptyset} VW^\omega$$

Finally we obtain an algorithm for complementation of Büchi automata
**Saturation**

**Definition 4** A congruence relation $R \subseteq \Sigma^* \times \Sigma^*$ saturates an $\omega$-language $L$ iff for all $R$-equivalence classes $V$ and $W$, if $VW^\omega \cap L \neq \emptyset$ then $VW^\omega \subseteq L$.

**Lemma 5** The congruence relation $\approx$ saturates $\mathcal{L}(A)$.
Every word belongs to some $VW^\omega$

Let $\alpha \in \Sigma^\omega$ be an infinite word.

Since $\sim$ is an equivalence relation, there exists a mapping $\varphi : \Sigma^+ \to \Sigma^+/\sim$ such that $\varphi(u) = [u]_{\sim}$, for all $u \in \Sigma^+$.

Then there exists a Ramseyan factorization of $\alpha = uv_0v_1v_2 \ldots$ such that $\varphi(v_i) = [v]_{\sim}$ for some $v \in \Sigma^+$ and for all $i \geq 0$.

Together with the saturation lemma, this proves

$$\mathcal{L}(A) = \bigcup_{VW^\omega \cap \mathcal{L}(A) \neq \emptyset} VW^\omega$$
Complementation of Büchi Automata

**Theorem 3**  For any Büchi automaton $A$ there exists a Büchi automaton $\overline{A}$ such that $\mathcal{L}(\overline{A}) = \Sigma^\omega \setminus \mathcal{L}(A)$.

$$\mathcal{L}(A) = \bigcup_{VW^\omega \cap \mathcal{L}(A) \neq \emptyset} VW^\omega$$

where $V, W$ are $\cong$-equivalence classes

$$\Sigma^\omega \setminus \mathcal{L}(A) = \bigcup_{VW^\omega \cap \mathcal{L}(A) = \emptyset} VW^\omega$$
Ramseyan Factorizations
Ramsey Theorem

Theorem 4 Let $X$ be some countably infinite set and colour the subsets of $X$ of size $n$ in $c$ different colours. Then there exists some infinite subset $M$ of $X$ such that the size $n$ subsets of $M$ all have the same colour.
A Particular Case of Ramsey Theorem

Let \( \alpha \in \Sigma^\omega \) be an infinite word.

A factorization of \( \alpha \) is an infinite sequence \( \{\alpha_i\}_{i=0}^{\infty} \) of finite words such that \( \alpha = \alpha_0\alpha_1 \ldots \)

Let \( E \) be a finite set of colors and \( \varphi : \Sigma^+ \to E \). A factorization \( \alpha = uv_0v_1v_2 \ldots \) is said to be Ramseyan for \( \varphi \) if there exists \( e \in E \) such that

\[
\varphi(v_iv_{i+1} \ldots v_{i+j}) = e
\]

for all \( i \leq j \).
A Particular Case of Ramsey Theorem

**Theorem 5** Let $\varphi : \Sigma^+ \to E$ be a map from $\Sigma^+$ into a finite set $E$. Then every infinite word of $\Sigma^\omega$ admits a Ramseyan factorization for $\varphi$.

Let $\{U_i\}_{i=0}^\infty$ be an infinite sequence of infinite subsets of $\mathbb{N}$ defined as:

\[
U_0 = \mathbb{N} \\
U_{i+1} = \{n \in U_i \mid \varphi(\alpha(\min U_i, n)) = e_i\}
\]

where $e_i \in E$ is chosen such that the set $U_{i+1}$ is infinite (show the existence of $e_i$)

Since $E$ is finite, there exists an infinite subsequence of integers $i_0, i_1, \ldots$ such that $e_{i_0} = e_{i_1} = \ldots = e$.

Then $v_j = \alpha(n_{ij}, n_{ij+1})$ is the required factorization.
A Particular Case of Ramsey Theorem

\[ n_0 = 0 \quad n_1 \quad n_2 \quad n_3 \quad n_4 \]

\[ U_1 = \{n_1, n_2, n_3, n_4, \ldots \} \]
A Particular Case of Ramsey Theorem

\[ U_2 = \{n'_2, n'_3, n'_4, \ldots \} \]
A Particular Case of Ramsey Theorem

\[ U_3 = \{n_3'', n_4'', n_5'' \ldots \} \]
A Particular Case of Ramsey Theorem

\[ U_3 = \{n''_3, n''_4, \ldots \} \]