Ultimately periodic sets, semi-linear sets, and Presburger arithmetic

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Logics over the integers

Examples:

$$\forall x\ \exists y\ \exists z\colon y > x \land y - x = 5z$$

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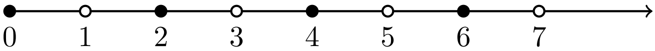
Motivation:

- Common framework/toolbox for problems from various domains
- Growing software support:
  SMT (satisfiability modulo theories) solvers
- Nice mathematics at the interface of several areas

Periodic and ultimately periodic sets of integers

Suppose $S \subseteq \mathbb{N}$.

S is periodic if there exists a $p > 0$ such that,
for all $x \in \mathbb{N}$: $\quad x \in S$ iff $x + p \in S$.



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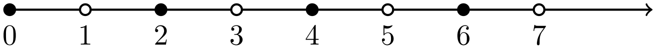
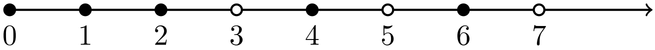
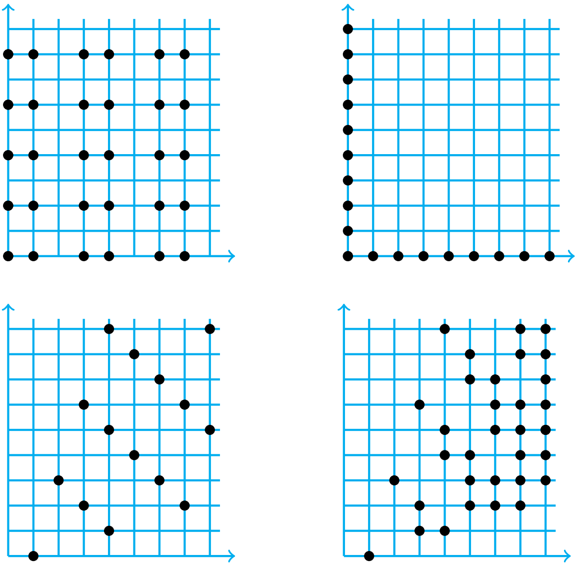


S is ultimately periodic if there exist N and $p > 0$ such that,
for all $x \geq N$: $\quad x \in S$ iff $x + p \in S$.



Ultimately periodic sets in higher dimension



Linear and semi-linear sets

[Parikh (1961)]

Vector $\boldsymbol{b}$: base vector
Vectors $P = \{\boldsymbol{p}_1, \ldots, \boldsymbol{p}_s\}$: period vectors

generators

Linear set: $|P| < \infty$

$$L(\boldsymbol{b}, P) = \{\boldsymbol{b} + \lambda_1 \boldsymbol{p}_1 + \ldots + \lambda_s \boldsymbol{p}_s : \lambda_1, \ldots, \lambda_s \in \mathbb{N}\}$$

Rohit J. Parikh

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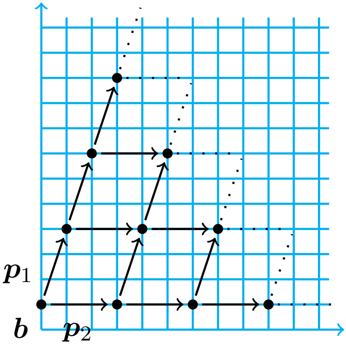

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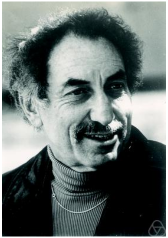
Semi-linear set: $|I|, |P_i| < \infty$

$$M = \bigcup_{i \in I} L(\boldsymbol{b}_i, P_i)$$

Theorem (Ginsburg and Spanier, 1964)

Semi-linear sets = sets definable in Presburger arithmetic.

Seymour Ginsburg



Edwin H. Spanier

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the first-order theory of integers with addition and order.

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Mojżesz Presburger

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$$\exists \boldsymbol{x}^1 \, \forall \boldsymbol{x}^2 \, \ldots \, \exists \boldsymbol{x}^k \, . \, \varphi(\boldsymbol{x})$$

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Semi-linear $\subseteq$ $\exists^*$-PA: by definition.

What about computational complexity?

Full Presburger arithmetic is elementary [Oppen, 1978]

$\forall^*\exists^*$-fragment is complete for $\mathbf{coNEXP}$ [Haase, 2014]

Integer programming $(A \cdot \boldsymbol{x} \geq \boldsymbol{c})$ in fixed dimension
is in $\mathbf{P}$ [Lenstra, 1984]

Quantified integer programming with k blocks
is complete for kth level of $\mathbf{PH}$ [C. & Haase, 2017]

. . . and many more results!

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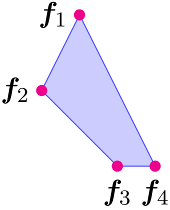
Geometry of
$A \cdot \boldsymbol{x} \geq \boldsymbol{c}$

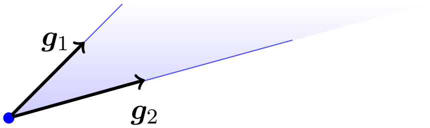
Linear Algebra	**Linear Arithmetic**
$=$	$\leq$
Equations	Inequalities
Points, lines, planes	Rays, segments, polygons
Subspaces	Polyhedra

Convex hulls and cones

Convex hull
$\mathrm{conv}\{\boldsymbol{f}_1, \boldsymbol{f}_2, \boldsymbol{f}_3, \boldsymbol{f}_4\}$



Finitely generated cone
$\mathrm{cone}\{\boldsymbol{g}_1, \boldsymbol{g}_2\}$



Integers	**Reals**
Linear sets	Cones
Semi-linear sets	Polyhedral sets

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The Minkowski–Weyl theorem (1896, 1935)

Theorem

The following are equivalent:

1. $P = \{\boldsymbol{x} \colon A \cdot \boldsymbol{x} \geq \boldsymbol{c}\}$ for some matrix A and vector $\boldsymbol{c}$; and
2. $P = \mathrm{conv}(F) + \mathrm{cone}(G)$ for some finite sets F, G.

"*This classical result is an outstanding example of a fact which is completely obvious to geometric intuition, but which wields important algebraic content and is not trivial to prove.*"
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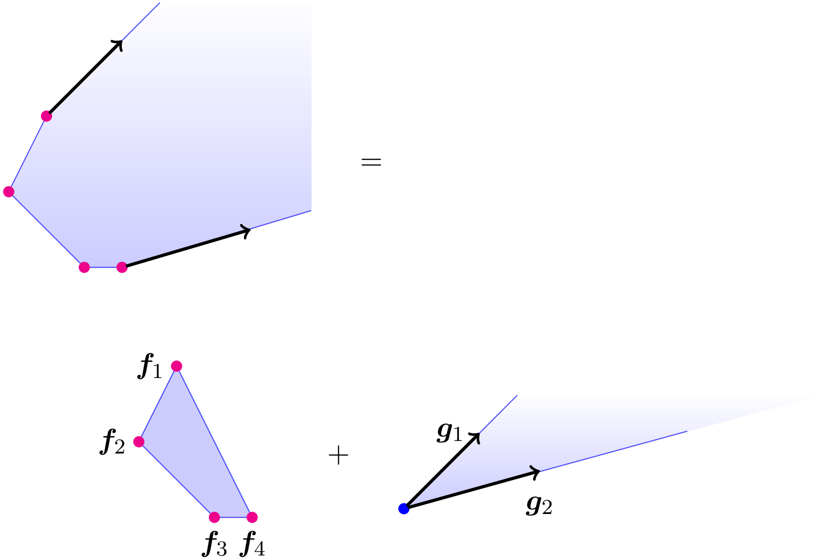

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In both translations $1 \Rightarrow 2$ and $2 \Rightarrow 1$:

- The blowup in size can be exponential.
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Integer programming

Input: matrix $A \in \mathbb{Z}^{m \times d}$, vector $\boldsymbol{c} \in \mathbb{Z}^m$
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$\mathbf{NP}$-hard: encode SAT
In $\mathbf{NP}$: **small model property**

Geometry of integer programming

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For any $S \subseteq \mathbb{Z}^d$, the following are equivalent:

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Linear, hybrid linear, and semi-linear sets

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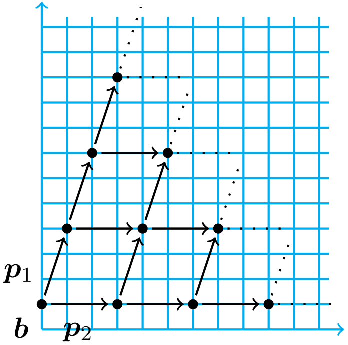

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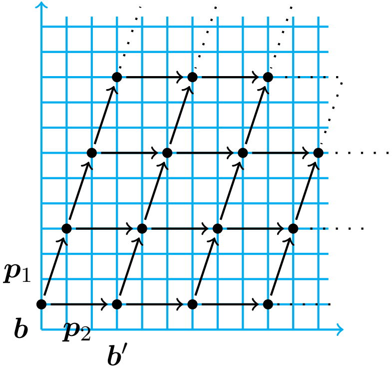

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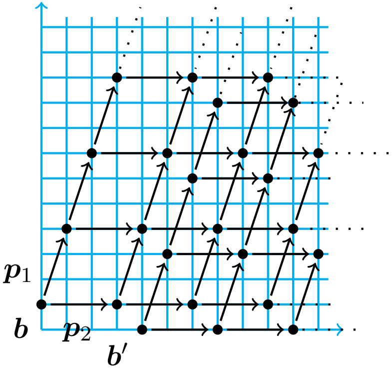

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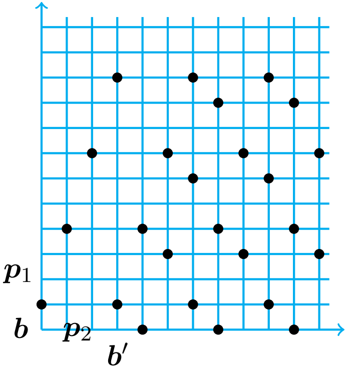

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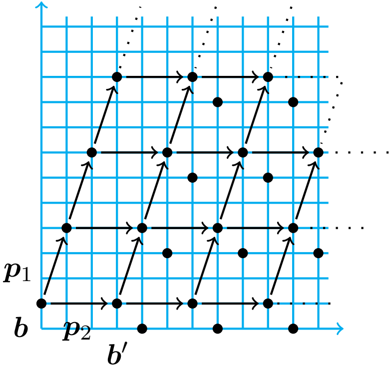

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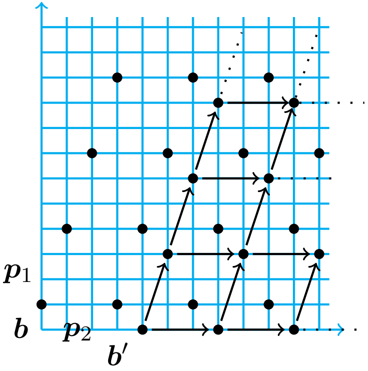

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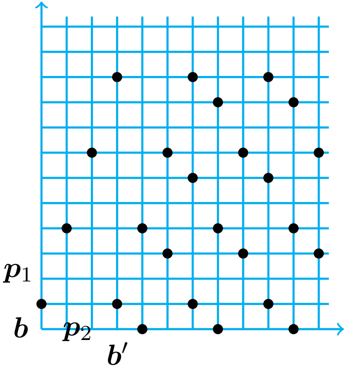

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On the Complexity of Integer Programming

CHRISTOS H. PAPADIMITRIOU

Massachusetts Institute of Technology, Cambridge, Massachusetts, and National Technical University, Athens, Greece

ABSTRACT. A simple proof that integer programming is in $\mathscr{NP}$ is given. The proof also establishes that there is a pseudopolynomial-time algorithm for integer programming with any (fixed) number of constraints.

KEY WORDS AND PHRASES: integer linear programming, $\mathscr{P}$, $\mathscr{NP}$, pseudopolynomial algorithms

CR CATEGORIES: 5 25, 5.3, 5.4

1. *Introduction*

The *knapsack problem* is the following one-line integer programming problem: Is there a 0–1 n-vector x such that

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b,$$

where $b, a_1, \ldots, a_n$ are given positive integers?

The knapsack problem is NP-complete [5, 7]. However, it is well known that it can be solved by a *pseudopolynomial* algorithm [4], that is, an algorithm with running time bounded by a polynomial in n *and* $a = \max\{a_1, \ldots, a_n, b\}$. Indeed, one can show quite easily that there is a pseudopolynomial-time algorithm for any one of the following extensions of the knapsack problem:

(a) The x_i are not restricted to be 0–1.
(b) Some of the a_i are negative.
(c) There are $m > 1$ equations to be satisfied (m fixed).

In fact, with a little care, pseudopolynomial algorithms can be developed for the combination of *any two* of these extensions. In this note we show that there is a pseudopolynomial algorithm for the problem that results by extending the knapsack problem in *all three* directions above.

Our proof settles another interesting question. It has been shown by many people (including [1, 2, 6]) that *integer programming* (i.e., the problem of deciding whether, for given $m \times n$ integer matrix A and m-vector b, the conditions

$$Ax = b, \qquad x \geq 0, \quad \text{integer},$$

are satisfied by some $x \in \mathbb{N}^n$) is in $\mathscr{NP}$. The proofs usually amount to showing

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A BOUND ON SOLUTIONS OF LINEAR INTEGER EQUALITIES AND INEQUALITIES

JOACHIM VON ZUR GATHEN AND MALTE SIEVEKING

ABSTRACT. Consider a system of linear equalities and inequalities with integer coefficients. We describe the set of rational solutions by a finite generating set of solution vectors. The entries of these vectors can be bounded by the absolute value of a certain subdeterminant. The smallest integer solution of the system has coefficients not larger than this subdeterminant times the number of indeterminates. Up to the latter factor, the bound is sharp.

Let A, B, C, D be $m \times n$-, $m \times 1$-, $p \times n$-, $p \times 1$-matrices respectively with integer entries. The rank of A is r, and s is the rank of the $(m + p) \times n$-matrix $\binom{A}{C}$. Let M be an upper bound on the absolute values of those $(s - 1) \times (s - 1)$- or $s \times s$-subdeterminants of the $(m + p) \times (n + 1)$-matrix $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$, which are formed with at least r rows from (A, B).

THEOREM. *If $Ax = B$ and $Cx \geqslant D$ have a common integer solution, then they have one with coefficients bounded by $(n + 1)M$.*

Let M_1, M_2, and M_3 be upper bounds on the absolute values of the $r \times r$-subdeterminants, the subdeterminants, and the entries of (A, B) respectively. Taking the $n \times n$-identity matrix for C and $D = 0$, we have the following

COROLLARY. *If $Ax = B$ has a nonnegative integer solution, then it has one with coefficients bounded by $(n + 1)M_1$.*

S. Cook [**4**] obtained a bound of the order of $M_3^{m^2}$ in this case. I. Borosh and L. B. Treybig conjecture that one can always have the bound M_1. For many cases, this bound would be sharp. They give an elegant proof for M_2^2 in [**2**]. In [**1**], [**3**] they obtain M_1 in the cases where $r = n - 1$ and for homogeneous systems (only nontrivial solutions being considered), and nrM_1 if the matrix has no $r \times r$-subdeterminants which are zero. Their work arose from topological questions, while Cook's and our aim was to prove that the solvable linear integer programs form a NP-complete set (see Remarks 2 and 3).

For the proof of the theorem we first note that it suffices to consider the case $s = n$. For if $s < n$, then choose an integer solution y, let e_i be 1 or -1 according to whether y_i is $\geqslant 0$ or < 0. To the given system add $n - s$

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Proof

[von zur Gathen and Sieveking, 1978]

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$$\boldsymbol{x} = \sum \lambda_i \boldsymbol{f}_i + \sum \mu_j \boldsymbol{g}_j$$

$$(\operatorname{conv} F) \quad (\operatorname{cone} G)$$

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$$\boldsymbol{x} = \underset{(\operatorname{conv} F)}{\sum \lambda_i \boldsymbol{f}_i} + \underset{(\operatorname{cone} G)}{\sum \mu_j \boldsymbol{g}_j}$$

$$= \left(\sum \lambda_i \boldsymbol{f}_i + \sum (\mu_j - \lfloor \mu_j \rfloor) \boldsymbol{g}_j\right) + \sum \lfloor \mu_j \rfloor \boldsymbol{g}_j$$

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$\boldsymbol{x} \in L(C, Q)$.

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Why geometry?

Example: The universality problem

Does the given set coincide with $\mathbb{N}^d$?

Motivation:

- Important special case of equivalence
- $\forall$ in logic

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Computational complexity:

- For linear set: trivial
- For hybrid linear set: easy
- For semi-linear set: hard

Beyond Presburger arithmetic

Additional operations (e.g., Kleene star)
[Piskac and Kuncak, 2008; Haase and Zetzsche, 2019]

Nonlinear predicates (e.g., divisibility)
[Lipshitz, 1978+, Lechner et al., 2015]

Counting problems and counting quantification (such as Härtig's quantifier)
[Schweikardt, 2005; Habermehl and Kuske, 2015]

Open questions

What is the computational complexity
of the following decision problems?

Quantified integer programming with unbounded alternation
(between $\mathbf{PSPACE}$ and $\mathbf{STA}(*, 2^{n^{O(1)}}, n)$)
[C. & Haase, ICALP'17]

Short Presburger arithmetic with quantifier prefix $\exists\forall\exists\exists$
[Nguyen and Pak, FOCS'17]

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Quantified integer programming with unbounded alternation
(between $\mathbf{PSPACE}$ and $\mathbf{STA}(*, 2^{n^{O(1)}}, n)$)

[C. & Haase, ICALP'17]

Short Presburger arithmetic with quantifier prefix $\exists\forall\exists\exists$

[Nguyen and Pak, FOCS'17]

Thank you!