Mid-term Exam INF242

Stéphane Devismes Benjamin Wack

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2 pages
Total: 120 points
Duration: 2h00

Authorized documents: a two-sided A4 sheet of handwritten notes.
The grading scale is indicative, the points correspond to the number of minutes needed to realize the exercises.
The exam will be graded on 120 points, the additionnal 15 points are treated as a bonus.
The result of an untreated question can be used in the remainder of the test. The exercises can be treated in any order you choose, but they have to be clearly labeled.

Exercice 1 (20 points) Consider the following two propositional logic formulae:
- \( A = (p \lor q \Rightarrow p \lor r) \Rightarrow p \land (q \Rightarrow r) \)
- \( B = p \land \neg q \lor \neg(p \lor \neg q) \Rightarrow (p \equiv \neg q) \)

1. (2 points) Write each of these formulae in its strict form and give their representation as a tree.
2. (10 points) Construct the truth table of each of these formulae
3. (3 points) For each of these formulae, indicate whether it is:
   - valid
   - contradictory
   - satisfiable
4. (1 point) If possible, give a model of \( A \). Otherwise, explain why.
5. (1 point) If possible, give a counter-model of \( B \). Otherwise, explain why.
6. (3 points) Determine a conjunctive normal form (CNF) and a disjunctive normal form (DNF) for each of these formulae.

Réponse:
1. \( A = (((p \lor q) \Rightarrow (p \lor r)) \Rightarrow (p \land (q \Rightarrow r))) \)
   - \( B = (((p \land \neg q) \lor \neg(p \lor \neg q)) \Rightarrow (p \equiv \neg q)) \)

2.

\[
\begin{array}{cccccccc}
\text{p} & \text{q} & r & p \lor q & p \lor r & p \lor q \Rightarrow p \lor r & q \Rightarrow r & p \land (q \Rightarrow r) & A \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
\end{array}
\]

3.

\[
\begin{array}{cccccccc}
\text{p} & \text{q} & \neg q & p \land \neg q & p \lor \neg q & \neg(p \lor \neg q) & p \land \neg q \lor \neg(p \lor \neg q) & p \equiv \neg q & B \\
1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\
\end{array}
\]
3. A is only satisfiable.
   B is valid, therefore a fortiori satisfiable.
   None of these formulae is contradictory.
4. A allows as model, for example \( p = 1, q = 1, r = 1 \).
5. B being valid, it does not allow any counter-model.
6. \( A = (\bar{p} + \bar{q} + r).(p + q + \bar{r}).(p + q + r) = p.q.r + p.q.r + p.q.r + \bar{p}.q.r \)
   \( B = 1 \) is simultaneously its CNF and its DNF.

**Exercice 2 (Formalization (17 points))** Brown, Jones and Smith are accused of tax fraud. They lend oath as follows:

**BROWN** : Jones is guilty and Smith est innocent.

**JONES** : if Brown is guilty, then Smith is also guilty.

**SMITH** : I am innocent but at least one of the other two are guilty.

Let B, J, S the statements : « Brown is innocent », « Jones is innocent », « Smith is innocent ».

1. (3 points) Formalize the testimony of every suspect using the logic formalism.
2. Answer the following questions :
   (a) (1 points) Are the testimonies compatible? (is there a model?)
   (b) (3 points) Which testimony is consequence of the other two? (Justify starting from the logic formalism.)
   (c) (2 points) Supposing that they are all innocent, who produced a false testimony? (Justify starting from the logic formalism.)
   (d) (3 points) Supposing that the testimony of every suspect is true, who is innocent and who is guilty? (Justify with a model)
   (e) (5 points) Supposing that every innocent says the truth and that every guilty person lies, who is innocent and who is guilty? Justify starting from the logic formalism. (Hint : the value of a testimony is equivalent to the status of the suspect.)

Réponse:
- \( j.s.(\bar{b} => \bar{s}).s.(\bar{b} + \bar{j}) \)
- \( b = 1, j = 0, s = 1 \), hence there is compatibility
- Brown : 1.1 = 0, Brown lies; Jones : \( \neg 1 => \neg 1 = 1 \), Jones says the truth; Smith : 1.(0+0) = 0, Smith lies.
- DNF : \( b,j,s \), hence Brown and Smith innocents; Jones guilty.
- under BDDC : \( \text{dnf} (b = -j.s).(j = -b => -s).(s = s.(-b+ji)) \); \( -b,j,-s \) hence Brown and Smith guilty, Jones innocent

**Exercice 3 (Completeness of the set \( \{\oplus, \land, 1\} \) (8 points))** We remind the truth table of the Xor operator, denoted \( \oplus \).

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
<th>x \oplus y</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Show that the set \( \{\oplus, \land, 1\} \) is complete.

Réponse:
- \( 0 = x \oplus x \)
- \( \neg x = x \oplus 1 \)
- \( x \land y = \neg(\neg x \land \neg y) \)
Exercice 4 (Resolution (20 points)) Show by resolution that the sets of the following formulae are unsatisfiable:

1. (5 points) \{ \bar{q} + m, \bar{q} + \bar{m} + \bar{r}, \bar{q} + \bar{m} + \bar{r}, \bar{p} + r, \bar{p} + \bar{r}, p + q \}
2. (15 points) \{ a \Rightarrow b + c, b \Leftrightarrow d, c \Rightarrow d + a, d \Rightarrow c + \bar{a}, \bar{a} \Rightarrow e, e \Rightarrow b + d, b \Rightarrow \bar{c} + \bar{d}, \bar{e} + a \}

Réponse:
1. \( \bar{q} + \bar{m} + \bar{r} \)
2. \( \bar{q} + \bar{m} + r \)
3. \( \bar{q} + m, \text{res } 1,2 \)
4. \( \bar{q} + m \)
5. \( q, \text{res } 3,4 \)
6. \( p + q \)
7. \( p, \text{res } 5,6 \)
8. \( \bar{p} + r \)
9. \( \bar{p} + \bar{r} \)
10. \( \bar{p}, \text{res } 8,9 \)
11. \( \bot, \text{res } 7,10 \)

1. \( \bar{a} + b + c \)
2. \( b + d \)
3. \( \bar{d} + b \)
4. \( \bar{c} + d + \bar{a} \)
5. \( d + c + \bar{a} \)
6. \( a + e \)
7. \( \bar{c} + b + d \)
8. \( b + \bar{c} + \bar{d} \)
9. \( \bar{e} + a \)
10. \( a, \text{res } 6,9 \)
11. \( b + c, \text{res } 1,10 \)
12. \( d + c, \text{res } 2,11 \)
13. \( \bar{a} + c, \text{res } 12,5 \)
14. \( c, \text{res } 10,13 \)
15. \( d + \bar{a}, \text{res } 4,14 \)
16. \( d, \text{res } 15,10 \)
17. \( b, \text{res } 16,3 \)
18. \( \bar{c} + \bar{d}, \text{res } 17,8 \)
19. \( \bar{d}, \text{res } 14,18 \)
20. \( \bot, \text{res } 16,19 \)

Exercice 5 (DPLL (20 points)) Determine, by applying the de DPLL algorithm, whether every the following sets of clauses is satisfiable. If so, give the model produced by DPLL.

1. (10 points) \{ a + b + \bar{d}, \bar{a} + c + \bar{d}, \bar{a} + \bar{b} + \bar{c}, \bar{a} + b + c, a + b + c + d, a + d + c, a + \bar{c} + d, \bar{c} + b, c + d + \bar{b} \}
2. (10 points) \{ a + b + d, b + c + f, b + \bar{e} + f, \bar{e} + e + f, e + f, \bar{c} + d, \bar{a}, \bar{e} + f \}

Réponse:
1. \{a + \bar{a} + c + \bar{c}, a + b + c, \bar{a} + \bar{c}, a + \bar{c} + \bar{d}, \bar{a} + b + \bar{c}, a + b + d, a + d + c, a + \bar{c} + d, \bar{c} + b, c + d + \bar{b}\}

Branch $a = 0$

\{\bar{b} + d, b + d + \bar{d}, \bar{d} + c + d, b + \bar{c}, \bar{c} + \bar{d}, \bar{d} + \bar{c}, \bar{c} + b, c + \bar{d} + \bar{b}\}

Branch $c = 0$

\{\bar{b} + d, \bar{b} + d, \bar{d} + \bar{b}\}

RE

\{b + d, \bar{d}, b + \bar{d} + b\}

UR $d = 0$

\{b, \bar{b}\}

UR

\perp

Branch $c = 1$

\{\bar{b} + \bar{d}, \bar{d}, b\}

UR $d = 1, b = 1$

\perp

Branch $a = 1$

\{c + \bar{d}, \bar{b} + \bar{d}, b + c, \bar{c} + b, c + d + \bar{b}\}

Branch $c = 0$

\{\bar{a}, \bar{c} + \bar{d}, \bar{b} + \bar{d}, \bar{d} + \bar{b}\}

UR $d = 0, b = 1$

\perp

Branch $c = 1$

\{\bar{b}, \bar{b}\}

UR

\perp

2. \{a + \bar{c} + d, \bar{b} + c + f, b + \bar{c} + f, \bar{b} + \bar{c} + f, e + \bar{c} + \bar{f}, e + \bar{f}, e + f, c + d + \bar{a}, \bar{c} + \bar{f}\}

ELLI $d = 1$

\{\bar{b} + c + f, \bar{b} + \bar{c} + f, \bar{b} + \bar{c} + f, e + \bar{c} + \bar{f}, e + \bar{f}, \bar{a}, \bar{c} + \bar{f}\}

ELLI $a = 0$

\{\bar{b} + c + f, b + \bar{c} + f, \bar{c} + e + \bar{f}, e + f, c + f, \bar{e} + \bar{f}\}

Branch $f = 0$
\{b + c, b + \bar{e}, e\}

ELLI \(c = 1\)

\{b + \bar{e}, e\}

ELLI \(b = 1\)

\{e\}

ELLI \(e = 1\)

\emptyset

**Exercice 6** (Natural Deduction (30 points))

Give a proof of the following formulae using the natural deduction in the table format:

- (10 points) \((p \lor q) \Rightarrow (\neg p \land \neg q) \Rightarrow r\).
- (10 points) \(((p \Rightarrow q) \land (q \Rightarrow r)) \land \neg r \Rightarrow \neg p\).
- (10 points) \((p \Rightarrow q) \Rightarrow ((p \land q) \lor \neg p)\).

Réponse:

\begin{center}
\begin{tabular}{|c|c|c|}
\hline
context & number & proof \\
\hline
1 & 1 & Suppose \(p \lor q\) \\
1.2 & 2 & Suppose \(\neg p \land \neg q\) \\
1.2 & 3 & \(\neg p\) & \&E1 2 \\
1.2 & 4 & \(\neg q\) & \&E2 2 \\
1.2 & 5 & \bot & \lor E 1,3,4 \\
1.2 & 6 & \(r\) & E/\lor 5 \\
1 & 7 & Hence \((\neg p \land \neg q) \Rightarrow r\) & \Rightarrow I 2,6 \\
1 & 8 & Hence \((p \lor q) \Rightarrow (\neg p \land \neg q) \Rightarrow r\) & \Rightarrow I 1,7 \\
\hline
\end{tabular}
\end{center}

\begin{center}
\begin{tabular}{|c|c|c|}
\hline
context & number & proof \\
\hline
1 & 1 & Suppose \(((p \Rightarrow q) \land (q \Rightarrow r)) \land \neg r\) \\
1.2 & 2 & Suppose \(p\) \\
1.2 & 3 & \((p \Rightarrow q) \land (q \Rightarrow r)\) & \&E1 1 \\
1.2 & 4 & \(\neg r\) & \&E2 1 \\
1.2 & 5 & \(p \Rightarrow q\) & \&E1 3 \\
1.2 & 6 & \(q \Rightarrow r\) & \&E2 3 \\
1.2 & 7 & \(q\) & \Rightarrow E 5,2 \\
1.2 & 8 & \(r\) & \Rightarrow E 6,7 \\
1 & 9 & \bot & \Rightarrow E 4,8 \\
1 & 10 & Hence \(\neg p\) & \Rightarrow I 2,9 \\
1 & 11 & Hence \(((p \Rightarrow q) \land (q \Rightarrow r)) \land \neg r \Rightarrow \neg p\) & \Rightarrow I 1,10 \\
\hline
\end{tabular}
\end{center}
Exercice 7 (Induction, exercise from the course support (Fr., polycopié) (20 points)) Show that every single-variable formula built without negation and only using $\lor$ and $\land$ is equivalent to a formula of height 0.

Réponse: We realize a proof by induction on the height of the formulae. Consider a formula $F$ with one variable $p$ without negation.

**Base case:** $|F| = 0$. The induction is trivially verified in this case.

**Inductive step:** $|F| = n + 1$ and suppose that all single-variable formulae constructed without negation of height less or equal to $n$ are equivalent to a formula of height 0. We have two dual cases, where $|A| \leq n$ and $|B| \leq n$, hence by induction hypothesis $A$ and $B$ are therefore equivalent to a formula of height 0, i.e., 0, 1, or $p$:

- $F = (A \lor B)$. We therefore have 9 cases to examine:
  - $A = 0$
    - $B = 0$, hence $F = 0$.
    - $B = 1$, hence $F = 1$.
    - $B = p$, hence $F = p$.
  - $A = 1$
    - $B = 0$, hence $F = 1$.
    - $B = 1$, hence $F = 1$.
    - $B = p$, hence $F = 1$.
  - $A = p$
    - $B = 0$, hence $F = p$.
    - $B = 1$, hence $F = p$.
    - $B = p$, hence $F = p$.

- $F = (A \land B)$. We therefore have 9 cases to examine:
  - $A = 0$
    - $B = 0$, hence $F = 0$.
    - $B = 1$, hence $F = 0$.
    - $B = p$, hence $F = 0$.
  - $A = 1$
    - $B = 0$, hence $F = 0$.
    - $B = 1$, hence $F = 1$.
    - $B = p$, hence $F = p$.
  - $A = p$
    - $B = 0$, hence $F = 0$.
    - $B = 1$, hence $F = p$.
    - $B = p$, hence $F = p$. 

\[\square\]