Natural Deduction: quantifiers, copy and equality

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Motivation
Motivation

Propositional Case

There exist algorithms to decide if a formula is valid or not valid.
Motivation

Propositional Case
There exist algorithms to decide if a formula is valid or not valid.

First-order Case (FO : First-order)
There is no algorithm to decide if a formula is valid or not.
Motivation

Propositional Case

There exist algorithms to decide if a formula is valid or not valid.

First-order Case (FO : First-order)

There is no algorithm to decide if a formula is valid or not.

By accepting the equivalence between provable (without environment) and valid, there is no algorithm which, given a formula, can construct the proof, or tell that this formula has no proof. (Church 1936 and Turing 1937)
Plan

Introduction

Rules

Examples

Copy rule

Equality rules

Proof tactics

Conclusion
Overview

Introduction

Rules

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Copy rule

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Conclusion
### Reminder: « Propositional » rules

**Table 3.1**

<table>
<thead>
<tr>
<th>Introduction</th>
<th>Elimination</th>
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<tbody>
<tr>
<td>([A]) . . .</td>
<td>( A \implies B )</td>
</tr>
<tr>
<td>( B ) ( \implies I )</td>
<td>( B ) ( \implies E )</td>
</tr>
<tr>
<td>( A \implies B ) ( A ) ( B ) ( \wedge I )</td>
<td>( A \wedge B ) ( A ) ( \wedge E_1 )</td>
</tr>
<tr>
<td>( A \wedge B ) ( B ) ( \wedge E_2 )</td>
<td>( A \lor B ) ( A \implies C ) ( B \implies C ) ( \lor E )</td>
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<tr>
<td>( A \lor B ) ( A \lor C ) ( B \lor C ) ( \lor E )</td>
<td>( A \lor B ) ( A \implies C ) ( B \implies C ) ( \lor E )</td>
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**Rule of the false**

\[ \bot \implies \text{Eq} \]

**Reductio ad absurdum**

\[ \neg \neg A \implies A \]

\[ [A] \text{ means that } A \text{ is a hypothesis} \]
Extension of the natural propositional deduction

Definitions of the scratch proof, environment, context, usable formula remain unchanged!
Extension of the natural propositional deduction

- Definitions of the **scratch proof, environment, context, usable formula** remain **unchanged**!
- A single rule to eliminate hypotheses: \( \Rightarrow \).
Extension of the natural propositional deduction

- Definitions of the **scratch proof**, **environment**, **context**, **usable formula** remain unchanged!
- A single rule to eliminate hypotheses: $\Rightarrow 1$.

Additional rules with respect to « propositional » ND.

- the quantifiers
- the copy
- the equality
Coherence and completeness
Coherence and completeness

- Coherence of the rules of our system.
  $(\Gamma \vdash A)$ implies $\Gamma \models A$.

Similar proofs in:

Coherence and completeness

- **Coherence of the rules of our system.**
  \((\Gamma \vdash A) \text{ implies } \Gamma \models A)\).

- **Completeness accepted without proof.**  \((\Gamma \models A \text{ implies } \Gamma \vdash A)\)
  Similar proofs in:
Overview

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Conclusion
Reminder

Definition 4.3.34

Let $x$ a variable, $t$ a term and $A$ a formula.

1. $A < x := t >$ is the formula obtained by replacing in the formula $A$ every free occurrence of $x$ by the term $t$.

2. The term $t$ is free for $x$ in $A$ if the variables of $t$ are not bound in the free occurrences of $x$. 
Exercise

In

\[ A = \forall y P(x, y) \]

- is \( x \) free for \( y \) in \( A \)?
Exercise

In

\[ A = \forall y P(x, y) \]

- is \( x \) free for \( y \) in \( A \)?
  yes

- is \( y \) free for \( x \) in \( A \)?
Exercise

In

\[ A = \forall y P(x, y) \]

- is \( x \) free for \( y \) in \( A \)?
  - yes

- is \( y \) free for \( x \) in \( A \)?
  - no
Quantification rules : ∀E

A is a formula, x is a variable, t is a term

Elimination ∀

$$\frac{\forall x A}{A < x := t >} \forall E$$

t is free for x in A
Quantification rules : $\forall I$

$A$ is a formula, $x$ is a variable.

**Introduction $\forall$**

\[
\frac{A}{\forall x A} \forall I
\]

$x$ must not be free
- neither within the **environment** of the proof,
- nor within the **context** of the rule’s premise
Quantification rules: $\exists I$

A is a formula, $x$ is a variable, $t$ is a term

### Introduction $\exists$

$$
A < x := t > \quad \exists I
$$

$\exists x A$

$t$ is free for $x$ in $A$
Quantification rules : $\exists E$

$A$ and $B$ are formulae, $x$ is a variable.

Elimination $\exists$

$$
\exists x A \quad (A \Rightarrow B) \quad \exists E
$$

$x$ must not be free
- in the environment,
- nor in $B$,
- nor in the context of the right premise of the rule.
### Summary of the quantification rules: Table 6.1

<table>
<thead>
<tr>
<th>Rule</th>
<th>Description</th>
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</table>
| \( \frac{A}{\forall x A} \) | \( \forall I \)  \( x \) must not be free  
  - in the proof environment,  
  - nor in the context of the rule’s premise |
| \( \frac{\forall x A}{A\langle x:=t \rangle} \) | \( \forall E \)  \( t \) is free for \( x \) in \( A \) |
| \( \frac{A\langle x:=t \rangle}{\exists x A} \) | \( \exists I \)  \( t \) is free for \( x \) in \( A \) |
| \( \frac{\exists x A (A \Rightarrow B)}{B} \) | \( \exists E \)  \( x \) must not be free  
  - in the environment  
  - nor in \( B \),  
  - nor in the context of the right premise of the rule |
Overview

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Introduction

How to use these rules with examples,

As well as errors caused by not respecting the conditions of employment of the rules.
Example 6.1.2 $\forall y P(y) \land \forall y Q(y) \Rightarrow \forall x (P(x) \land Q(x))$
Example 6.1.2 \( \forall y P(y) \land \forall y Q(y) \Rightarrow \forall x (P(x) \land Q(x)) \)

1 1  suppose \( \forall y P(y) \land \forall y Q(y) \)
Example 6.1.2 $\forall y P(y) \land \forall y Q(y) \Rightarrow \forall x (P(x) \land Q(x))$

1 1 suppose $\forall y P(y) \land \forall y Q(y)$
1 2 $\forall y P(y)$ $\land E 1 1$
Example 6.1.2 \( \forall y P(y) \land \forall y Q(y) \Rightarrow \forall x (P(x) \land Q(x)) \)

\[
\begin{array}{c|c}
1 & 1 \text{ suppose } \forall y P(y) \land \forall y Q(y) \\
1 & 2 \forall y P(y) & \land E 1 1 \\
1 & 3 \forall y Q(y) & \land E 2 1 \\
\end{array}
\]
Example 6.1.2 $\forall y P(y) \land \forall y Q(y) \Rightarrow \forall x (P(x) \land Q(x))$

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<tbody>
<tr>
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<td>1</td>
<td>suppose $\forall y P(y) \land \forall y Q(y)$</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>$\forall y P(y)$ $\land E 1$ 1</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>$\forall y Q(y)$ $\land E 2$ 1</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>$P(x)$ $\forall E 2, x$</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>$Q(x)$ $\forall E 3, x$</td>
</tr>
</tbody>
</table>

**Remark:** Note that in using the instantiation rule on lines 4 and 5, we indicated that $y$ is replaced by $x$. 
Example 6.1.2 \( \forall y P(y) \land \forall y Q(y) \Rightarrow \forall x (P(x) \land Q(x)) \)

1 1 suppose \( \forall y P(y) \land \forall y Q(y) \)
1 2 \( \forall y P(y) \) \( \land E 1 1 \)
1 3 \( \forall y Q(y) \) \( \land E 2 1 \)
1 4 \( P(x) \) \( \forall E 2, x \)
1 5 \( Q(x) \) \( \forall E 3, x \)
1 6 \( P(x) \land Q(x) \) \( \land I 4, 5 \)

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Example 6.1.2 $\forall y P(y) \land \forall y Q(y) \Rightarrow \forall x (P(x) \land Q(x))$

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1 2 $\forall y P(y)$ $\land E 1 1$
1 3 $\forall y Q(y)$ $\land E 2 1$
1 4 $P(x)$ $\forall E 2, x$
1 5 $Q(x)$ $\forall E 3, x$
1 6 $P(x) \land Q(x)$ $\land I 4, 5$
1 7 $\forall x (P(x) \land Q(x))$ $\forall I 6$

Remark : Note that in using the instantiation rule on lines 4 and 5, we indicated that $y$ is replaced by $x$. 
Example 6.1.2 \( \forall y P(y) \land \forall y Q(y) \Rightarrow \forall x(P(x) \land Q(x)) \)

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<tr>
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<td></td>
<td>suppose ( \forall y P(y) \land \forall y Q(y) )</td>
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<tr>
<td>1</td>
<td>1</td>
<td>( \forall y P(y) ) ( \land E ) 1 1</td>
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<tr>
<td>1</td>
<td>2</td>
<td>( \forall y Q(y) ) ( \land E ) 2 1</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>( P(x) ) ( \forall ) 2, ( x )</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>( Q(x) ) ( \forall ) 3, ( x )</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>( P(x) \land Q(x) ) ( \land I ) 4, 5</td>
</tr>
<tr>
<td>1</td>
<td>6</td>
<td>( \forall x(P(x) \land Q(x)) ) ( \forall I ) 6</td>
</tr>
<tr>
<td>1</td>
<td>7</td>
<td>hence ( \forall x P(x) \land \forall x Q(x) \Rightarrow \forall x(P(x) \land Q(x)) ) ( \Rightarrow I ) 1, 7</td>
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**Remark:** Note that in using the instantiation rule on lines 4 and 5, we indicated that \( y \) is replaced by \( x \).
Example 6.1.1

Misuse of the rule $\forall E$ : where is the error?

1 1  suppose $\forall x \exists y P(x,y)$
1 2  $\exists y P(y,y)$  $\forall E$ 1, $y$
3  hence $\forall x \exists y P(x,y) \implies \exists y P(y,y)$
Example 6.1.1

Misuse of the rule $\forall E$: where is the error?

1 1 suppose $\forall x \exists y P(x, y)$
1 2 $\exists y P(y, y)$ $\forall E 1, y$ ERROR
3 hence $\forall x \exists y P(x, y) \Rightarrow \exists y P(y, y)$

On line 2, we have not met the conditions of application of the rule $\forall E$ since the term $y$ is not free for $x$ in the formula $\exists y P(x, y)$. 
Example 6.1.1

Misuse of the rule $\forall E$ : where is the error?

1 1 suppose $\forall x \exists y P(x, y)$
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3 hence $\forall x \exists y P(x, y) \Rightarrow \exists y P(y, y)$

On line 2, we have not met the conditions of application of the rule $\forall E$ since the term $y$ is not free for $x$ in the formula $\exists y P(x, y)$.

Let $I$ the interpretation of domain $\{0, 1\}$ with $P_I = \{(0, 1), (1, 0)\}$. This interpretation makes false the « conclusion ».
Example 6.1.3

Misuse of the rule $\forall I$

1 1 suppose $P(x)$
1 2 $\forall x P(x)$       $\forall I$ 1
3 hence $P(x) \Rightarrow \forall x P(x)$       $\Rightarrow I$ 1, 2
Example 6.1.3

Misuse of the rule $\forall I$

1 1 suppose $P(x)$
1 2 $\forall x P(x)$ $\forall I$ 1 ERROR
3 hence $P(x) \Rightarrow \forall x P(x)$ $\Rightarrow I$ 1, 2

On line 2, we have not met the conditions of application of the rule $\forall I$ since the premise $P(x)$ is made in the context $P(x)$, which precludes generalizations of $x$. 
Example 6.1.3

Misuse of the rule $\forall I$

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<tr>
<td>1</td>
<td>1 \quad \text{suppose $P(x)$}</td>
</tr>
<tr>
<td>1</td>
<td>2 \quad \forall x P(x) \quad \forall I 1 \ \text{ERROR}</td>
</tr>
<tr>
<td>3</td>
<td>hence $P(x) \Rightarrow \forall x P(x) \Rightarrow I 1, 2$</td>
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On line 2, we have not met the conditions of application of the rule $\forall I$ since the premise $P(x)$ is made in the context $P(x)$, which precludes generalizations of $x$.

Let $I$ the interpretation of domain $\{0, 1\}$ with $P_I = \{0\}$. Let $e$ a state where $x = 0$. The assignment $(I, e)$ makes false the « conclusion ».
Example 6.1.4

Misuse of the rule $\exists E$

1 1  suppose $\exists x P(x)$
1, 2 2  suppose $P(x)$
1 3  hence $P(x) \Rightarrow P(x)$  $\Rightarrow I 2, 2$
1 4  $P(x)$  $\exists E 1, 3$
1 5  $\forall x P(x)$  $\forall I 4$
6  hence $\exists x P(x) \Rightarrow \forall x P(x)$

The conclusion of the rule $\exists E$ is $P(x)$, contrary to the condition of application of this rule which imposes that the conclusion must not depend on $x$. Let $I$ the interpretation of domain $\{0, 1\}$ with $P_I = \{0\}$. $I$ makes false the conclusion.
Example 6.1.4

Misuse of the rule $\exists E$

1. 1. suppose $\exists x P(x)$
2. 1, 2. suppose $P(x)$
3. 1. hence $P(x) \Rightarrow P(x) \Rightarrow I\ 2, 2$
4. 1. $P(x)$
5. 1. $\forall x P(x)$
6. hence $\exists x P(x) \Rightarrow \forall x P(x)$

The conclusion of the rule $\exists E$ is $P(x)$, contrary to the condition of application of this rule which imposes that the conclusion must not depend on $x$.

Let $I$ the interpretation of domain $\{0, 1\}$ with $P_I = \{0\}$. $I$ makes false the « conclusion ».
Example 6.1.5

Misuse of the rule $\exists E$

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<td>Suppose $\exists x P(x) \land (P(x) \Rightarrow \forall y Q(y))$</td>
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<td></td>
<td>Hence $\exists x P(x) \land (P(x) \Rightarrow \forall y Q(y)) \Rightarrow \forall y Q(y)$</td>
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$\Rightarrow I 1, 4$

Has not complied with the condition that the context of the premise $P(x) \Rightarrow \forall y Q(y)$ must not depend on $x$.

Let $I$ the interpretation of domain $\{0, 1\}$ with $P_I = \{0\}$ and the state $e$ where $x = 1$. The assignment $(I, e)$ makes this $\conclusion$ false.
Example 6.1.5

Misuse of the rule $\exists E$

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<tr>
<td>1</td>
<td>2</td>
<td>$\exists x P(x)$</td>
<td>$\land E 1$</td>
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<tr>
<td>1</td>
<td>3</td>
<td>$P(x) \Rightarrow \forall y Q(y)$</td>
<td>$\land E 2$</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>$\forall y Q(y)$</td>
<td>$\exists E 2, 3$ ERROR</td>
<td></td>
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</tr>
<tr>
<td>5</td>
<td></td>
<td>Hence $\exists x P(x) \land (P(x) \Rightarrow \forall y Q(y)) \Rightarrow \forall y Q(y)$</td>
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Has not complied with the condition that the context of the premise $P(x) \Rightarrow \forall y Q(y)$ must not depend on $x$. 
Example 6.1.5

Misuse of the rule $\exists E$

1 1 Suppose $\exists x P(x) \land (P(x) \Rightarrow \forall y Q(y))$
1 2 $\exists x P(x)$ $\land\!\! E 1 1$
1 3 $P(x) \Rightarrow \forall y Q(y)$ $\land\!\! E 2 1$
1 4 $\forall y Q(y)$ $E 2, 3$ ERROR
5 Hence $\exists x P(x) \land (P(x) \Rightarrow \forall y Q(y)) \Rightarrow \forall y Q(y) \Rightarrow I 1,4$

Has not complied with the condition that the context of the premise $P(x) \Rightarrow \forall y Q(y)$ must not depend on $x$.

Let $I$ the interpretation of domain $\{0, 1\}$ with $P_I = Q_I = \{0\}$ and the state $e$ where $x = 1$. The assignment $(I, e)$ makes this « conclusion »false.
Example 6.1.6 \( \neg \forall x A \Rightarrow \exists x \neg A \) (De Morgan law)
Example 6.1.6 \( \neg \forall x A \Rightarrow \exists x \neg A \) (De Morgan law)

1 1 Suppose \( \neg \forall x A \)
Example 6.1.6 $\neg \forall x A \Rightarrow \exists x \neg A$ (De Morgan law)

1 1 Suppose $\neg \forall x A$
1, 2 2 Suppose $\neg \exists x \neg A$
Example 6.1.6 \( \neg \forall x A \Rightarrow \exists x \neg A \) (De Morgan law)

1  1  Suppose \( \neg \forall x A \)
1, 2  2  Suppose \( \neg \exists x \neg A \)
1, 2, 3  3  Suppose \( \neg A \)
Example 6.1.6  \( \neg \forall x A \Rightarrow \exists x \neg A \) (De Morgan law)

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<td>1, 2, 3</td>
<td>3</td>
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<tr>
<td>1, 2, 3</td>
<td>4</td>
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</table>

1. Suppose \( \neg \forall x A \)
2. Suppose \( \neg \exists x \neg A \)
3. Suppose \( \neg A \)
4. \( \exists x \neg A \) \( \exists / 3, x \)

- On line 4, we used the fact that \( \neg A \) can be seen as the result of the substitution of \( x \) by \( x \) in \( \neg A \) and that a variable is always free wrt. itself in a formula.
Example 6.1.6 \( \neg \forall x A \Rightarrow \exists x \neg A \) (De Morgan law)

1. Suppose \( \neg \forall x A \)
2. Suppose \( \neg \exists x \neg A \)
3. Suppose \( \neg A \)
4. \( \exists x \neg A \) \( \exists I \ 3, x \)
5. \( \bot \) \( \Rightarrow E 2, 4 \)

- On line 4, we used the fact that \( \neg A \) can be seen as the result of the substitution of \( x \) by \( x \) in \( \neg A \) and that a variable is always free wrt. itself in a formula.
Example 6.1.6 \( \neg \forall x A \Rightarrow \exists x \neg A \) (De Morgan law)

1 1 Suppose \( \neg \forall x A \)
1, 2 2 Suppose \( \neg \exists x \neg A \)
1, 2, 3 3 Suppose \( \neg A \)
1, 2, 3 4 \( \exists x \neg A \) \( \exists I \, 3, x \)
1, 2, 3 5 \( \bot \) \( \Rightarrow E \, 2, 4 \)
1, 2 6 Hence \( \neg \neg A \) \( \Rightarrow I \, 3, 5 \)

On line 4, we used the fact that \( \neg A \) can be seen as the result of the substitution of \( x \) by \( x \) in \( \neg A \) and that a variable is always free wrt. itself in a formula.
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1  1  Suppose  \( \neg \forall x A \)
1, 2  2  Suppose  \( \neg \exists x \neg A \)
1, 2, 3  3  Suppose  \( \neg A \)
1, 2, 3  4  \( \exists x \neg A \)  \( \exists I \ 3, x \)
1, 2, 3  5  \( \bot \)  \( \Rightarrow E \ 2, 4 \)
1, 2  6  Hence  \( \neg \neg A \)  \( \Rightarrow I \ 3, 5 \)
1, 2  7  \( A \)  Raa 6

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2. Suppose \( \neg \exists x \neg A \)
3. Suppose \( \neg A \)
4. \( \exists x \neg A \) \( \exists I \ 3, x \)
5. \( \bot \) \( \Rightarrow \ E \ 2, 4 \)
6. Hence \( \neg \neg A \) \( \Rightarrow \ I \ 3, 5 \)
7. \( A \) \( \text{Raa} \ 6 \)
8. \( \forall x A \) \( \forall I \ 7 \)

- On line 4, we used the fact that \( \neg A \) can be seen as the result of the substitution of \( x \) by \( x \) in \( \neg A \) and that a variable is always free wrt. itself in a formula.
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1, 2, 3 3 Suppose \( \neg A \)
1, 2, 3 4 \( \exists x \neg A \) \( \exists I 3, x \)
1, 2, 3 5 \( \bot \) \( \Rightarrow E 2, 4 \)
1, 2 6 Hence \( \neg \neg A \) \( \Rightarrow I 3, 5 \)
1, 2 7 \( A \) \( \text{Raa 6} \)
1, 2 8 \( \forall x A \) \( \forall I 7 \)
1, 2 9 \( \bot \) \( \Rightarrow E 1, 8 \)

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3. Suppose \( \neg A \)
4. \( \exists x \neg A \) \( \exists I 3, x \)
5. \( \bot \) \( \Rightarrow E 2, 4 \)
6. Hence \( \neg \neg A \) \( \Rightarrow I 3, 5 \)
7. \( A \) \( \text{Raa 6} \)
8. \( \forall x A \) \( \forall I 7 \)
9. \( \bot \) \( \Rightarrow E 1, 8 \)
10. Hence \( \neg \exists x \neg A \) \( \Rightarrow I 2, 9 \)

- On line 4, we used the fact that \( \neg A \) can be seen as the result of the substitution of \( x \) by \( x \) in \( \neg A \) and that a variable is always free wrt. itself in a formula.
Example 6.1.6 $\neg\forall x A \Rightarrow \exists x \neg A$ (De Morgan law)

1. Suppose $\neg\forall x A$
2. Suppose $\neg\exists x \neg A$
3. Suppose $\neg A$
4. $\exists x \neg A$ ($\exists I$ 3, $x$
5. $\bot$ $\Rightarrow$ $E$ 2, 4
6. Hence $\neg \neg A$ $\Rightarrow$ 1, 3, 5
7. $A$ Raa 6
8. $\forall x A$ ($\forall I$ 7
9. $\bot$ $\Rightarrow$ $E$ 1, 8
10. Hence $\neg \exists x \neg A$ $\Rightarrow$ 1, 2, 9
11. $\exists x \neg A$ Raa 10

- On line 4, we used the fact that $\neg A$ can be seen as the result of the substitution of $x$ by $x$ in $\neg A$ and that a variable is always free wrt. itself in a formula.
Example 6.1.6 \( \neg \forall x A \Rightarrow \exists x \neg A \) (De Morgan law)

1  1  Suppose \( \neg \forall x A \)
1, 2  2  Suppose \( \neg \exists x \neg A \)
1, 2, 3  3  Suppose \( \neg A \)
1, 2, 3  4  \( \exists x \neg A \)  \( \exists I \ 3, \ x \)
1, 2, 3  5  \( \bot \)  \( \Rightarrow E \ 2, \ 4 \)
1, 2  6  Hence \( \neg \neg A \)  \( \Rightarrow I \ 3, \ 5 \)
1, 2  7  \( A \)  \( \text{Raa 6} \)
1, 2  8  \( \forall x A \)  \( \forall I \ 7 \)
1, 2  9  \( \bot \)  \( \Rightarrow E \ 1, \ 8 \)
1  10  Hence \( \neg \exists x \neg A \)  \( \Rightarrow I \ 2, \ 9 \)
1  11  \( \exists x \neg A \)  \( \text{Raa 10} \)
12  Hence \( \neg \forall x A \Rightarrow \exists x \neg A \)  \( \Rightarrow I \ 1, \ 11 \)

- On line 4, we used the fact that \( \neg A \) can be seen as the result of the substitution of \( x \) by \( x \) in \( \neg A \) and that a variable is always free wrt. itself in a formula.
Overview

Introduction

Rules

Examples

Copy rule

Equality rules

Proof tactics

Conclusion
Definition

The copy rule deduces a formula from another formula so that they are equal with respect to a change of bound variables.

\[
\begin{array}{c}
A' \\
A
\end{array} \quad \text{copy}
\]

Reminder: Two formulae are equal with respect to a change of bound variables if we can obtain one starting from the other by replacing sub-formulae of the form \(QxA\) by \(QyA < x := y >\) where \(Q\) is a quantifier and \(y\) is a variable which is not in \(QxA\).
Overview

Introduction

Rules

Examples

Copy rule

Equality rules

Proof tactics

Conclusion
Reflexivity and congruence

The equality is defined by two rules:

- a term is equal to itself
- if two terms are equal, one can replace the other.
Reflexivity and congruence

The equality is defined by two rules:

- a term is equal to itself
- if two terms are equal, one can replace the other.

<table>
<thead>
<tr>
<th></th>
<th>reflexivity</th>
<th>t is a term</th>
</tr>
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<tbody>
<tr>
<td>$t = t$</td>
<td></td>
<td>$t$ is a term</td>
</tr>
<tr>
<td>$s = t$</td>
<td>$A &lt; x := s &gt;$</td>
<td>$A &lt; x := t &gt;$</td>
</tr>
<tr>
<td></td>
<td>congruence</td>
<td>s and t are two free terms for the variable x in the formula A</td>
</tr>
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</table>
Example 6.1.7

Let us prove that $s = t \Rightarrow t = s$ (symmetry)
Example 6.1.7

Let us prove that $s = t \Rightarrow t = s$ (symmetry)

1  1  suppose $s = t$
Example 6.1.7

Let us prove that $s = t \Rightarrow t = s$ (symmetry)

1. 1  suppose $s = t$
1. 2  $s = s$  reflexivity
Example 6.1.7

Let us prove that $s = t \Rightarrow t = s$ (symmetry)

<table>
<thead>
<tr>
<th></th>
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<td>$s = s$</td>
<td>reflexivity</td>
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<td>1</td>
<td>3</td>
<td>$t = s$</td>
<td>congruence 1, 2</td>
</tr>
</tbody>
</table>

$(s = s) = (x = s) < x ::= s >$

$(t = s) = (x = s) < x ::= t >$
Example 6.1.7

Let us prove that $s = t \Rightarrow t = s$ (symmetry)

1. Suppose $s = t$
2. $s = s$ (reflexivity)
3. $t = s$ (congruence 1, 2)

Hence $s = t \Rightarrow t = s \Rightarrow I \ 1, 3$
Example 6.1.7

Let us prove that $s = t \Rightarrow t = s$ (symmetry)

1. 1. Suppose $s = t$
2. 2. $s = s$ (reflexivity)
3. 3. $t = s$ (congruence 1, 2)

Hence $s = t \Rightarrow t = s \Rightarrow I 1, 3$

Remark: Note that the variable $x$ does not appear in the proof, it only indicates the place where $s$ is replaced by $t$. In the following examples, this place will simply be underlined.
Example 6.1.8

Let us prove that $s = t \land t = u \Rightarrow s = u$ (transitivity)
Example 6.1.8

Let us prove that $s = t \land t = u \Rightarrow s = u$ (transitivity)

1  1  suppose $s = t \land t = u$
Example 6.1.8

Let us prove that $s = t \land t = u \Rightarrow s = u$ (transitivity)

\[
\begin{array}{c|cl}
1 & 1 & \text{suppose} \ s = t \land t = u \\
1 & 2 & s = t \\
& & \land \text{E1 1}
\end{array}
\]
Example 6.1.8

Let us prove that $s = t \land t = u \Rightarrow s = u$ (transitivity)

1 1 suppose $s = t \land t = u$
1 2 $s = t$ \qquad \land E1 1
1 3 $t = u$ \qquad \land E2 1
Example 6.1.8

Let us prove that $s = t \land t = u \Rightarrow s = u$ (transitivity)

<table>
<thead>
<tr>
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<td>1</td>
<td>3</td>
<td>$t = u$</td>
<td>$\land E2$ 1</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>$s = u$</td>
<td>congruence 2, 3</td>
</tr>
</tbody>
</table>
Example 6.1.8

Let us prove that $s = t \land t = u \Rightarrow s = u$ (transitivity)

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<tr>
<td>1</td>
<td>1</td>
<td>$s = t$</td>
<td>$t = u$</td>
<td>$s = u$</td>
<td>hence $s = t \land t = u \Rightarrow s = u \Rightarrow I 1, 4$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\land E1 1$</td>
<td>$\land E2 1$</td>
<td>congruence 2, 3</td>
<td></td>
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<td></td>
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<tr>
<td>1</td>
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<td>$s = t \land t = u$</td>
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</tbody>
</table>
Overview

Introduction

Rules

Examples

Copy rule

Equality rules

Proof tactics

Conclusion
Introduction

1. Two proof tactics for the rules $\forall I$ and $\exists E$:
   1.1 forward chaining with an existence hypothesis,
   1.2 backward chaining for generalizing.

2. Application to an example.
Forward chaining with an existence hypothesis

Let $\Gamma$ a set of formulae, $x$ a variable, $A$ and $C$ formulae.

Suppose that we look for a proof of $C$ in the environment $\Gamma, \exists x A$. 
Forward chaining with an existence hypothesis

Let \( \Gamma \) a set of formulae, \( x \) a variable, \( A \) and \( C \) formulae.

Suppose that we look for a proof of \( C \) in the environment \( \Gamma, \exists x A \).

Two possible cases :

- \( x \) is not free neither in \( \Gamma \), nor in \( C \).
- \( x \) is free in \( \Gamma \) or \( C \).
1\textsuperscript{st} case: \(x\) is not free neither in \(\Gamma\), nor in \(C\)

In this case, the proof can always be written:

\begin{align*}
\text{suppose} & \quad A \\
\text{proof of} \ C & \quad \text{in the environment} \ \Gamma, \ A \\
\text{hence} & \quad A \Rightarrow C \quad \Rightarrow I \ 1, \_ \\
C & \quad \exists E
\end{align*}
2\textsuperscript{nd} case: $x$ is free in $\Gamma$ or $C$

We choose a "new" variable $y$, i.e., not free in $\Gamma$, $C$ and absent from $A$, then we fall in the previous case, via the copy rule.

The proof is written:

$$\exists y A < x := y >$$  \text{copy of $\exists x A$}

suppose $A < x := y >$

proof of $C$ in the environment $\Gamma, A < x := y >$

hence $A < x := y > \Rightarrow C$  $\Rightarrow I$ 1, -

$C$  $\exists E$
Remarks

The search of the initial proof was reduced to the search of a proof in a simpler environment.

Mode of reasoning applied in Mathematics when searching for a proof of a formula $C$ with the hypothesis $\exists x P(x)$.

We introduce a « new » constant $a$ verifying $P(a)$ and we prove $C$ under the hypothesis $P(a)$. 
Backward chaining for generalizing

Suppose that we search for a proof of $\forall x A$ in the environment $\Gamma$. 
Suppose that we search for a proof of $\forall x A$ in the environment $\Gamma$. Two possible cases:

- $x$ is not free in $\Gamma$.
- $x$ is free in $\Gamma$. 
1\textsuperscript{st} case: \( x \) is not free in \( \Gamma \)

\[
\begin{array}{ll}
\forall x A & \text{\( \forall I \)}
\end{array}
\]

proof of \( A \) in the environment \( \Gamma \)
2\textsuperscript{nd} case: $x$ is free in $\Gamma$

We choose a «new» variable $y$, i.e. not free in $\Gamma$, then we fall in the previous case, via the copy rule.

The proof is written:

- proof of $A < x := y >$ in the environment $\Gamma$
- $\forall y A < x := y > \ \forall I$
- $\forall x A$ (copy of the previous formula)
Remark

The search of the initial proof was reduced to the search of a proof in a simpler environment.

Mode of reasoning applied in Mathematics when searching for a proof of $\forall x P(x)$.

We introduce a « new » constant $a$ and we prove $P(a)$. Then we add: since the choice of $a$ is arbitrary, we have $\forall x P(x)$. 
An example of application of tactics

Notation « there is one and only one x » \((\exists! x)\) by:

\[
1 \quad \exists! x P(x) \equiv \exists x(P(x) \land \forall y(P(y) \Rightarrow x = y)).
\]
An example of application of tactics

Notation « there is one and only one x » (∃!x) by :

1  ∃!xP(x) ≡ ∃x(P(x) ∧ ∀y(P(y) ⇒ x = y)).

By separating the existence of x and its uniqueness, it can also be defined by :

2  ∃!xP(x) ≡ ∃xP(x) ∧ ∀x∀y(P(x) ∧ P(y) ⇒ x = y).

These two definitions are obviously equivalent and we formally show here that the first one implies the second one.
An example of application of tactics

Notation « there is one and only one x » (∃!x) by :

1 \( \exists ! x P(x) \equiv \exists x (P(x) \land \forall y (P(y) \Rightarrow x = y)) \).

By separating the existence of x and its uniqueness, it can also be defined by :

2 \( \exists ! x P(x) \equiv \exists x P(x) \land \forall x \forall y (P(x) \land P(y) \Rightarrow x = y) \).

These two definitions are obviously equivalent and we formally show here that \textbf{the first one implies the second one}.

As the proof is long, we decompose it.
6.2.3 Proof overview

\[ \exists x (P(x) \land \forall y (P(y) \Rightarrow x = y)) \Rightarrow \exists x P(x) \land \forall x \forall y (P(x) \land P(y) \Rightarrow x = y) \]

We apply the following two tactics:

- To prove \( A \Rightarrow B \), suppose \( A \) and deduce \( B \)
- To prove \( A \land B \), prove \( A \) and prove \( B \).
6.2.3 Proof overview

\[ \exists x(P(x) \land \forall y(P(y) \Rightarrow x = y)) \Rightarrow \exists xP(x) \land \forall x \forall y(P(x) \land P(y) \Rightarrow x = y) \]

We apply the following two tactics:

- To prove \( A \Rightarrow B \), suppose \( A \) and deduce \( B \)
- To prove \( A \land B \), prove \( A \) and prove \( B \).

<table>
<thead>
<tr>
<th>1</th>
<th>suppose ( \exists x(P(x) \land \forall y(P(y) \Rightarrow x = y)) )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>proof of ( \exists xP(x) ) in the environment 1</td>
</tr>
<tr>
<td></td>
<td>proof of ( \forall x \forall y(P(x) \land P(y) \Rightarrow x = y) ) in the environment 1</td>
</tr>
<tr>
<td></td>
<td>( \exists xP(x) \land \forall x \forall y(P(x) \land P(y) \Rightarrow x = y) )</td>
</tr>
<tr>
<td></td>
<td>( \Rightarrow \exists x(P(x) \land \forall y(P(y) \Rightarrow x = y)) \Rightarrow \exists xP(x) \land \forall x \forall y(P(x) \land P(y) \Rightarrow x = y) )</td>
</tr>
</tbody>
</table>
6.2.3 Application of the tactic using an existence

\[ \exists x P(x) \] in the environment of \[ \exists x (P(x) \land \forall y (P(y) \Rightarrow x = y)) \]
6.2.3 Application of the tactic using an existence

\( \exists x P(x) \) in the environment of \( \exists x(P(x) \land \forall y(P(y) \Rightarrow x = y)) \)

<table>
<thead>
<tr>
<th>reference</th>
<th>formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>i</td>
<td>( \exists x(P(x) \land \forall y(P(y) \Rightarrow x = y)) )</td>
</tr>
<tr>
<td>1</td>
<td>suppose ( P(x) \land \forall y(P(y) \Rightarrow x = y) )</td>
</tr>
<tr>
<td>2</td>
<td>( P(x) ) ( \wedge E1 \ 1 )</td>
</tr>
<tr>
<td>3</td>
<td>( \exists x P(x) ) ( \exists I \ 2, \ x )</td>
</tr>
<tr>
<td>4</td>
<td>hence ( P(x) \land \forall y(P(y) \Rightarrow x = y) \Rightarrow \exists x P(x) ) ( \Rightarrow I \ 1, 2 )</td>
</tr>
<tr>
<td>5</td>
<td>( \exists x P(x) ) ( \exists E \ i, \ 4 )</td>
</tr>
</tbody>
</table>
6.2.3 Application of the tactic to obtain a general conclusion : Proof overview

\( \forall x \forall y (P(x) \land P(y) \Rightarrow x = y) \) in the environment
\( \exists x (P(x) \land \forall y (P(y) \Rightarrow x = y)) \)

The following tactics are applied :

1. « forward chaining using an existential hypothesis ».
2. To prove \( A \Rightarrow B \), suppose \( A \) and deduce \( B \)
3. « backward chaining to obtain a general solution ».
6.2.3 Application of the tactic to obtain a general conclusion: Proof

<table>
<thead>
<tr>
<th>reference</th>
<th>formula</th>
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<tr>
<td>i</td>
<td>( \exists x(P(x) \land \forall y(P(y) \Rightarrow x = y)) )</td>
</tr>
<tr>
<td>1</td>
<td>suppose ( P(x) \land \forall y(P(y) \Rightarrow x = y) )</td>
</tr>
<tr>
<td>2</td>
<td>suppose ( P(u) \land P(y) )</td>
</tr>
<tr>
<td>3</td>
<td>( \forall y(P(y) \Rightarrow x = y) )</td>
</tr>
<tr>
<td>4</td>
<td>( P(u) )</td>
</tr>
<tr>
<td>5</td>
<td>( P(u) \Rightarrow x = u )</td>
</tr>
<tr>
<td>6</td>
<td>( x = u )</td>
</tr>
<tr>
<td>7</td>
<td>( P(y) )</td>
</tr>
<tr>
<td>8</td>
<td>( P(y) \Rightarrow x = y )</td>
</tr>
<tr>
<td>9</td>
<td>( x = y )</td>
</tr>
<tr>
<td>10</td>
<td>( u = y )</td>
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<td>11</td>
<td>hence ( P(u) \land P(y) \Rightarrow u = y )</td>
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<tr>
<td>12</td>
<td>( \forall y(P(u) \land P(y) \Rightarrow u = y) )</td>
</tr>
<tr>
<td>13</td>
<td>( \forall u \forall y(P(u) \land P(y) \Rightarrow u = y) )</td>
</tr>
<tr>
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<td>( \forall x \forall y(P(x) \land P(y) \Rightarrow x = y) )</td>
</tr>
<tr>
<td>15</td>
<td>hence ( (P(x) \land \forall y(P(y) \Rightarrow x = y)) \Rightarrow \forall x \forall y(P(x) \land P(y) \Rightarrow x = y) )</td>
</tr>
<tr>
<td>16</td>
<td>( \forall x \forall y(P(x) \land P(y) \Rightarrow x = y) )</td>
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</tbody>
</table>

\( \& E 1 \)
\( \& E 2 \)
\( \forall E 3, u \)
\( \Rightarrow E 4, 5 \)
\( \forall E 3, y \)
\( \Rightarrow E 7, 8 \)
congruence \( 6, 9 \)
\( \Rightarrow I 2, 10 \)
\( \forall I 11 \)
\( \forall I 12 \)
copy of \( 13 \)
\( \Rightarrow I 1, 14 \)
\( \exists E i, 15 \)
Conclusion

As we can see in the preceding example, the difficulty of all proofs is concentrated around the rules $\forall E$ and $\exists I$:

- via forward chaining, the correct instantiations of the formulae starting with an existential quantifier must be found
- via backward chaining, the correct instance must be found allowing to deduce a formula starting with an universal quantifier
Overview

Introduction

Rules

Examples

Copy rule

Equality rules

Proof tactics

Conclusion
Today

- first-order ND
Next course

- Coherence of the system
Conclusion

Thank you for your attention.

Questions?