Exercice 1 (Formalisation in propositional logic, 15 points) There is a club in Scotland which is very closed and which obeys to the following rules :

1. every non-Scottish member wears red socks,
2. married members do not go out on Sundays,
3. a member goes out on Sundays if and only if he is Scottish,
4. every member wearing a kilt is Scottish and married,
5. every member wearing red socks wears a kilt,
6. every Scottish member wears a kilt.

The rules of this club therefore apply to a possible member of the club, let us call this member \( x \).

1. You are first asked to model this statement in propositional logic, i.e., to transform every rule into a formula that uses the following atoms (6 points) :
   - \( c \) : \( x \) is a Scottish member,
   - \( k \) : \( x \) wears a kilt,
   - \( m \) : \( x \) is married,
   - \( c \) : \( x \) wears red socks,
   - \( d \) : \( x \) goes out on Sundays.

2. Show by transformation into sum of monomials that the rules of this club are so restrictive that it cannot accept any member. You will detail your calculus. (9 points)

Exercice 2 (Expansion and counter-model, exercise from the course support (poly), 20 points) Find, by the expansion method, counter-models of the following formulae :

1. \( \forall x (P(x) \Rightarrow Q(x)) \Rightarrow \exists x Q(x) \). (10 points)
2. \( \forall x \exists y R(x, y) \Rightarrow \exists x R(x, x) \). (10 points)

Hint : simply build 1- or 2-expansions.

Exercice 3 (Unification, 15 points) For every of the following equations, determine if it admits a solution, and if yes determine a most general unifier. You will use the algorithm from the course where you detail every step.

1. \( f(g(x, a), h(z), y) = f(g(a, y), x, a) \) (5 points)
2. \( f(g(x, y), h(z), a) = f(g(h(b), a), x, a) \) (5 points)
3. \( f(x, z, y) = f(h(y), h(x), h(x)) \) (5 points)
Exercice 4 (Skolemisation and Clause Form, 20 points)  Let $A$ the following formula :

$$\neg(\neg\forall x P(x) \lor \neg\forall x Q(x) \Rightarrow \neg(\forall x P(x) \land \forall x Q(x)))$$

1. Skolemise $A$ then give its clause form. (10 points)
2. Find contradictory instances of the obtained clauses and show by propositional resolution that these instances are contradictory. (5 points)
3. Give a direct proof of this contradiction by factoring, copy and binary resolution. It is possible that only the last rule is used. (5 points)

Exercice 5 (First order resolution, 15 points)  By using a proof by factoring, copy and binary resolution, prove that the universal closure of the following set of clauses is unsatisfiable :

1. $\neg E(x) \lor Q(f(x))$
2. $\neg Q(x) \lor E(f(x))$
3. $f(x) > x$
4. $\neg(x > y) \lor \neg(y > z) \lor x > z$
5. $Q(a)$
6. $\neg(y > a) \lor \neg Q(y)$

Exercice 6 (Proof, 15 points)  Let $x_1 \ldots x_n$, $n$ different variables.
- $\oplus$ is the « exclusive or » operator, i.e., $x \oplus y = \overline{x \underline{\lor} y}$. Recall the truth table of $\oplus$.
- Prove by induction that for every state $e$, $[x_1 \oplus \ldots \oplus x_n]_e = (\sum_{i \in \{1..n\}}[x_i]_e) \mod 2$, i.e., $[x_1 \oplus \ldots \oplus x_n]_e$ gives the parity of the number of variables assigned to 1 in $e$.

Exercice 7 (Natural Deduction, 20 points)  Prove the following formulae by first order natural deduction.
1. $\exists x(Q(x) \Rightarrow P(x)) \land \forall x Q(x) \Rightarrow \exists x P(x)$. (10 points)
2. $\forall x \forall y(R(x,y) \Rightarrow \neg R(y,x)) \Rightarrow \forall x \neg R(x,x)$ (10 points)