First-order logic

Part one:
Language and Truth Value of Formulae

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Course given by Ioana Ciuciu (ciuciu@imag.fr)

Université Joseph Fourier, Grenoble I

22 February 2013
Homework: solution using ND

\((p \Rightarrow \neg j) \land (\neg p \Rightarrow j) \land (j \Rightarrow m) \Rightarrow m \lor p\)
Homework : solution using ND

\[(p \Rightarrow \neg j) \land (\neg p \Rightarrow j) \land (j \Rightarrow m) \Rightarrow m \lor p\]

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\[(p \implies \neg j) \land (\neg p \implies j) \land (j \implies m) \implies m \lor p\]

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S. Devismes et al (Grenoble I)
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\[(p \Rightarrow \neg j) \land (\neg p \Rightarrow j) \land (j \Rightarrow m) \Rightarrow m \lor p\]

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\[(p \Rightarrow \neg j) \land (\neg p \Rightarrow j) \land (j \Rightarrow m) \Rightarrow m \lor p\]

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\[(p \Rightarrow \neg j) \land (\neg p \Rightarrow j) \land (j \Rightarrow m) \Rightarrow m \lor p\]

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\[(p \Rightarrow \neg j) \land (\neg p \Rightarrow j) \land (j \Rightarrow m) \Rightarrow m \lor p\]

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\[(p \implies \neg j) \land (\neg p \implies j) \land (j \implies m) \implies m \lor p\]

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<td>&amp;E 2 1</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>((\neg p \implies j))</td>
<td>&amp;E 2 2</td>
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<tr>
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<td>suppose ((\neg p \lor p) \implies \bot)</td>
<td>\lor I 1 6</td>
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<tr>
<td>1, 5, 6</td>
<td>6</td>
<td>suppose (p)</td>
<td></td>
</tr>
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<td>1, 5, 6</td>
<td>7</td>
<td>(\neg p \lor p)</td>
<td>\implies I 1 9</td>
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<td>1, 5, 6</td>
<td>8</td>
<td>(\bot)</td>
<td>\implies I 5, 7</td>
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<td>9</td>
<td>hence (\neg p)</td>
<td>\lor I 6, 8</td>
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<td>1, 5</td>
<td>10</td>
<td>(\neg p \lor p)</td>
<td>\lor I 1 9</td>
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<td>1, 5</td>
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<td>(\bot)</td>
<td>\implies I 5, 10</td>
</tr>
<tr>
<td>1</td>
<td>12</td>
<td>hence (\neg \neg (\neg p \lor p))</td>
<td>\implies I 5, 11</td>
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<td>13</td>
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<td>\implies E 4, 14</td>
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<tr>
<td>1, 14</td>
<td>15</td>
<td>(j)</td>
<td>\implies E 3, 15</td>
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<td>16</td>
<td>(m)</td>
<td>\lor I 1 16</td>
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<td>17</td>
<td>(m \lor p)</td>
<td>\implies I 17, 14</td>
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<td>18</td>
<td>hence (\neg p \implies p \lor m)</td>
<td>\implies I 17, 14</td>
</tr>
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<td>1, 19</td>
<td>19</td>
<td>suppose (p)</td>
<td>\lor I 1 19</td>
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<td>1, 19</td>
<td>20</td>
<td>(p \lor m)</td>
<td>\lor I 20, 19</td>
</tr>
<tr>
<td>1</td>
<td>21</td>
<td>hence (p \implies p \lor m)</td>
<td>\lor E 21, 18, 13</td>
</tr>
<tr>
<td>1</td>
<td>22</td>
<td>(m \lor p)</td>
<td></td>
</tr>
</tbody>
</table>
Homework: solution using ND

\[(p \Rightarrow \neg j) \land (\neg p \Rightarrow j) \land (j \Rightarrow m) \Rightarrow m \lor p\]

<table>
<thead>
<tr>
<th>context</th>
<th>number</th>
<th>proof</th>
<th>justification</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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<td>1</td>
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<td>&amp;E2 1</td>
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<td>1</td>
<td>3</td>
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<td>4</td>
<td>((\neg p \Rightarrow j))</td>
<td>&amp;I 9</td>
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<tr>
<td>1, 5</td>
<td>5</td>
<td>suppose ((\neg p \lor p) \Rightarrow \bot)</td>
<td>&amp;E 5, 7</td>
</tr>
<tr>
<td>1, 5, 6</td>
<td>6</td>
<td>suppose (p)</td>
<td>&amp;I 6, 8</td>
</tr>
<tr>
<td>1, 5, 6</td>
<td>7</td>
<td>(\neg p \lor p)</td>
<td>&amp;I 1 6</td>
</tr>
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<td>1, 5, 6</td>
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<td>(\bot)</td>
<td>&amp;I 1 9</td>
</tr>
<tr>
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<td>hence (\neg p)</td>
<td>&amp;E 5, 10</td>
</tr>
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<tr>
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<td>&amp;E 5, 13</td>
</tr>
<tr>
<td>1</td>
<td>13</td>
<td>(\neg p \lor p)</td>
<td>&amp;I 1 14</td>
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<td>1, 14</td>
<td>14</td>
<td>suppose (\neg p)</td>
<td>&amp;E 3, 15</td>
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<tr>
<td>1</td>
<td>15</td>
<td>(j)</td>
<td>&amp;I 1 16</td>
</tr>
<tr>
<td>1, 14</td>
<td>16</td>
<td>(m)</td>
<td>&amp;I 1 17</td>
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<tr>
<td>1, 14</td>
<td>17</td>
<td>(m \lor p)</td>
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<td>1</td>
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<td>hence (\neg p \Rightarrow p \lor m)</td>
<td>&amp;I 2 19</td>
</tr>
<tr>
<td>1, 19</td>
<td>19</td>
<td>suppose (p)</td>
<td>&amp;E 2 20</td>
</tr>
<tr>
<td>1, 19</td>
<td>20</td>
<td>(p \lor m)</td>
<td>&amp;E 2 21</td>
</tr>
<tr>
<td>1</td>
<td>21</td>
<td>hence (p \Rightarrow p \lor m)</td>
<td>&amp;E 2 22</td>
</tr>
<tr>
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<td>22</td>
<td>(m \lor p)</td>
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</tr>
<tr>
<td>1</td>
<td>23</td>
<td>hence ((p \Rightarrow \neg j) \land (\neg p \Rightarrow j) \land (j \Rightarrow m) \Rightarrow m \lor p)</td>
<td>&amp;I 22, 1</td>
</tr>
</tbody>
</table>
Structure of the first-order logic

A non-empty domain (more than two elements)
Structure of the first-order logic

A non-empty domain (more than two elements)

Three categories:

- **Terms** representing the elements of the domain or functions on these elements
- **Relations**
- **Formulae** describing the interactions between the relations thanks to connectives and quantifiers
Structure of the first-order logic

A non-empty domain (more than two elements)

Three categories:

- the terms representing the elements of the domain or functions on these elements
- the relations
- the formulae describing the interactions between the relations thanks to connectives and quantifiers

Remark:

Two particular symbols (quantifiers): $\forall$ (universal quantification) and $\exists$ (existential quantification).
Structure of the first-order logic

Examples:

- the term $parent(x)$ is the parent of $x$,
- the formula $\forall x \exists y parent(y, x)$ indicates that every individual has a parent.
Syllogism

A cheap horse is rare.
Everything that is rare is expensive.
Hence a cheap horse is expensive.
Syllogism

A cheap horse is rare.
Everything that is rare is expensive.
Hence a cheap horse is expensive.

\[ \forall x(\text{horse}(x) \land \text{cheap}(x) \Rightarrow \text{rare}(x)) \]
\[ \forall x(\text{rare}(x) \Rightarrow \text{expensive}(x)) \]
\[ \forall x(\text{horse}(x) \land \text{cheap}(x) \Rightarrow \text{expensive}(x)) \]
Syllogism

A cheap horse is rare.
Everything that is rare is expensive.
Hence a cheap horse is expensive.

Nothing bothers you?
Syllogism

A cheap horse is rare.
Everything that is rare is expensive.
Hence a cheap horse is expensive.

Nothing bothers you?

Everything that is expensive is not cheap and vice versa.
A cheap horse is rare.
Everything that is rare is expensive.
Hence a cheap horse is expensive.

Nothing bothers you?

Everything that is expensive is not cheap and vice versa.

\[ \forall x (\text{cheap}(x) \iff \neg \text{expensive}(x)) \]
Syllogism

A cheap horse is rare.
Everything that is rare is expensive.
Hence a cheap horse is expensive.

Nothing bothers you?

Everything that is expensive is not cheap and vice versa.

\[ \forall x (\text{cheap}(x) \iff \neg \text{expensive}(x)) \]

This time the reasoning is contradictory.
The first-order logic allows to model:

- one non-empty domain,
- functions over the domain, and
- relations over the domain.
Overview

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Vocabulary

- Two propositional constants: \(\bot\) and \(\top\)
- Variables: sequence of letters and digits starting with one of the following lower case letters: \(u,v,w,x,y,z\).
- Connectives: \(\neg, \land, \lor, \Rightarrow, \Leftrightarrow\)
- Quantifiers: \(\forall\) the universal quantification and \(\exists\) the existential quantification
- Punctuation: the comma \(\langle, \rangle\) and the open \(\langle(\rangle\) and closing \(\langle)\rangle\) parenthesis.
- Ordinary and special symbols:
  - An ordinary symbol is a sequence of letters and digits not starting by one of the following lower case letters: \(u,v,w,x,y,z\).
  - A special symbol is \(+, -, \ast, /, =, \neq, <, \leq, >, \geq, \ldots\)
Example 4.1.1

- $x, x_1, x_2, y$ are variables,
- $\text{man, parent, succ, 12, 24, } f_1$ are ordinary symbols, the ordinary symbols will represent:
  - **functions** (numerical constants or multiple argument functions) or
  - **relations** (propositional variables or multiple argument relations).
- $x = y, z > 3$ are application examples of the special symbols
Term

Definition 4.1.2

- an ordinary symbol is a term,
- a variable is a term,
- if $t_1, \ldots, t_n$ are terms and if $s$ is a (ordinary or special) symbol then $s(t_1, \ldots, t_n)$ is a term.
Term

Definition 4.1.2

- an ordinary symbol is a term,
- a variable is a term,
- if $t_1, \ldots, t_n$ are terms and if $s$ is a (ordinary or special) symbol then $s(t_1, \ldots, t_n)$ is a term.

Example 4.1.3

$x, a, f(x_1, x_2, g(y)), + (x, \ast (y, z)), + (5, 42)$ are terms

on the contrary, $f(\perp, 2, y)$ is not a term.
Term

Definition 4.1.2

- an ordinary symbol is a term,
- a variable is a term,
- if $t_1, \ldots, t_n$ are terms and if $s$ is a (ordinary or special) symbol
  then $s(t_1, \ldots, t_n)$ is a term.

Example 4.1.3

$x, a, f(x_1, x_2, g(y)), + (x, \ast (y, z)), + (5, 42)$ are terms

on the contrary, $f(\bot, 2, y)$ is not a term.

Note that $42(1, y, 3)$ is also a term, but by convention function and relation names are ordinary symbols starting with letters.
Atomic formula

Definition 4.1.4 atomic formulae

- $\top$ and $\bot$ are atomic formulae
- an ordinary symbol is an atomic formula
- if $t_1, \ldots, t_n$ are terms and if $s$ is a (ordinary or special) symbol then $s(t_1, \ldots, t_n)$ is an atomic formula.

Example 4.1.5: $f(1, + (5, 42), g(z)), a$, and $+ (x, \ast (y, z))$ are atomic formulae.

$x$ and $A \lor f(4, 2, 6)$ are not atomic formulae.
Atomic formula

Definition 4.1.4 atomic formulae

- \( \top \) and \( \bot \) are atomic formulae
- an ordinary symbol is an atomic formula
- if \( t_1, \ldots, t_n \) are terms and if \( s \) is a (ordinary or special) symbol then \( s(t_1, \ldots, t_n) \) is an atomic formula.

Example 4.1.5 :

- \( f(1, +(5, 42), g(z)), a, \) and \( +(x, *(y, z)) \) are atomic formulae
- \( x \) and \( A \lor f(4, 2, 6) \) are not atomic formulae
Syntax v.s. Semantics

The set of terms and the set of atomic formulae are not disjoint.

For example $p(x)$ is a term and an atomic formula.

When $t$ is a term and an atomic formula simultaneously, we distinguish $[[t]]$, the value of $t$ seen as a term of $[t]$, value of $t$ seen as a formula.
(Strict) Formula

Definition 4.1.6

- an atomic formula is a formula,
- if $A$ is a formula then $\neg A$ is a formula,
- if $A$ and $B$ are formulae and if $\circ$ one of the following operations $\lor, \land, \Rightarrow, \iff$ then $(A \circ B)$ is a formula,
- if $A$ is a formula and if $x$ is any variable then $\forall x A$ and $\exists x A$ are formulae.
Example 4.1.7

- \( man(x) \), \( parents(son(y), mother(Alice)) \), \( = (x, +(f(x), g(y))) \)
  are atomic formulae, hence formulae.
- On the contrary

\[
\forall x \ (man(x) \Rightarrow man(Socrate))
\]

is a non-atomic formula.
(Strict) Formula : Examples

Among these expressions, which ones are strict formulae :

- $\forall x$
- $(a(x) \Rightarrow b) \land a(x) \Rightarrow b$
- $\exists x \exists y \langle -(x, y), + (a, y) \rangle$
- $(a < b \Rightarrow (2 \ast b) > (2 \ast a))$
(Strict) Formula : Examples

Among these expressions, which ones are strict formulae :

- $x$
  - no

- $a$
  - yes
(Strict) Formula : Examples

Among these expressions, which ones are strict formulae :

- $x$
  - no
- $a$
  - yes
- $(a(x) \Rightarrow b) \land a(x) \Rightarrow b$
  - no
(Strict) Formula : Examples

Among these expressions, which ones are strict formulae :

- $x$
  - no
- $a$
  - yes
- $(a(x) \Rightarrow b) \land a(x) \Rightarrow b$
  - no, missing ()
- $\exists x((\perp \Rightarrow a(x)) \land b(x))$
(Strict) Formula : Examples

Among these expressions, which ones are strict formulae :

- ▶ $x$
  - no

- ▶ $a$
  - yes

- ▶ $(a(x) \Rightarrow b) \land a(x) \Rightarrow b$
  - no, missing ()

- ▶ $\exists x ((\bot \Rightarrow a(x)) \land b(x))$
  - yes

- ▶ $\exists x \exists y < (-(x,y), +(a,y))$

Among these expressions, which ones are strict formulae:

- ▶ $x$
  - no

- ▶ $a$
  - yes

- ▶ $(a(x) \Rightarrow b) \land a(x) \Rightarrow b$
  - no, missing ()

- ▶ $\exists x((\bot \Rightarrow a(x)) \land b(x))$
  - yes

- ▶ $\exists x \exists y < (-(x,y),+(a,y))$
  - yes

- ▶ $((a < b) \Rightarrow ((2 \times b) > (2 \times a)))$
(Strict) Formula : Examples

Among these expressions, which ones are strict formulae :

- $x$
  - no

- $a$
  - yes

- $(a(x) \Rightarrow b) \land a(x) \Rightarrow b$
  - no, missing ()

- $\exists x((\bot \Rightarrow a(x)) \land b(x))$
  - yes

- $\exists x \exists y < (-(x, y), +(a, y))$
  - yes

- $((a < b) \Rightarrow ((2 \ast b) > (2 \ast a)))$
  - no
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Abbreviate the writing

**Prioritized formulae**: the symbols of the functions $+,-,\times,\div$ and the symbols of the relations $=,\neq,<,>,\leq,\geq$ are written in the usual manner.
Abbreviate the writing

**Prioritized formulae** : the symbols of the functions $+, -, *, /$ and the symbols of the relations $=, \neq, <, >, \leq, \geq$ are written in the usual manner.

**Example 4.1.8**

$\leq (\ast (3, x), + (y, 5))$ is abbreviated as
Abbreviate the writing

**Prioritized formulae:** the symbols of the functions $+, -, *, /$ and the symbols of the relations $=, \neq, <, >, \leq, \geq$ are written in the usual manner.

**Example 4.1.8**

- $\leq (\ast(3, x), +(y, 5))$ is abbreviated as $3 \ast x \leq y + 5$
- $+(x, \ast(y, z))$ is abbreviated as
Abbreviate the writing

**Prioritized formulae**: the symbols of the functions $+, -, *, /$ and the symbols of the relations $=, \neq, <, >, \leq, \geq$ are written in the usual manner.

### Example 4.1.8

- $\leq (\ast(3, x), + (y, 5))$ is abbreviated as $3 \ast x \leq y + 5$
- $+(x, \ast(y, z))$ is abbreviated as $x + y \ast z$
The inverse transformation

Prioritize

- the connectives have a lower priority than the relations
- the priority of the quantifiers is identical to that of the negation.
- $=, \neq, <, \leq, >, \geq$ have a lower priority than $+, -, *, /$
Table 4.1 summary of priorities

Priorities decreasing from top to bottom.

<table>
<thead>
<tr>
<th>OPERATIONS</th>
<th>RELATIONS</th>
<th>NEGATION, QUANTIFIERS</th>
<th>BINARY CONNECTIVES</th>
</tr>
</thead>
<tbody>
<tr>
<td>−, + unary</td>
<td>=, ≠, &lt;, ≤, &gt;, ≥</td>
<td>¬, ∀, ∃</td>
<td>left associative</td>
</tr>
<tr>
<td>*, /</td>
<td></td>
<td></td>
<td>left associative</td>
</tr>
<tr>
<td>+, − binary</td>
<td></td>
<td></td>
<td>right associative</td>
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<tr>
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<td>right associative</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>left associative</td>
</tr>
</tbody>
</table>
Prioritized formulae

Definition 4.1.9 prioritized formulae

A prioritized formula is:

- An atomic formula.
- If $A$ is a prioritized formula then $\neg A$ is a prioritized formula.
- If $A$ and $B$ are prioritized formulae then $A \circ B$ is a prioritized formula.
- If $A$ is a prioritized formula and if $x$ is any variable then $\forall x\ A$ and $\exists x\ A$ are prioritized formulae.
- If $A$ is a prioritized formula (A) is a prioritized formula.
Examples

Example 4.1.10

\[ \forall x P(x) \land \forall x Q(x) \iff \forall x (P(x) \land Q(x)) \]

is an abbreviation of
Examples

**Example 4.1.10**

- \( \forall x P(x) \land \forall x Q(x) \iff \forall x (P(x) \land Q(x)) \) is an abbreviation of

  \[
  (((\forall x P(x)) \land \forall x Q(x)) \iff \forall x (P(x) \land Q(x)))
  \]

- \( \forall x \forall y \forall z (x \leq y \land y \leq z \Rightarrow x \leq z) \) is an abbreviation of ?
Examples

Example 4.1.10

- $\forall x P(x) \land \forall x Q(x) \iff \forall x (P(x) \land Q(x))$ is an abbreviation of

  $((\forall x P(x) \land \forall x Q(x)) \iff \forall x (P(x) \land Q(x)))$

- $\forall x \forall y \forall z (x \leq y \land y \leq z \Rightarrow x \leq z)$ is an abbreviation of

  $\forall x \forall y \forall z ((\leq (x, y) \land \leq (y, z)) \Rightarrow \leq (x, z))$
Tree representation

Example 4.1.11 \( \forall x P(x) \Rightarrow Q(x) \)

the left-hand side operand of the implication is \( \forall x P(x) \).
Tree representation

Example 4.1.11 $\forall x P(x) \Rightarrow Q(x)$

the left-hand side operand of the implication is $\forall x P(x)$. 

$\Rightarrow$

$\forall x$

$\Rightarrow$

$Q$

$\Rightarrow$

$P$

$\Rightarrow$

$x$

$\Rightarrow$

$x$
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Idea

- The truth value of the formula $x + 2 = 4$ depends on $x$. The formula is not true (in arithmetics) unless $x = 2$. $x$ is free in the formula.
The truth value of the formula $x + 2 = 4$ depends on $x$. The formula is not true (in arithmetics) unless $x = 2$. $x$ is free in the formula.

- $\forall x (x + 2 = 4)$ is false
- $\forall x (x + 0 = x)$ is true

Regardless of the values of $x$, these two formulae do not contain free variables.
Free and bound occurrences

Definition 4.2.1

- In $\forall x A$ or $\exists x A$, the **scope of the binding of** $x$ is $A$.  
Free and bound occurrences

Definition 4.2.1

- In $\forall x \ A$ or $\exists x \ A$, the **scope of the binding** of $x$ is $A$.
- An occurrence of $x$ in $A$ is **free** if it is not in the scope of a binding of $x$, otherwise it is said to be **bound**.
Free and bound occurrences

**Definition 4.2.1**

- In $\forall x A$ or $\exists x A$, the **scope of the binding** of $x$ is $A$.
- An occurrence of $x$ in $A$ is **free** if it is not in the scope of a binding of $x$, otherwise it is said to be **bound**.

If we represent a formula by a tree:

- A bound occurrence of $x$ is
Free and bound occurrences

Definition 4.2.1

- In $\forall x \ A$ or $\exists x \ A$, the **scope of the binding** of $x$ is $A$.
- An occurrence of $x$ in $A$ is **free** if it is not in the scope of a binding of $x$, otherwise it is said to be **bound**

If we represent a formula by a tree:

- A bound occurrence of $x$ is
  
  below a node $\exists x$ or $\forall x$.

- An occurrence of $x$ is free if
Free and bound occurrences

Definition 4.2.1

- In $\forall x A$ or $\exists x A$, the **scope of the binding** of $x$ is $A$.
- An occurrence of $x$ in $A$ is **free** if it is not in the scope of a binding of $x$, otherwise it is said to be **bound**.

If we represent a formula by a tree:

- A bound occurrence of $x$ is below a node $\exists x$ or $\forall x$.
- An occurrence of $x$ is free if is not below such a node.
Example 4.2.2

$$\forall x P(x, y) \land \exists z R(x, z)$$
Example 4.2.2

\( \forall x P(x, y) \land \exists z R(x, z) \)
Example 4.2.2

\[ \forall x P(x, y) \land \exists z R(x, z) \]

- The bold occurrence of \( x \) is bound.
- The underlined occurrence of \( x \) is free.
- The occurrence of \( z \) is bound.
Free, bound variables

Definition 4.2.3

- The variable $x$ is a **free variable** of a formula if and only if there is a free occurrence of $x$ in the formula.
- A variable $x$ is a **bound variable** of a formula if and only if there is a bound occurrence of $x$ in the formula.
- A formula without free variable is also called a **closed formula**.
Free, bound variables

Definition 4.2.3

- The variable $x$ is a **free variable** of a formula if and only if there is a free occurrence of $x$ in the formula.
- A variable $x$ is a **bound variable** of a formula if and only if there is a bound occurrence of $x$ in the formula.
- A formula without free variable is also called a **closed formula**.

Remark 4.2.4

A variable can be simultaneously free and bound. For example, in the formula $\forall x P(x) \lor Q(x)$, $x$ is both free and bound.

Remark 4.2.5

By definition, a variable which does not appear in a formula (0 occurrence) is by definition a **NON free variable** of the formula.
Free, bound variables

Definition 4.2.3

- The variable $x$ is a **free variable** of a formula if and only if there is a free occurrence of $x$ in the formula.
- A variable $x$ is a **bound variable** of a formula if and only if there is a bound occurrence of $x$ in the formula.
- A formula without free variable is also called a **closed formula**.

Remark 4.2.4

A variable can be simultaneously free and bound. For example, in the formula $\forall x P(x) \lor Q(x)$, $x$ is both free and bound.

Remark 4.2.5

By definition, a variable which does not appear in a formula (0 occurrence) is by definition a **NON** free variable of the formula.

Example 4.2.6

The free variables of the formula of example 4.2.2 are $x$ and $y$. 
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Declaring a symbol

Definition 4.3.1

A symbol declaration is a triple denoted $s^{gn}$ where:

- $s$ is a symbol
- $g$ one of the letters $f$ (meaning function) or $r$ (meaning relation)
- $n$ is a natural number.
Declaring a symbol

**Definition 4.3.1**

A symbol declaration is a triple denoted $s^{gn}$ where:

- $s$ is a symbol
- $g$ one of the letters $f$ (meaning function) or $r$ (meaning relation)
- $n$ is a natural number.

**Remark 4.3.3**

If the context gives an implicit declaration of a symbol, we omit the exponent.

*Example:* `equal` is always a 2 arguments relation, we abbreviate the declaration $=^{r2}$ as $=.$
Symbol declaration: Example

Example 4.3.2

- $parent^{r_2}$ is a relation (r) with 2 arguments
- $^\ast f_2$ is function (f) with 2 arguments
- $man^{r_1}$ a unary relation
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Definition 4.3.4

A signature is a set of symbol declarations.

Let $n > 0$ and $\Sigma$ a signature, the symbol $s$ is:

1. a constant of the signature if and only if $s^{f0} \in \Sigma$

2. one symbol of the function of $n$ arguments of the signature, if and only if $s^{fn} \in \Sigma$

3. a propositional variable of the signature if and only if $s^{r0} \in \Sigma$

4. a symbol of the relation of $n$ arguments of the signature, if and only if $s^{rn} \in \Sigma$
Examples in mathematics (1/2)

$0^{f_0}, 1^{f_0}, +^{f_2}, -^{f_2}, \ast^{f_2}, =^{r_2}$ is a signature for the arithmetics.
Examples in mathematics (1/2)

$0^f_0, 1^f_0, +^f_2, ^f_2, ^f_2, ^r_2$ is a signature for the arithmetics.

Remark:

- We write: 0, 1, + and − (with two arguments), * and =.
- Note that the plus and the minus have two arguments, (since we can meet the symbol plus with only one argument).
Examples in mathematics (2/2)

Example 4.3.5 set theory

A possible signature is $\in, =$. 
Examples in mathematics (2/2)

Example 4.3.5 set theory

A possible signature is $\in$, $=\$

the other operations can be defined starting from these symbols.
Overload

Definition 4.3.6
A symbol is overloaded in a signature, when this signature has two distinct declarations of the same symbol.
Definition 4.3.6

A symbol is overloaded in a signature, when this signature has two distinct declarations of the same symbol.

Example 4.3.7: the minus sign is often overloaded.

- the opposite of a number
- the subtraction of two numbers
Overload

Definition 4.3.6
A symbol is overloaded in a signature, when this signature has two distinct declarations of the same symbol.

Example 4.3.7: the minus sign is often overloaded.
  ▶ the opposite of a number
  ▶ the subtraction of two numbers

In what follows, in this course, we will prohibit the use of signatures having overloaded symbols.
Term over a signature

Definition 4.3.8

Let $\Sigma$ a signature, a term over $\Sigma$ is:

- either a variable,
- either a constant $s$ where $s^{f_0} \in \Sigma$,
- either a term of the form $s(t_1, \ldots, t_n)$, where $n \geq 1$, $s^{f_n} \in \Sigma$ and $t_1, \ldots, t_n$ are terms over $\Sigma$. 
Term over a signature

Definition 4.3.8

Let $\Sigma$ a signature, a term over $\Sigma$ is:

- either a variable,
- either a constant $s$ where $s^{f_0} \in \Sigma$,
- either a term of the form $s(t_1, \ldots, t_n)$, where $n \geq 1$, $s^{f_n} \in \Sigma$ and $t_1, \ldots, t_n$ are terms over $\Sigma$.

The set of terms over the signature $\Sigma$ is denoted $T_\Sigma$. 
Atomic formula over a signature

Definition 4.3.9

Let $\Sigma$ a signature, an atomic formula over $\Sigma$ is:

- either one of the constants $\top, \bot$,
- either a propositional variable $s$ where $s^{r_0} \in \Sigma$,
- either of the form $s(t_1, \ldots, t_n)$ where $n \geq 1$, $s^{r_n} \in \Sigma$ and where $t_1, \ldots, t_n$ are terms over $\Sigma$. 

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Formula over a signature

Definition 4.3.10

A formula over a signature $\Sigma$ is a formula, whose atomic sub-formulae are atomic formulae over $\Sigma$ (according to definition 4.3.9).
Formula over a signature

**Definition 4.3.10**

A formula over a signature \( \Sigma \) is a formula, whose atomic sub-formulae are atomic formulae over \( \Sigma \) (according to definition 4.3.9).

We denote the set of formulae over the signature \( \Sigma \) by \( F_\Sigma \).
Example 4.3.11

\[ \forall x \ (p(x) \Rightarrow \exists y \ q(x, y)) \] is a formula over the signature \( \Sigma = \{ p^{r_1}, q^{r_2}, h^{f_1}, c^{f_0} \}. \]
Example 4.3.11

\( \forall x \ (p(x) \Rightarrow \exists y \ q(x, y)) \) is a formula over the signature \( \Sigma = \{ p^{r_1}, q^{r_2}, h^{f_1}, c^{f_0} \} \).

But it is also a formula over the signature \( \Sigma' = \{ p^{r_1}, q^{r_2} \} \), since the symbols \( h \) and \( c \) are not in the formula.
Associated signature

Definition 4.3.12
The associated signature of a formula is the smallest signature \( \Sigma \) so that the formula is element of \( F_\Sigma \), it is the smallest signature allowing to write the formula.
Associated signature

**Definition 4.3.12**

The associated signature of a formula is the smallest signature $\Sigma$ so that the formula is element of $F_\Sigma$, it is the smallest signature allowing to write the formula.

**Example 4.3.13**

The associated signature of the formula $\forall x (p(x) \Rightarrow \exists y q(x, y))$ is
Associated signature

Definition 4.3.12

The associated signature of a formula is the smallest signature $\Sigma$ so that the formula is element of $F_{\Sigma}$, it is the smallest signature allowing to write the formula.

Example 4.3.13

The associated signature of the formula $\forall x (p(x) \Rightarrow \exists y q(x, y))$ is $p^{r_1}, q^{r_2}$.
Associated signature

Definition 4.3.14

The associated signature to a set of formulae is the union of the associated signatures of every formula of the set.
Definition 4.3.14

The **associated signature** to a set of formulae is the union of the associated signatures of every formula of the set.

Example 4.3.15

The associated signature of a set of two formulae

\[ \forall x (p(x) \Rightarrow \exists y q(x, y)), \forall u \forall v (u + s(v) = s(u) + v) \]

is

\[ \sum = \{ p^{r_1}, q^{r_2}, +^{f_2}, s^{f_1}, =^{r_2} \}. \]
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Definition 4.3.16

An interpretation $I$ over a signature $\Sigma$ is defined by a non-empty domain $D$ and an application which associates to every symbol declaration $s_{gn} \in \Sigma$ its value $s^I_{gn}$ as follows:

1. $s^I_{f0}$ is an element of $D$.
2. $s^I_{fn}$ where $n \geq 1$ is a function of $D^n$ in $D$, in other words a function of $n$ arguments.
3. $s^I_{r0}$ equals 0 or 1.
4. $s^I_{rn}$ where $n \geq 1$ is a subset of $D^n$, in other words a relation of $n$ arguments.
Example 4.3.17

Let $I$ the interpretation of domain $D = \{1, 2, 3\}$ where the binary relation $friend$ is true for the pairs $(1, 2)$, $(1, 3)$ and $(2, 3)$, i.e., $friend^r_I = \{(1, 2), (1, 3), (2, 3)\}$.

$friend(2, 3)$ is true in the interpretation $I$. On the other hand, $friend(2, 1)$ is false in the interpretation $I$. 
Example 4.3.17

Let $I$ the interpretation of domain $D = \{1, 2, 3\}$ where the binary relation $\text{friend}$ is true for the pairs $(1, 2)$, $(1, 3)$ and $(2, 3)$, i.e., $\text{friend}^I = \{(1, 2), (1, 3), (2, 3)\}$.

$\text{friend}(2, 3)$ is true in the interpretation $I$. On the other hand, $\text{friend}(2, 1)$ is false in the interpretation $I$.

Remark 4.3.18

In every interpretation $I$, the value of the symbol $=$ is the set $\{(d, d) \mid d \in D\}$, in other words in every interpretation the truth value of the equality is the identity over the domain of interpretation.
Example 4.3.19

Let us consider the following signature.

- \( \text{Anne}^0 \), \( \text{Bernard}^0 \) and \( \text{Claude}^0 \): the first names Anne, Bernard, and Claude denote constants,
- \( a^2 \): the letter \( a \) denotes a two-argument relation (we read \( a(x, y) \) as \( x \) likes \( y \)) and
- \( c^1 \): the letter \( c \) denotes a single argument function (we read \( c(x) \) as the friend of \( x \)).
Example 4.3.19

Let us consider the following signature.

- $Anne^f_0$, $Bernard^f_0$ and $Claude^f_0$: the first names Anne, Bernard, and Claude denote constants,
- $a^{r_2}$: the letter $a$ denotes a two-argument relation (we read $a(x, y)$ as $x$ likes $y$) and
- $c^{f_1}$: the letter $c$ denotes a single-argument function (we read $c(x)$ as the friend of $x$).

A possible interpretation over this signature is the interpretation $I$ of domain $D = \{0, 1, 2\}$ where:

- $Anne^f_I_0 = 0$, $Bernard^f_I_0 = 1$, and $Claude^f_I_0 = 2$.
- $a^{r_2}_I = \{(0, 1), (1, 0), (2, 0)\}$.
- $c^{f_1}_I(0) = 1$, $c^{f_1}_I(1) = 0$, $c^{f_1}_I(2) = 2$. Note that the function $c^{f_1}_I$ has $D$ as domain, which makes it necessary to artificially define $c^{f_1}_I(2)$: Claude, denoted by 2, has no friend.
Interpretation of a set of formulae

Definition 4.3.20

The interpretation of a set of formulae is an interpretation which only defines the truth value of the signature associated to the set of formulae.
State, assignment

**Definition 4.3.21**

A state $e$ of an interpretation is an application from the set of variables to the interpretation domain.
State, assignment

**Definition 4.3.21**

A state $e$ of an interpretation is an application from the set of variables to the interpretation domain.

**Definition 4.3.22**

An assignment is a pair $(I, e)$ composed of an interpretation $I$ and a state $e$. 
Example 4.3.23

Let the domain $D = \{1, 2, 3\}$ and the interpretation $I$ where the binary relation $\text{friend}$ is true only for the pairs $(1, 2)$, $(1, 3)$ and $(2, 3)$, i.e., $\text{friend}_I^r = \{(1, 2), (1, 3), (2, 3)\}$.

Let $e$ the state which associates 2 to $x$ and 1 to $y$.

The assignment $(I, e)$ makes the relation $\text{friend}(x, y)$ false.
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Remark 4.3.24

The value of a formula depends only on its free variables and on its symbols, also in order to evaluate a formula without any free variables, the state of the variables is useless. There are two possible cases:

▶ For a formula with no free variables, simply give an interpretation \( I \) of the symbols of the formula. In this case, the assignments \((I, e)\) and \((I, e')\) will give the formula the same value for all the states \(e\) and \(e'\). Thus for any state \(e\), we’ll identify \((I, e)\) and \(I\). Depending on the context, \(I\) will be considered either as an interpretation or as an assignment of any state.

▶ For a formula with free variables, we therefore need an assignment.
Terms

Definition 4.3.25 Evaluation

The evaluation of a term $t$ is inductively defined as:

1. If $t$ is a variable, then $\llbracket t \rrbracket_{(I,e)} = e(t)$,
2. If $t$ is a constant, then $\llbracket t \rrbracket_{(I,e)} = t^0_I$,
3. If $t = s(t_1, \ldots, t_n)$ where $s$ is a symbol and $t_1, \ldots, t_n$ are terms, then $\llbracket t \rrbracket_{(I,e)} = s^f_n(\llbracket t_1 \rrbracket_{(I,e)}, \ldots, \llbracket t_n \rrbracket_{(I,e)})$. 
Example 4.3.26

Let $I$ the interpretation of domain $\mathbb{N}$ which gives to the declaration of the symbols $1^f_0, *^f_2, +^f_2$ their usual value over the integers.
Let $e$ the state such that $x = 2, y = 3$.
Compute $\langle x \ast (y + 1) \rangle_{(I,e)}$. 
Example 4.3.26

Let $I$ the interpretation of domain $\mathbb{N}$ which gives to the declaration of the symbols $1^f, \ast^f, +^f$ their usual value over the integers. Let $e$ the state such that $x = 2, y = 3$. Compute $[x \ast (y + 1)]_{(I, e)}$.

$$[x \ast (y + 1)]_{(I, e)} = [x]_{(I, e)} \ast [(y + 1)]_{(I, e)} = [x]_{(I, e)} \ast ([y]_{(I, e)} + [1]_{(I, e)}) = e(x) \ast (e(y) + 1) = 2 \ast (3 + 1) = 8.$$
Definition 4.3.27 truth value of atomic formulae

The truth value of atomic formulae is given by the following inductive rules:

1. \([\top]_{(l,e)} = 1, [\bot]_{(l,e)} = 0\). In the example, we allow the replacement of \(\top\) by its value 1 and \(\bot\) by its value 0.

2. Let \(s\) a propositional variable, \([s]_{(l,e)} = s^0_{l}\).

3. Let \(A = s(t_1, \ldots, t_n)\) where \(s\) is a symbol and \(t_1, \ldots, t_n\) are terms. If \(([t_1]_{(l,e)}, \ldots, [t_n]_{(l,e)}) \in s^{rn}_l\) then \([A]_{(l,e)} = 1\) else \([A]_{(l,e)} = 0\).
Example 4.3.31

Let the interpretation $I$ of domain $D = \{1, 2, 3\}$ where the binary relation $friend$ is true for the pairs $(1, 2), (1, 3)$ and $(2, 3)$, i.e., $friend_I^2 = \{(1, 2), (1, 3), (2, 3)\}$. 
Example 4.3.31

Let the interpretation $I$ of domain $D = \{1, 2, 3\}$ where the binary relation $friend$ is true for the pairs $(1, 2)$, $(1, 3)$ and $(2, 3)$, i.e., $friend^I = \{(1, 2), (1, 3), (2, 3)\}$.

The formula $friend(1, 2) \land friend(2, 3) \Rightarrow friend(1, 3)$ is true in the interpretation $I$, i.e., $[friend(1, 2) \land friend(2, 3) \Rightarrow friend(1, 3)]_I = 1$. 
Example 4.3.29

Let us consider the following signature.

- $\text{Anne}^{f_0}$, $\text{Bernard}^{f_0}$ and $\text{Claude}^{f_0}$: the first names Anne, Bernard, and Claude denote constants,
- $a^{r_2}$: letter $a$ denotes a two-argument relation (we read $a(x, y)$ as $x$ likes $y$) and
- $c^{f_1}$: the letter $c$ denotes a one-argument function (we read $c(x)$ as the friend of $x$).

Let $I$ the interpretation of domain $D = \{0, 1, 2\}$ over this signature where:

- $\text{Anne}_I^{f_0} = 0$, $\text{Bernard}_I^{f_0} = 1$, and $\text{Claude}_I^{f_0} = 2$.
- $a_I^{r_2} = \{(0, 1), (1, 0), (2, 0)\}$.
- $c_I^{f_1}(0) = 1$, $c_I^{f_1}(1) = 0$, $c_I^{f_1}(2) = 2$. Note that the function $c_I^{f_1}$ has $D$ as domain, which makes it necessary to artificially define $c_I^{f_1}(2)$: Claude, denoted by 2, has no friend.
Example 4.3.29

We obtain:

- \([a(\text{Anne}, \text{Bernard})]_I = 1\) since \((J_{\text{Anne}} K_I, J_{\text{Bernard}} K_I) = (0, 1) \in a_{\text{arb}}^2 I_2\).

- \([a(\text{Anne}, \text{Claude})]_I = 0\) since \((J_{\text{Anne}} K_I, J_{\text{Claude}} K_I) = (0, 2) \notin a_{\text{arb}}^2 I_2\).
Example 4.3.29

We obtain:

- \([a(Anne, Bernard)]_I = 1\) since \((\llbracket Anne \rrbracket_I, \llbracket Bernard \rrbracket_I) = (0, 1) \in a_I^2\).

- \([a(Anne, Claude)]_I = \)
Example 4.3.29

We obtain:

1. \([a(Anne, Bernard)]_I = 1\) since \((\llbracket Anne \rrbracket_I, \llbracket Bernard \rrbracket_I) = (0, 1) \in a^r_2\).

2. \([a(Anne, Claude)]_I = 0\) since \((\llbracket Anne \rrbracket_I, \llbracket Claude \rrbracket_I) = (0, 2) \notin a^r_2\).
Example 4.3.29

Let $e$ the state $x = 0, y = 2$. We have:

- $[a(x, c(x))]_{(I,e)} =$
Example 4.3.29

Let $e$ the state $x = 0, y = 2$. We have:

- $[a(x, c(x))]_{(l,e)} = 1$ since $(\llbracket x \rrbracket_{(l,e)}, \llbracket c(x) \rrbracket_{(l,e)}) = (0, c^f_1(\llbracket x \rrbracket_{(l,e)})) = (0, c^f_1(0)) = (0, 1) \in a^2_f$.

- $[a(y, c(y))]_{(l,e)} =$
Example 4.3.29

Let $e$ the state $x = 0$, $y = 2$. We have:

- $[a(x, c(x))]_{(l,e)} =$
  
  $1$ since \( ([x]_{(l,e)}, [c(x)]_{(l,e)}) = (0, c^f_1([x]_{(l,e)})) = (0, c^f_1(0)) = (0, 1) \in a^r_2 \).

- $[a(y, c(y))]_{(l,e)} =$
  
  $0$ since \( ([y]_{(l,e)}, [c(y)]_{(l,e)}) = (2, c^f_1([y]_{(l,e)})) = (2, c^f_1(2)) = (2, 2) \notin a^r_2 \).

Make sure to distinguish (depending on the context), the elements of the domain $0, 1$ and the truth values $0, 1$. 
Example 4.3.29

We have:

- \([\langle (Anne = Bernard) \rangle]_I = \)
Example 4.3.29

We have:

- $[(\text{Anne} = \text{Bernard})]_I = 0$, since $(\llbracket \text{Anne} \rrbracket_I, \llbracket \text{Bernard} \rrbracket_I) = (0, 1) \notin r^2_I$.

- $[(c(\text{Anne}) = \text{Anne})]_I = $
Example 4.3.29

We have:

- $[(Anne = Bernard)]_I =$
  
  0, since $(\llbracket Anne \rrbracket_I, \llbracket Bernard \rrbracket_I) = (0, 1) \not\in r^2_I$.

- $[(c(Anne) = Anne)]_I =$
  
  0, since $(\llbracket c(Anne) \rrbracket_I, \llbracket Anne \rrbracket_I) = (c^f_1(\llbracket Anne \rrbracket_I), 0) = (c^f_1(0), 0) = (1, 0) \not\in r^2_I$.

- $[(c(c(Anne)) = Anne)]_I =$
  
  0, since $(\llbracket c(c(Anne)) \rrbracket_I, \llbracket Anne \rrbracket_I) = (c^f_1(\llbracket c(Anne) \rrbracket_I), 0) = (c^f_1(c^f_1(0)), 0) = (c^f_1(1), 0) = (0, 0) \in r^2_I$.
Example 4.3.29

We have:

- $\left[ (\text{Anne} = \text{Bernard}) \right]_I = 0$, since $(\llbracket \text{Anne} \rrbracket_I, \llbracket \text{Bernard} \rrbracket_I) = (0, 1) \notin =_I^2$.

- $\left[ (c(\text{Anne}) = \text{Anne}) \right]_I = 0$, since $(\llbracket c(\text{Anne}) \rrbracket_I, \llbracket \text{Anne} \rrbracket_I) = (c^f_1(\llbracket \text{Anne} \rrbracket_I), 0) = (c^f_1(0), 0) = (1, 0) \notin =_I^2$.

- $\left[ (c(c(\text{Anne})) = \text{Anne}) \right]_I = 1$, since $(\llbracket c(c(\text{Anne})) \rrbracket_I, \llbracket \text{Anne} \rrbracket_I) = (c^f_1(\llbracket c(\text{Anne}) \rrbracket_I), 0) = (c^f_1(0), 0) = (1, 0) \in =_I^2$. 
1. The propositional connectives have the same meaning as in propositional logic.

2. Let $x$ a variable and $B$ a formula. $[\forall xB]_{(I,e)} = 1$ if and only if $[B]_{(I,f)} = 1$ for all state $f$ identical to $e$, except for $x$. Let $d \in D$. Let us denote $e[x=d]$ the state identical to the $e$, except for the variable $x$, whose state $e[x=d]$ associates the value $d$. The above definition can be written as:

$$[\forall xB]_{(I,e)} = \min_{d \in D} [B]_{(I,e[x=d])} = \prod_{d \in D} [B]_{(I,e[x=d])},$$

where the product is the boolean product.

3. $[\exists xB]_{(I,e)} = 1$ if and only if there is a state $f$ identical to $e$, except for $x$, so that $[B]_{(I,f)} = 1$. The above definition can be written as:

$$[\exists xB]_{(I,e)} = \max_{d \in D} [B]_{(I,e[x=d])} = \sum_{d \in D} [B]_{(I,e[x=d])},$$

where the sum is the boolean sum.
Example 4.3.32

Let us use the interpretation $I$ given in example 4.3.19.
(Reminder $D = \{0, 1, 2\}$)

$[\exists x \ a(x, x)]_I = \text{max}\{[a(0, 0)]_I, [a(1, 1)]_I, [a(2, 2)]_I\} = 0$ since $(0, 0), (1, 1), (2, 2) \notin a^2_I$.

According to the definition, we have:

$[\exists x \ a(x, x)]_I = [a(0, 0)]_I + [a(1, 1)]_I + [a(2, 2)]_I = 0 + 0 + 0 = 0$.

$[\forall x \ \exists y \ a(x, y)]_I = \text{min}\{\text{max}\{[a(0, 0)]_I, [a(0, 1)]_I, [a(0, 2)]_I\}, \text{max}\{[a(1, 0)]_I, [a(1, 1)]_I, [a(1, 2)]_I\}, \text{max}\{[a(2, 0)]_I, [a(2, 1)]_I, [a(2, 2)]_I\}\}$

$= \text{min}\{1, 1, 1\} = 1$.

According to the definition, we have:

$[\forall x \ \exists y \ a(x, y)]_I = ([a(0, 0)]_I + [a(0, 1)]_I + [a(0, 2)]_I, [a(1, 0)]_I + [a(1, 1)]_I + [a(1, 2)]_I, [a(2, 0)]_I + [a(2, 1)]_I + [a(2, 2)]_I) = (0 + 1 + 0, 1 + 0 + 0, 1 + 0 + 0) = (1, 1, 1) = 1$. 

$[\forall x \ a(x, x)]_I = 1$. 

$[\forall x \ \exists y \ a(x, y)]_I = \text{min}\{\text{max}\{[a(0, 0)]_I, [a(0, 1)]_I, [a(0, 2)]_I\}, \text{max}\{[a(1, 0)]_I, [a(1, 1)]_I, [a(1, 2)]_I\}, \text{max}\{[a(2, 0)]_I, [a(2, 1)]_I, [a(2, 2)]_I\}\}$

$= \text{min}\{1, 1, 1\} = 1$.
Example 4.3.32

Let us use the interpretation $I$ given in example 4.3.19. (Reminder $D = \{0, 1, 2\}$)

- $[\exists x \ a(x, x)]_I =$
  
  $\max\{[a(0, 0)]_I, [a(1, 1)]_I, [a(2, 2)]_I\} = 0$ since $(0, 0), (1, 1), (2, 2) \not\in a^2_I$. According to the definition, we have: $[\exists x \ a(x, x)]_I = [a(0, 0)]_I + [a(1, 1)]_I + [a(2, 2)]_I = 0 + 0 + 0 = 0$. 

- $[\forall x \exists y \ a(x, y)]_I = $
Example 4.3.32

Let us use the interpretation $I$ given in example 4.3.19.
(Reminder $D = \{0, 1, 2\}$)

$\exists x \ a(x, x) \implies$

$max\{[a(0, 0)]_I, [a(1, 1)]_I, [a(2, 2)]_I\} = 0 \text{ since } (0, 0), (1, 1), (2, 2) \notin a^2_I.$

According to the definition, we have: $[\exists x \ a(x, x)]_I = [a(0, 0)]_I + [a(1, 1)]_I + [a(2, 2)]_I = 0 + 0 + 0 = 0.$

$\forall x \exists y \ a(x, y) \implies$

$\min\{\max\{[a(0, 0)]_I, [a(0, 1)]_I, [a(0, 2)]_I\}, \max\{[a(1, 0)]_I, [a(1, 1)]_I, [a(1, 2)]_I\}, \max\{[a(2, 0)]_I, [a(2, 1)]_I, [a(2, 2)]_I\}\} = \min\{\max\{0, 1, 0\}, \max\{1, 0, 0\}\} = \min\{1, 1, 1\} = 1.$

According to the definition, we have: $[\forall x \exists y \ a(x, y)]_I = ([a(0, 0)]_I + [a(0, 1)]_I + [a(0, 2)]_I). ([a(1, 0)]_I + [a(1, 1)]_I + [a(1, 2)]_I). ([a(2, 0)]_I + [a(2, 1)]_I + [a(2, 2)]_I) = (0 + 1 + 0). (1 + 0 + 0). (1 + 0 + 0) = 1.1.1 = 1.$
Example 4.3.32

\[\exists y \forall x \, a(x, y)\] \(= \)

According to the definition, we have:

\[\exists y \forall x \, a(x, y)\] \(=\) \([a(0, 0)]_I\), \([a(1, 0)]_I\), \([a(2, 0)]_I\) + \([a(0, 1)]_I\), \([a(1, 1)]_I\), \([a(2, 1)]_I\) + \([a(0, 2)]_I\), \([a(1, 2)]_I\), \([a(2, 2)]_I\) = \([0, 0, 0]_I\) = 0.

Remark 4.3.33

The formulae \(\forall x \exists y \, a(x, y)\) and \(\exists y \forall x \, a(x, y)\) do not have the same value. So by interchanging an existential quantification and an universal quantification, the truth value of formulae is not preserved.
Example 4.3.32

\[ \exists y \forall x \ a(x, y) \] \]

\[
\begin{align*}
\max \{ \min \{ & [a(0, 0)], [a(1, 0)], [a(2, 0)] \}, \min \{ [a(0, 1)], [a(1, 1)], [a(2, 1)] \}, \min \{ [a(0, 2)], [a(1, 2)], [a(2, 2)] \} \} = \\
\max \{ \min \{ & 0, 1, 1 \}, \min \{1, 0, 0\}, \min \{0, 0, 0\} \} = \max \{0, 0, 0\} = 0.
\end{align*}
\]

According to the definition, we have:

\[ \exists y \forall x \ a(x, y) \] \]

\[
\begin{align*}
[a(0, 0)] \cdot [a(1, 0)] \cdot [a(2, 0)] + [a(0, 1)] \cdot [a(1, 1)] \cdot [a(2, 1)] + [a(0, 2)] \cdot [a(1, 2)] \cdot [a(2, 2)] = 0.1.1 + 1.0.0 + 0.0.0 = 0 + 0 + 0 = 0.
\end{align*}
\]
Example 4.3.32

\[ \exists y \forall x a(x, y) \]_I =

\[
\max\{\min\{[a(0,0)]_I, [a(1,0)]_I, [a(2,0)]_I\}, \min\{[a(0,1)]_I, [a(1,1)]_I, [a(2,1)]_I\}, \min\{[a(0,2)]_I, [a(1,2)]_I, [a(2,2)]_I\}\} = \max\{\min\{0, 1, 1\}, \min\{1, 0, 0\}, \min\{0, 0, 0\}\} = \max\{0, 0, 0\} = 0.
\]

According to the definition, we have:

\[ \exists y \forall x a(x, y) \]_I =

\[ [a(0,0)]_I, [a(1,0)]_I, [a(2,0)]_I + [a(0,1)]_I, [a(1,1)]_I, [a(2,1)]_I + [a(0,2)]_I, [a(1,2)]_I, [a(2,2)]_I \] = 0.1.1 + 1.0.0 + 0.0.0 = 0 + 0 + 0 = 0.

Remark 4.3.33

The formulae \( \forall x \exists y a(x, y) \) and \( \exists y \forall x a(x, y) \) do not have the same value. So by interchanging an existential quantification and an universal quantification, the truth value of formulae is not preserved.
Overview

Introduction

Language
  (Strict) Formulae
  Prioritized formulae

Free vs. bound

Truth value of formulae
  Declaring a symbol
  Signature

Interpretation

Truth value of formulae

Conclusion
Conclusion : Next course

- Interpret a first order formula (contd.)
- Important equivalences
Conclusion

Thank you for your attention.

Questions?