Natural Deduction

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Intuition

When you write proofs in math courses,

when you decompose a reasoning in elementary obvious steps,

you somehow practice Natural Deduction.

Goals of Natural Deduction

Gerhard Gentzen (1934)

The goal is to provide a formal system to write proofs that are **close to the "natural" way of reasoning**.

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The goal is to provide a formal system to write proofs that are **close to the "natural" way of reasoning**.

Two orthogonal subgoals:

1. Proofs should be "readable enough" to be easily checked by a human being.
2. Proofs should be "formal enough" to prevent bugs (and to be mechanically checked/generated by a computer).

Models for Natural Deduction

Gerhard Gentzen introduced two models of Natural Deduction for classical logic:

- **NK**: a proof is a tree of formulas.
- **LK**: a proof is a tree of sequents.

(**NJ** et **LJ** for intuitionistic logic.)

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We try to take advantages from the two previous models!

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1. Try to convince you that it is easy and safe to write proofs by yourself in **Natural Deduction**.

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1. Try to convince you that it is easy and safe to write proofs by yourself in **Natural Deduction**.
2. Present a tool that automatically constructs proofs in **Natural Deduction**:

   `http://teachinglogic.liglab.fr/DN/`

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Propositional Logic

Definition 1

Propositional logic is a logic *without quantifiers*.

The only logical operations used are:

- $\neg$ (negation),
- $\wedge$ (conjunction, also known as logical "and"),
- $\vee$ (disjunction, also known as logical "or"),
- $\Rightarrow$ (implication)
- $\Leftrightarrow$ (equivalence)

Syntax: Vocabulary of the language

- The constants: $\top$ (*true*) and $\bot$ (*false*)
- The variables: for example, x, y_1
- The parentheses: left (and right).
- The connectives: $\neg, \vee, \wedge, \Rightarrow, \Leftrightarrow$

(Strict) Formula

Definition 2

A strict formula is defined inductively as:

- $\top$ and $\bot$ are strict formulae.
- A variable is a strict formula.
- If A is a strict formula then $\neg A$ is a strict formula.
- If A and B are strict formulae and if $\circ$ is one of the following operations $\vee, \wedge, \Rightarrow, \Leftrightarrow$ then $(A \circ B)$ is a strict formula.

Example 1

$(a \vee (\neg b \wedge c))$ is a **strict formula**,
but neither $a \vee (\neg b \wedge c)$, nor $(a \vee (\neg(b) \wedge c))$.

Canonical Decomposition

Strict formulae decompose uniquely in their sub-formulae.

Theorem 1

For every formula A, there is one and only one of the following cases:

- A is a variable,
- A is a constant,
- A can be written **in a unique manner** as $\neg B$ where B is a formula,
- A can be written **in a unique manner** as $(B \circ C)$ where B and C are formulae.

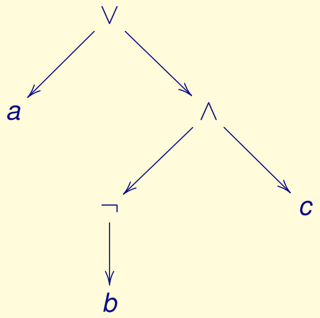
This will allow us to:

- prove properties *by cases*
- perform *structural induction* on the formulae: we will **ALWAYS consider STRICT formulae in proofs**

Tree

Example 2

The structure of the formula $(a \vee (\neg b \wedge c))$ is illustrated by the following tree:



Subtrees = sub-formulae

Prioritized formula

Definition 3

A prioritized formula is inductively defined in a similar way, but:

- if A and B are prioritized formulae, then $A \circ B$ is a prioritized formula,
- if A is a prioritized formula then (A) is a prioritized formula.

Example 3

$a \vee \neg b \wedge c$ is a prioritized formula, but not a (strict) formula.

Connective precedence

Definition 4

By decreasing precedence, the connectives are: $\neg$, $\wedge$, $\vee$, $\Rightarrow$ and $\Leftrightarrow$.

Left associativity

For identical connectives, the left-hand side connective has higher precedence:
$A \circ B \circ C \ = \ (A \circ B) \circ C$
except for the implication: $A \Rightarrow B \Rightarrow C \ = \ A \Rightarrow (B \Rightarrow C)$

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We usually write prioritized formulae, but we should reason with their corresponding strict formulae

Basic tables

0 indicates false and 1 indicates true.
The value of the constant $\top$ is 1 and the value of the constant $\perp$ is 0

Table 1

x	y	$\neg x$	$x \vee y$	$x \wedge y$	$x \Rightarrow y$	$x \Leftrightarrow y$
0	0	1	0	0	1	1
0	1	1	1	0	1	0
1	0	0	1	0	0	0
1	1	0	1	1	1	1

Assignment

Definition 5

A truth assignment (assignment, for short) is a function from the set of variables of a formula to the set $\{0, 1\}$.

$[A]_v$ denotes the truth value of the formula A for the assignment v.

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A truth assignment (assignment, for short) is a function
from the set of variables of a formula to the set $\{0, 1\}$.

$[A]_v$ denotes the truth value of the formula A for the assignment v.

Example: Let v be an assignment such that $v(x) = 0$ and $v(y) = 1$.

Applying v to $x \vee y$ is written as $[x \vee y]_v$

$[x \vee y]_v = 0 \vee 1 = 1$

Conclusion: $x \vee y$ is true for the truth assignment v

Example

x	y	$x \vee y \Rightarrow \neg x$
1	0	**0**
1	1	0
0	0	1
0	1	1

Assignment v: line 1, $v(x) = 1$ and $v(y) = 0$

Value of the assignment: $[x \vee y \Rightarrow \neg x]_v = \mathbf{0}$

Model for a formula

Definition 6

An assignment v for which a formula has truth value equal to 1 is a model for that formula.

v satisfies A or v makes A true.

Example 4

A model for $x \Rightarrow y$ is $x = 1, y = 1$ (among others)

Conversely, $x = 1$, $y = 0$ is not a model for $x \Rightarrow y$.

Model for a set of formulae

Definition 7

v is a model for a set of formulae $\{A_1, \ldots, A_n\}$
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Example 5

A model of $\{a \Rightarrow b, b \Rightarrow c\}$ is $a = 0, b = 0$ (for any c).

Validity, Contradiction

Definition 8

- A formula is valid (resp., a tautology) if its value is 1 for all truth assignments.
- A formula is unsatisfiable (resp., contradictory or a contradiction) if its value is 0 for all truth assignments.

Example 6

- $(x \Rightarrow y) \Leftrightarrow (\neg x \vee y)$ is valid.
- $x \Rightarrow y$ is not valid since it is false for $x = 1$ and $y = 0$, but is not a contradiction since it is true for $x = 0$ and $y = 0$.
- $x \wedge \neg x$ is a contradiction.

Logical consequence (entailment)

Definition 9

A is a consequence of the set Γ of formulae (called set of hypotheses or environment) if
every model of Γ is a model of A.

The fact that A is a consequence of Γ is noted $\Gamma \models A$.

Example of a consequence

Example 7

$a \Rightarrow b, b \Rightarrow c \models a \Rightarrow c$.

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a	b	c	$a \Rightarrow b$	$b \Rightarrow c$	$a \Rightarrow c$
0	0	0	1	1	1
0	0	1	1	1	1
0	1	0	1	0	1
0	1	1	1	1	1
1	0	0	0	1	0
1	0	1	0	1	1
1	1	0	1	0	0
1	1	1	1	1	1

Property

Property 1

The following two formulations are equivalent:

1. $A_1, \ldots, A_n \models B$
2. $A_1 \wedge \ldots \wedge A_n \Rightarrow B$ is valid.

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Remark: Let Γ be a set of formulae and A be a formula.

1. $\Gamma \models \bot$ means that Γ is contradictory (always false),
2. $\emptyset \models A$ ($\models A$, for short) means that Γ is valid (always true).

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Two main specificities

- System with several rules.
- During a proof, we can add and remove hypotheses.
- During a proof, we use abbreviations of formulae.

Abbreviations

$\top$, negation and equivalence are abbreviations defined as:

- $\top$ abbreviates $\bot \Rightarrow \bot$.
- $\neg A$ abbreviates $A \Rightarrow \bot$.
- $A \Leftrightarrow B$ abbreviates $(A \Rightarrow B) \wedge (B \Rightarrow A)$.

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E.g., the formulae $\neg\neg a$, $\neg a \Rightarrow \bot$ and $(a \Rightarrow \bot) \Rightarrow \bot$ are equal.

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Two equal formulae are equivalent!

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E.g., the formulae $\neg\neg a$, $\neg a \Rightarrow \bot$ and $(a \Rightarrow \bot) \Rightarrow \bot$ are equal.

Two equal formulae are equivalent!

In the proof, **we usually write every formula in its abbreviated form**, *e.g.*, $\neg A$, but we keep in mind that we can use its unabbreviated form, *e.g.*, $A \Rightarrow \bot$, when applying a rule.

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Rule

Definition 10

A rule consists of:

- some formulae $H_1, \ldots, H_n$ called **premises** (or hypotheses)
- a unique **conclusion** C
- a name R for the rule (optional)

$$\frac{H_1 \ldots H_n}{C}\ R$$

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Example 8

$$\frac{A \quad B}{A \wedge B}\ \wedge I$$

Classification of rules

- **Introduction rules** for introducing a connective in the conclusion.

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- **Elimination rules** for removing a connective from one of the premises.
- $+$ **two special rules**

The rules (system NK of Gentzen)

Table 2

	Introduction	Elimination
Implication		
Conjunction		
Disjunction		
$\bot$	Ex falso quodlibet	
	Reductio ad absurdum	

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	Introduction	Elimination
Implication	$\dfrac{\begin{array}{c}[A]\\ \ldots\\ B\end{array}}{A \Rightarrow B} \Rightarrow I$	
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Disjunction		
$\perp$	Ex falso quodlibet	
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$[A]$ means that A is a hypothesis

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Conjunction	$\dfrac{A \quad B}{A\wedge B} \wedge I$	$\dfrac{A\wedge B}{A} \wedge E1$ $\dfrac{A\wedge B}{B} \wedge E2$
Disjunction		
	Ex falso quodlibet	
$\perp$		
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	Ex falso quodlibet	
$\bot$		
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Disjunction	$\dfrac{A}{A \vee B} \vee I1$ $\dfrac{B}{A \vee B} \vee I2$	$\dfrac{A \vee B \quad A \Rightarrow C \quad B \Rightarrow C}{C} \vee E$
$\perp$	Ex falso quodlibet	
	Reductio ad absurdum	

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	Ex falso quodlibet	
$\perp$	$\dfrac{\perp}{A}$ *Efq*	
	Reductio ad absurdum	

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The rules (system NK of Gentzen)

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	Ex falso quodlibet	
$\bot$	$\dfrac{\bot}{A}$ *Efq*	
	Reductio ad absurdum	
	$\dfrac{\neg\neg A}{A}$ *RAA*	

$[A]$ means that A is a hypothesis

A “simple” example

$$\dfrac{\dfrac{A \qquad A \Rightarrow B}{B} \Rightarrow E \qquad \dfrac{A \qquad A \Rightarrow C}{C} \Rightarrow E}{B \wedge C} \wedge I$$

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What have we proven here exactly?

A "simple" example

$$\dfrac{\dfrac{A \qquad A \Rightarrow B}{B} \Rightarrow E \qquad \dfrac{A \qquad A \Rightarrow C}{C} \Rightarrow E}{B \wedge C} \wedge I$$

What have we proven here exactly? $B \wedge C$

A "simple" example

$$\dfrac{\dfrac{A \qquad A \Rightarrow B}{B} \Rightarrow E \qquad \dfrac{A \qquad A \Rightarrow C}{C} \Rightarrow E}{B \wedge C} \wedge I$$

What have we proven here exactly? $B \wedge C$
under the hypotheses $A,\ A \Rightarrow B,\ A \Rightarrow C$

i.e., $A,\ A \Rightarrow B,\ A \Rightarrow C \models B \wedge C$

Fundamental rule of Natural Deduction

Implies-introduction:

In order to prove $A \Rightarrow B$,
just derive B under the additional hypothesis A and then remove this assumption.

$$\text{If } A \models B \text{ then } \models A \Rightarrow B$$

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Implies-introduction:

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just derive B under the additional hypothesis A and then remove this assumption.

$$\text{If } A \models B \text{ then } \models A \Rightarrow B$$

$$\cfrac{\begin{matrix}[A] & H_1 & \dots & H_n \\ \ddots & \vdots & \iddots & \\ & B & & \end{matrix}}{A \Rightarrow B} \Rightarrow I$$

proves that $H_1, \dots, H_n \models A \Rightarrow B$.

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Proof line

Definition 11

A proof line is one of the three following:

- `Assume` formula
- formula
- `Therefore` formula

Proof line

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A proof line is one of the three following:

- `Assume` formula (to add an hypothesis)
- formula (derived from previous lines using the rules)
- `Therefore` formula (to remove the last hypothesis)

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This last case is the rule of implies-introduction.

Proof line

Definition 11

A proof line is one of the three following:

- `Assume` formula   (to add an hypothesis)
- formula   (derived from previous lines using the rules)
- `Therefore` formula   (to remove the last hypothesis)

This last case is the rule of implies-introduction.

Examples:

- `Assume` $A \wedge B$
- A
- `Therefore` $A \wedge B \Rightarrow A$

$$\dfrac{\dfrac{[A \wedge B]}{A}\wedge E}{A \wedge B \Rightarrow A} \Rightarrow I$$

Proof sketch

Definition 12

A proof sketch is a sequence of lines such that, in every prefix of the sequence, there are at least as many `Assume` as `Therefore`.

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Example 9

number	line
1	Assume a
2	$a \vee b$
3	Therefore $a \Rightarrow a \vee b$
4	Therefore $\neg a$
5	Assume b

Proof sketch

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A proof sketch is a sequence of lines such that, in every prefix of the sequence, there are at least as many `Assume` as `Therefore`.

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1	`Assume` a
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Proof sketch: examples

Where are the sketches?

number	line
1	`Assume` $a \wedge b$
2	b
3	$b \vee c$
4	`Therefore` $a \wedge b \Rightarrow b \vee c$
5	`Therefore` $\neg a$
6	`Assume` b

number	line
1	`Assume` a
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3	`Therefore` $a \Rightarrow a \vee b$
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3	`Therefore` $a \Rightarrow a \vee b$
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Context (1/2)

- Each line of a proof sketch has a context
- The context is the sequence of hypotheses introduced (using `Assume` lines) until the current line (included) and not removed in `Therefore` lines.

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Example:

context	number	line	rule
1	1	`Assume` a	
1,2	2	`Assume` b	
1,2	3	$a \wedge b$	$\wedge$I 1,2
1	4	`Therefore` $b \Rightarrow a \wedge b$	$\Rightarrow$I 2,3
1,5	5	`Assume` e	

Context (2/2)

The context of a formula represents the hypotheses from which it has been derived.

Definition 13

Formally: Γ_i is the context of the line i.

$\Gamma_0 = \emptyset$

If the line i is:

- `Assume` A
  then $\Gamma_i = \Gamma_{i-1}, i$
- `Therefore` A
  then Γ_i is obtained by deleting the last formula in Γ_{i-1}
- A
  then $\Gamma_i = \Gamma_{i-1}$

Example of context

Write down the **contexts** of the following proof sketch:

context	number	line
	1	Assume a
	2	$a \vee b$
	3	Therefore $a \Rightarrow a \vee b$
	4	Assume b
	5	Therefore b

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Write down the **contexts** of the following proof sketch:

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Usable formulae, *i.e.*, formulae on which can be applied rules (1/2)

Definition 14

- A formula appearing on a line of a proof sketch is its conclusion.
- The conclusion of a line is usable as long as its context (*i.e.*, the hypotheses from which it has been derived) is present.

Usable formulae, *i.e.*, formulae on which can be applied rules (1/2)

Definition 14

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Example 10

context	number	line
1	1	`Assume` a
1	2	$a \vee b$
	3	`Therefore` $a \Rightarrow b$
	4	a
	5	$b \vee a$

The conclusion of line 2 is usable on line 2 and not beyond.

Usable formulae (2/2)

On which lines are formulae 1 and 3 **usable**?

context	number	line
1	1	`Assume` a
1,2	2	`Assume` b
1,2	3	c
1	4	`Therefore` d
1,5	5	`Assume` e

Definition of a Proof

Definition 15

Let Γ be a set of formulae.
A proof in the environment Γ is a proof sketch such that:

1. For every "`Therefore`" line, the formula is $B \Rightarrow C$, where:
   - B is the last hypothesis we've removed (from the context of the previous line)
   - C is either a formula usable on the previous line, or belongs to Γ.

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2. For every "A" line, the formula A is:
   - the conclusion of a rule (other than $\Rightarrow I$)
   - whose premises are usable on the previous line, or belong to Γ.

Beware:

- The context Γ_i changes during the proof.
- The environment Γ remains the same.

Proof of formulae

Definition 16

A proof of formula A within the environment Γ is:

- either the empty proof (when A is an element of Γ),
- or a proof whose last line is A with an empty context.

Proof of formulae

Definition 16

A proof of formula A within the environment Γ is:

- either the empty proof (when A is an element of Γ),
- or a proof whose last line is A with an empty context.

We note:

- $\Gamma \vdash A$ the fact that there is a proof of A within the environment Γ,
- $\Gamma \vdash P : A$ the fact that P is a proof of A within Γ.
- When the environment is empty, we abbreviate $\emptyset \vdash A$ by $\vdash A$.
- When we ask for a proof without indicating the environment, we mean that $\Gamma = \emptyset$.

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Introduction

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Examples

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First Example

Prove $(a \Rightarrow b) \Rightarrow (\neg b \Rightarrow \neg a)$, *i.e.*, $\models (a \Rightarrow b) \Rightarrow (\neg b \Rightarrow \neg a)$

context	number	line	justification

First Example

Prove $(a \Rightarrow b) \Rightarrow (\neg b \Rightarrow \neg a)$, *i.e.*, $\models (a \Rightarrow b) \Rightarrow (\neg b \Rightarrow \neg a)$

context	number	line	justification
1	1	Assume $a \Rightarrow b$	

First Example

Prove $(a \Rightarrow b) \Rightarrow (\neg b \Rightarrow \neg a)$, *i.e.*, $\models (a \Rightarrow b) \Rightarrow (\neg b \Rightarrow \neg a)$

context	number	line	justification
1	1	Assume $a \Rightarrow b$	
1,2	2	Assume $\neg b$	

First Example

Prove $(a \Rightarrow b) \Rightarrow (\neg b \Rightarrow \neg a)$, *i.e.*, $\models (a \Rightarrow b) \Rightarrow (\neg b \Rightarrow \neg a)$

context	number	line	justification
1	1	Assume $a \Rightarrow b$	
1,2	2	Assume $\neg b$	
1,2,3	3	Assume a	

First Example

Prove $(a \Rightarrow b) \Rightarrow (\neg b \Rightarrow \neg a)$, *i.e.*, $\models (a \Rightarrow b) \Rightarrow (\neg b \Rightarrow \neg a)$

context	number	line	justification
1	1	Assume $a \Rightarrow b$	
1,2	2	Assume $\neg b$	
1,2,3	3	Assume a	
1,2,3	4	b	$\Rightarrow E$ 1, 3

First Example

Prove $(a \Rightarrow b) \Rightarrow (\neg b \Rightarrow \neg a)$, *i.e.*, $\models (a \Rightarrow b) \Rightarrow (\neg b \Rightarrow \neg a)$

context	number	line	justification
1	1	`Assume` $a \Rightarrow b$	
1,2	2	`Assume` $\neg b$	
1,2,3	3	`Assume` a	
1,2,3	4	b	$\Rightarrow E$ 1, 3
1,2,3	5	$\bot$	$\Rightarrow E$ 2, 4

Remark: line 2, $\neg b$ is an abbreviation of $b \Rightarrow \bot$.

So, applying $\Rightarrow E$ on b and $\neg b$ (*i.e.*, $b \Rightarrow \bot$), we obtain $\bot$!

$$\frac{A \quad A \Rightarrow B}{B} \Rightarrow E$$

First Example

Prove $(a \Rightarrow b) \Rightarrow (\neg b \Rightarrow \neg a)$, *i.e.*, $\models (a \Rightarrow b) \Rightarrow (\neg b \Rightarrow \neg a)$

context	number	line	justification
1	1	Assume $a \Rightarrow b$	
1,2	2	Assume $\neg b$	
1,2,3	3	Assume a	
1,2,3	4	b	$\Rightarrow E$ 1, 3
1,2,3	5	$\bot$	$\Rightarrow E$ 2, 4
1,2	6	Therefore $\neg a$	$\Rightarrow I$ 3, 5

Remark: $\Rightarrow I$ on a and $\bot$ gives $a \Rightarrow \bot$, which is abbreviated as $\neg a$.

$$\frac{\begin{array}{c}[A]\\ \cdots \\ B\end{array}}{A \Rightarrow B} \quad \Rightarrow I$$

First Example

Prove $(a \Rightarrow b) \Rightarrow (\neg b \Rightarrow \neg a)$, *i.e.*, $\models (a \Rightarrow b) \Rightarrow (\neg b \Rightarrow \neg a)$

context	number	line	justification
1	1	Assume $a \Rightarrow b$	
1,2	2	Assume $\neg b$	
1,2,3	3	Assume a	
1,2,3	4	b	$\Rightarrow E$ 1, 3
1,2,3	5	$\perp$	$\Rightarrow E$ 2, 4
1,2	6	Therefore $\neg a$	$\Rightarrow I$ 3, 5
1	7	Therefore $\neg b \Rightarrow \neg a$	$\Rightarrow I$ 2, 6

First Example

Prove $(a \Rightarrow b) \Rightarrow (\neg b \Rightarrow \neg a)$, *i.e.*, $\models (a \Rightarrow b) \Rightarrow (\neg b \Rightarrow \neg a)$

context	number	line	justification
1	1	Assume $a \Rightarrow b$	
1,2	2	Assume $\neg b$	
1,2,3	3	Assume a	
1,2,3	4	b	$\Rightarrow E$ 1, 3
1,2,3	5	$\perp$	$\Rightarrow E$ 2, 4
1,2	6	Therefore $\neg a$	$\Rightarrow I$ 3, 5
1	7	Therefore $\neg b \Rightarrow \neg a$	$\Rightarrow I$ 2, 6
	8	Therefore $(a \Rightarrow b) \Rightarrow (\neg b \Rightarrow \neg a)$	$\Rightarrow I$ 1,7

Second Example

Prove $a \wedge \neg a \Rightarrow b$, *i.e.*, $\models a \wedge \neg a \Rightarrow b$

context	number	line	justification

Second Example

Prove $a \wedge \neg a \Rightarrow b$, *i.e.*, $\models a \wedge \neg a \Rightarrow b$

context	number	line	justification
1	1	Assume $a \wedge \neg a$	

Second Example

Prove $a \land \neg a \Rightarrow b$, *i.e.*, $\models a \land \neg a \Rightarrow b$

context	number	line	justification
1	1	Assume $a \land \neg a$	
1	2	a	$\land E1$ 1

Second Example

Prove $a \wedge \neg a \Rightarrow b$, *i.e.*, $\models a \wedge \neg a \Rightarrow b$

context	number	line	justification
1	1	$\texttt{Assume}\ a \wedge \neg a$	
1	2	a	$\wedge E1\ 1$
1	3	$\neg a$	$\wedge E2\ 1$

Second Example

Prove $a \wedge \neg a \Rightarrow b$, *i.e.*, $\models a \wedge \neg a \Rightarrow b$

context	number	line	justification
1	1	$\texttt{Assume}\ a \wedge \neg a$	
1	2	a	$\wedge E1\ 1$
1	3	$\neg a$	$\wedge E2\ 1$
1	4	$\perp$	$\Rightarrow E\ 2,3$

Remark: $\neg a$ is the abbreviation of $a \Rightarrow \perp$.

Second Example

Prove $a \wedge \neg a \Rightarrow b$, *i.e.*, $\models a \wedge \neg a \Rightarrow b$

context	number	line	justification
1	1	Assume $a \wedge \neg a$	
1	2	a	$\wedge E1\ 1$
1	3	$\neg a$	$\wedge E2\ 1$
1	4	$\bot$	$\Rightarrow E\ 2,3$
1	5	b	$Efq\ 4$

Second Example

Prove $a \wedge \neg a \Rightarrow b$, *i.e.*, $\models a \wedge \neg a \Rightarrow b$

context	number	line	justification
1	1	Assume $a \wedge \neg a$	
1	2	a	$\wedge E1$ 1
1	3	$\neg a$	$\wedge E2$ 1
1	4	$\perp$	$\Rightarrow E$ 2, 3
1	5	b	Efq 4
	6	Therefore $a \wedge \neg a \Rightarrow b$	$\Rightarrow I$ 1, 5

Third Example

$$(p \Rightarrow \neg j) \wedge (\neg p \Rightarrow j) \wedge (j \Rightarrow m) \Rightarrow m \vee p$$

Third Example

$$(p \Rightarrow \neg j) \land (\neg p \Rightarrow j) \land (j \Rightarrow m) \Rightarrow m \lor p$$

context	number	line	justification
1	1	`assume` $(p \Rightarrow \neg j) \land (\neg p \Rightarrow j) \land (j \Rightarrow m)$	

Third Example

$$(p \Rightarrow \neg j) \wedge (\neg p \Rightarrow j) \wedge (j \Rightarrow m) \Rightarrow m \vee p$$

context	number	line	justification
1	1	assume $(p \Rightarrow \neg j) \wedge (\neg p \Rightarrow j) \wedge (j \Rightarrow m)$	
1	2	$(\neg p \Rightarrow j) \wedge (j \Rightarrow m)$	$\wedge$E2 1

Third Example

$$(p \Rightarrow \neg j) \wedge (\neg p \Rightarrow j) \wedge (j \Rightarrow m) \Rightarrow m \vee p$$

context	number	line	justification
1	1	assume $(p \Rightarrow \neg j) \wedge (\neg p \Rightarrow j) \wedge (j \Rightarrow m)$	
1	2	$(\neg p \Rightarrow j) \wedge (j \Rightarrow m)$	$\wedge$E2 1
1	3	$(j \Rightarrow m)$	$\wedge$E2 2

Third Example

$$(p \Rightarrow \neg j) \wedge (\neg p \Rightarrow j) \wedge (j \Rightarrow m) \Rightarrow m \vee p$$

context	number	line	justification
1	1	`assume` $(p \Rightarrow \neg j) \wedge (\neg p \Rightarrow j) \wedge (j \Rightarrow m)$	
1	2	$(\neg p \Rightarrow j) \wedge (j \Rightarrow m)$	$\wedge$E2 1
1	3	$(j \Rightarrow m)$	$\wedge$E2 2
1	4	$(\neg p \Rightarrow j)$	$\wedge$E1 2

Third Example

$$(p \Rightarrow \neg j) \wedge (\neg p \Rightarrow j) \wedge (j \Rightarrow m) \Rightarrow m \vee p$$

context	number	line	justification
1	1	assume $(p \Rightarrow \neg j) \wedge (\neg p \Rightarrow j) \wedge (j \Rightarrow m)$	
1	2	$(\neg p \Rightarrow j) \wedge (j \Rightarrow m)$	$\wedge$E2 1
1	3	$(j \Rightarrow m)$	$\wedge$E2 2
1	4	$(\neg p \Rightarrow j)$	$\wedge$E1 2
1,5	5	assume $\neg(m \vee p)$	

Third Example

$$(p \Rightarrow \neg j) \wedge (\neg p \Rightarrow j) \wedge (j \Rightarrow m) \Rightarrow m \vee p$$

context	number	line	justification
1	1	assume $(p \Rightarrow \neg j) \wedge (\neg p \Rightarrow j) \wedge (j \Rightarrow m)$	
1	2	$(\neg p \Rightarrow j) \wedge (j \Rightarrow m)$	$\wedge$E2 1
1	3	$(j \Rightarrow m)$	$\wedge$E2 2
1	4	$(\neg p \Rightarrow j)$	$\wedge$E1 2
1,5	5	assume $\neg(m \vee p)$	
1,5,6	6	assume p	

Third Example

$$(p \Rightarrow \neg j) \land (\neg p \Rightarrow j) \land (j \Rightarrow m) \Rightarrow m \lor p$$

context	number	line	justification
1	1	assume $(p \Rightarrow \neg j) \land (\neg p \Rightarrow j) \land (j \Rightarrow m)$	
1	2	$(\neg p \Rightarrow j) \land (j \Rightarrow m)$	$\land$E2 1
1	3	$(j \Rightarrow m)$	$\land$E2 2
1	4	$(\neg p \Rightarrow j)$	$\land$E1 2
1,5	5	assume $\neg(m \lor p)$	
1,5,6	6	assume p	
1,5,6	7	$m \lor p$	$\lor$I2 6

Third Example

$$(p \Rightarrow \neg j) \wedge (\neg p \Rightarrow j) \wedge (j \Rightarrow m) \Rightarrow m \vee p$$

context	number	line	justification
1	1	assume $(p \Rightarrow \neg j) \wedge (\neg p \Rightarrow j) \wedge (j \Rightarrow m)$	
1	2	$(\neg p \Rightarrow j) \wedge (j \Rightarrow m)$	$\wedge$E2 1
1	3	$(j \Rightarrow m)$	$\wedge$E2 2
1	4	$(\neg p \Rightarrow j)$	$\wedge$E1 2
1,5	5	assume $\neg(m \vee p)$	
1,5,6	6	assume p	
1,5,6	7	$m \vee p$	$\vee$I2 6
1,5,6	8	$\perp$	$\Rightarrow$E 5,7

Remark: Line 5, $\neg(m \vee p)$ is the abbreviation of $(m \vee p) \Rightarrow \perp$.

Third Example

$$(p \Rightarrow \neg j) \wedge (\neg p \Rightarrow j) \wedge (j \Rightarrow m) \Rightarrow m \vee p$$

context	number	line	justification
1	1	assume $(p \Rightarrow \neg j) \wedge (\neg p \Rightarrow j) \wedge (j \Rightarrow m)$	
1	2	$(\neg p \Rightarrow j) \wedge (j \Rightarrow m)$	$\wedge$E2 1
1	3	$(j \Rightarrow m)$	$\wedge$E2 2
1	4	$(\neg p \Rightarrow j)$	$\wedge$E1 2
1,5	5	assume $\neg(m \vee p)$	
1,5,6	6	assume p	
1,5,6	7	$m \vee p$	$\vee$I2 6
1,5,6	8	$\perp$	$\Rightarrow$E 5,7
1,5	9	therefore $\neg p$	$\Rightarrow$I 6,8

Remark: $\neg p$ is the abbreviation of $p \Rightarrow \perp$.

Third Example

$$(p \Rightarrow \neg j) \wedge (\neg p \Rightarrow j) \wedge (j \Rightarrow m) \Rightarrow m \vee p$$

context	number	line	justification
1	1	assume $(p \Rightarrow \neg j) \wedge (\neg p \Rightarrow j) \wedge (j \Rightarrow m)$	
1	2	$(\neg p \Rightarrow j) \wedge (j \Rightarrow m)$	$\wedge$E2 1
1	3	$(j \Rightarrow m)$	$\wedge$E2 2
1	4	$(\neg p \Rightarrow j)$	$\wedge$E1 2
1,5	5	assume $\neg(m \vee p)$	
1,5,6	6	assume p	
1,5,6	7	$m \vee p$	$\vee$I2 6
1,5,6	8	$\perp$	$\Rightarrow$E 5,7
1,5	9	therefore $\neg p$	$\Rightarrow$I 6,8
1,5	10	j	$\Rightarrow$E 4,9

Third Example

$$(p \Rightarrow \neg j) \wedge (\neg p \Rightarrow j) \wedge (j \Rightarrow m) \Rightarrow m \vee p$$

context	number	line	justification
1	1	assume $(p \Rightarrow \neg j) \wedge (\neg p \Rightarrow j) \wedge (j \Rightarrow m)$	
1	2	$(\neg p \Rightarrow j) \wedge (j \Rightarrow m)$	$\wedge$E2 1
1	3	$(j \Rightarrow m)$	$\wedge$E2 2
1	4	$(\neg p \Rightarrow j)$	$\wedge$E1 2
1,5	5	assume $\neg(m \vee p)$	
1,5,6	6	assume p	
1,5,6	7	$m \vee p$	$\vee$I2 6
1,5,6	8	$\bot$	$\Rightarrow$E 5,7
1,5	9	therefore $\neg p$	$\Rightarrow$I 6,8
1,5	10	j	$\Rightarrow$E 4,9
1,5	11	m	$\Rightarrow$E 3,10

Third Example

$$(p \Rightarrow \neg j) \wedge (\neg p \Rightarrow j) \wedge (j \Rightarrow m) \Rightarrow m \vee p$$

context	number	line	justification
1	1	`assume` $(p \Rightarrow \neg j) \wedge (\neg p \Rightarrow j) \wedge (j \Rightarrow m)$	
1	2	$(\neg p \Rightarrow j) \wedge (j \Rightarrow m)$	$\wedge$E2 1
1	3	$(j \Rightarrow m)$	$\wedge$E2 2
1	4	$(\neg p \Rightarrow j)$	$\wedge$E1 2
1,5	5	`assume` $\neg(m \vee p)$	
1,5,6	6	`assume` p	
1,5,6	7	$m \vee p$	$\vee$I2 6
1,5,6	8	$\bot$	$\Rightarrow$E 5,7
1,5	9	`therefore` $\neg p$	$\Rightarrow$I 6,8
1,5	10	j	$\Rightarrow$E 4,9
1,5	11	m	$\Rightarrow$E 3,10
1,5	12	$m \vee p$	$\vee$I1 11

Third Example

$$(p \Rightarrow \neg j) \wedge (\neg p \Rightarrow j) \wedge (j \Rightarrow m) \Rightarrow m \vee p$$

context	number	line	justification
1	1	assume $(p \Rightarrow \neg j) \wedge (\neg p \Rightarrow j) \wedge (j \Rightarrow m)$	
1	2	$(\neg p \Rightarrow j) \wedge (j \Rightarrow m)$	$\wedge$E2 1
1	3	$(j \Rightarrow m)$	$\wedge$E2 2
1	4	$(\neg p \Rightarrow j)$	$\wedge$E1 2
1,5	5	assume $\neg(m \vee p)$	
1,5,6	6	assume p	
1,5,6	7	$m \vee p$	$\vee$I2 6
1,5,6	8	$\perp$	$\Rightarrow$E 5,7
1,5	9	therefore $\neg p$	$\Rightarrow$I 6,8
1,5	10	j	$\Rightarrow$E 4,9
1,5	11	m	$\Rightarrow$E 3,10
1,5	12	$m \vee p$	$\vee$I1 11
1,5	13	$\perp$	$\Rightarrow$E 5,12

Remark: Line 5, $\neg(m \vee p)$ is the abbreviation of $(m \vee p) \Rightarrow \perp$.

Third Example

$$(p \Rightarrow \neg j) \wedge (\neg p \Rightarrow j) \wedge (j \Rightarrow m) \Rightarrow m \vee p$$

context	number	line	justification
1	1	assume $(p \Rightarrow \neg j) \wedge (\neg p \Rightarrow j) \wedge (j \Rightarrow m)$	
1	2	$(\neg p \Rightarrow j) \wedge (j \Rightarrow m)$	$\wedge$E2 1
1	3	$(j \Rightarrow m)$	$\wedge$E2 2
1	4	$(\neg p \Rightarrow j)$	$\wedge$E1 2
1,5	5	assume $\neg(m \vee p)$	
1,5,6	6	assume p	
1,5,6	7	$m \vee p$	$\vee$I2 6
1,5,6	8	$\bot$	$\Rightarrow$E 5,7
1,5	9	therefore $\neg p$	$\Rightarrow$I 6,8
1,5	10	j	$\Rightarrow$E 4,9
1,5	11	m	$\Rightarrow$E 3,10
1,5	12	$m \vee p$	$\vee$I1 11
1,5	13	$\bot$	$\Rightarrow$E 5,12
1	14	therefore $\neg\neg(m \vee p)$	$\Rightarrow$I 5,13

Remark: $\neg\neg(m \vee p)$ is an abbreviation of $\neg(m \vee p) \Rightarrow \bot$.

Third Example

$$(p \Rightarrow \neg j) \wedge (\neg p \Rightarrow j) \wedge (j \Rightarrow m) \Rightarrow m \vee p$$

context	number	line	justification
1	1	`assume` $(p \Rightarrow \neg j) \wedge (\neg p \Rightarrow j) \wedge (j \Rightarrow m)$	
1	2	$(\neg p \Rightarrow j) \wedge (j \Rightarrow m)$	$\wedge$E2 1
1	3	$(j \Rightarrow m)$	$\wedge$E2 2
1	4	$(\neg p \Rightarrow j)$	$\wedge$E1 2
1,5	5	`assume` $\neg(m \vee p)$	
1,5,6	6	`assume` p	
1,5,6	7	$m \vee p$	$\vee$I2 6
1,5,6	8	$\perp$	$\Rightarrow$E 5,7
1,5	9	`therefore` $\neg p$	$\Rightarrow$I 6,8
1,5	10	j	$\Rightarrow$E 4,9
1,5	11	m	$\Rightarrow$E 3,10
1,5	12	$m \vee p$	$\vee$I1 11
1,5	13	$\perp$	$\Rightarrow$E 5,12
1	14	`therefore` $\neg\neg(m \vee p)$	$\Rightarrow$I 5,13
1	15	$m \vee p$	*RAA* 14

Third Example

$$(p \Rightarrow \neg j) \wedge (\neg p \Rightarrow j) \wedge (j \Rightarrow m) \Rightarrow m \vee p$$

context	number	line	justification
1	1	assume $(p \Rightarrow \neg j) \wedge (\neg p \Rightarrow j) \wedge (j \Rightarrow m)$	
1	2	$(\neg p \Rightarrow j) \wedge (j \Rightarrow m)$	$\wedge$E2 1
1	3	$(j \Rightarrow m)$	$\wedge$E2 2
1	4	$(\neg p \Rightarrow j)$	$\wedge$E1 2
1,5	5	assume $\neg(m \vee p)$	
1,5,6	6	assume p	
1,5,6	7	$m \vee p$	$\vee$I2 6
1,5,6	8	$\perp$	$\Rightarrow$E 5,7
1,5	9	therefore $\neg p$	$\Rightarrow$I 6,8
1,5	10	j	$\Rightarrow$E 4,9
1,5	11	m	$\Rightarrow$E 3,10
1,5	12	$m \vee p$	$\vee$I1 11
1,5	13	$\perp$	$\Rightarrow$E 5,12
1	14	therefore $\neg\neg(m \vee p)$	$\Rightarrow$I 5,13
1	15	$m \vee p$	*RAA* 14
	16	therefore $(p \Rightarrow \neg j) \wedge (\neg p \Rightarrow j) \wedge (j \Rightarrow m) \Rightarrow m \vee p$	$\Rightarrow$I 1,15

Fourth Example: with an environment

Prove $\neg A$ in the environment $\neg(A \vee B)$, *i.e.*, $\neg(A \vee B) \models \neg A$

<table>
<tr><th colspan="4">environment</th></tr>
<tr><th colspan="2">reference</th><th colspan="2">formula</th></tr>
<tr><td colspan="2">(i)</td><td colspan="2">$\neg(A \vee B)$</td></tr>
<tr><td>context</td><td>number</td><td>line</td><td>justification</td></tr>
</table>

Fourth Example: with an environment

Prove $\neg A$ in the environment $\neg(A \vee B)$, *i.e.*, $\neg(A \vee B) \models \neg A$

environment			
reference		formula	
(i)		$\neg(A \vee B)$	
context	number	line	justification
1	1	`Assume` A	

Fourth Example: with an environment

Prove $\neg A$ in the environment $\neg(A \lor B)$, *i.e.*, $\neg(A \lor B) \models \neg A$

environment			
reference		formula	
(i)		$\neg(A \lor B)$	
context	number	line	justification
1	1	`Assume` A	
1	2	$A \lor B$	$\lor I1$ 1

Fourth Example: with an environment

Prove $\neg A$ in the environment $\neg(A \vee B)$, *i.e.*, $\neg(A \vee B) \models \neg A$

environment			
reference		formula	
(i)		$\neg(A \vee B)$	
context	number	line	justification
1	1	`Assume` A	
1	2	$A \vee B$	$\vee I1$ 1
1	3	$\perp$	$\Rightarrow E\ i, 2$

Remark: $\neg(A \vee B)$ is the abbreviation of $(A \vee B) \Rightarrow \perp$.

Fourth Example: with an environment

Prove $\neg A$ in the environment $\neg(A \vee B)$, *i.e.*, $\neg(A \vee B) \models \neg A$

environment			
reference		formula	
(i)		$\neg(A \vee B)$	
context	number	line	justification
1	1	Assume A	
1	2	$A \vee B$	$\vee I1$ 1
1	3	$\bot$	$\Rightarrow E\ i, 2$
	4	Therefore $\neg A$	$\Rightarrow I\ 1, 3$

Remark: $\neg A$ is the abbreviation of $A \Rightarrow \bot$.

Fifth example

Prove $\neg A \vee B$ in the environment $A \Rightarrow B$, *i.e.*, $A \Rightarrow B \models \neg A \vee B$.

environment			
reference		formula	
(i)		$A \Rightarrow B$	
context	number	line	justification

Fifth example

Prove $\neg A \vee B$ in the environment $A \Rightarrow B$, *i.e.*, $A \Rightarrow B \models \neg A \vee B$.

environment			
reference		formula	
(i)		$A \Rightarrow B$	
context	number	line	justification
1	1	`Assume` $\neg(\neg A \vee B)$	

Fifth example

Prove $\neg A \vee B$ in the environment $A \Rightarrow B$, *i.e.*, $A \Rightarrow B \models \neg A \vee B$.

environment			
reference		formula	
(i)		$A \Rightarrow B$	
context	number	line	justification
1	1	`Assume` $\neg(\neg A \vee B)$	
1,2	2	`Assume` A	

Fifth example

Prove $\neg A \vee B$ in the environment $A \Rightarrow B$, *i.e.*, $A \Rightarrow B \models \neg A \vee B$.

environment			
reference		formula	
(i)		$A \Rightarrow B$	
context	number	line	justification
1	1	Assume $\neg(\neg A \vee B)$	
1,2	2	Assume A	
1,2	3	B	$\Rightarrow E\ i, 2$

Fifth example

Prove $\neg A \vee B$ in the environment $A \Rightarrow B$, *i.e.*, $A \Rightarrow B \models \neg A \vee B$.

environment			
reference		formula	
(i)		$A \Rightarrow B$	
context	number	line	justification
1	1	Assume $\neg(\neg A \vee B)$	
1,2	2	Assume A	
1,2	3	B	$\Rightarrow E\ i, 2$
1,2	4	$\neg A \vee B$	$\vee I2\ 3$

Fifth example

Prove $\neg A \vee B$ in the environment $A \Rightarrow B$, *i.e.*, $A \Rightarrow B \models \neg A \vee B$.

environment			
reference		formula	
(i)		$A \Rightarrow B$	
context	number	line	justification
1	1	`Assume` $\neg(\neg A \vee B)$	
1,2	2	`Assume` A	
1,2	3	B	$\Rightarrow E\ i, 2$
1,2	4	$\neg A \vee B$	$\vee I2\ 3$
1,2	5	$\perp$	$\Rightarrow E\ 1, 4$

Remark: $\neg(\neg A \vee B)$ is the abbreviation of $(\neg A \vee B) \Rightarrow \perp$

Fifth example

Prove $\neg A \vee B$ in the environment $A \Rightarrow B$, *i.e.*, $A \Rightarrow B \models \neg A \vee B$.

environment			
reference		formula	
(i)		$A \Rightarrow B$	
context	number	line	justification
1	1	`Assume` $\neg(\neg A \vee B)$	
1,2	2	`Assume` A	
1,2	3	B	$\Rightarrow E\ i, 2$
1,2	4	$\neg A \vee B$	$\vee I2\ 3$
1,2	5	$\bot$	$\Rightarrow E\ 1, 4$
1	6	`Therefore` $\neg A$	$\Rightarrow I\ 2, 5$

Remark: $\neg A$ is the abbreviation of $A \Rightarrow \bot$

Fifth example

Prove $\neg A \lor B$ in the environment $A \Rightarrow B$, *i.e.*, $A \Rightarrow B \models \neg A \lor B$.

environment			
reference		formula	
(i)		$A \Rightarrow B$	
context	number	line	justification
1	1	Assume $\neg(\neg A \lor B)$	
1,2	2	Assume A	
1,2	3	B	$\Rightarrow E$ i, 2
1,2	4	$\neg A \lor B$	$\lor I2$ 3
1,2	5	$\bot$	$\Rightarrow E$ 1, 4
1	6	Therefore $\neg A$	$\Rightarrow I$ 2, 5
1	7	$\neg A \lor B$	$\lor I1$ 6

Fifth example

Prove $\neg A \vee B$ in the environment $A \Rightarrow B$, *i.e.*, $A \Rightarrow B \models \neg A \vee B$.

environment			
reference		formula	
(i)		$A \Rightarrow B$	
context	number	line	justification
1	1	`Assume` $\neg(\neg A \vee B)$	
1,2	2	`Assume` A	
1,2	3	B	$\Rightarrow E\ i, 2$
1,2	4	$\neg A \vee B$	$\vee I2\ 3$
1,2	5	$\bot$	$\Rightarrow E\ 1, 4$
1	6	`Therefore` $\neg A$	$\Rightarrow I\ 2, 5$
1	7	$\neg A \vee B$	$\vee I1\ 6$
1	8	$\bot$	$\Rightarrow E\ 1, 7$

Remark: line 1, $\neg(\neg A \vee B)$ is the abbreviation of $(\neg A \vee B) \Rightarrow \bot$

Fifth example

Prove $\neg A \vee B$ in the environment $A \Rightarrow B$, *i.e.*, $A \Rightarrow B \models \neg A \vee B$.

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(i)		$A \Rightarrow B$	
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1	7	$\neg A \vee B$	$\vee I1\ 6$
1	8	$\bot$	$\Rightarrow E\ 1, 7$
	9	`Therefore` $\neg\neg(\neg A \vee B)$	$\Rightarrow I\ 1, 8$

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Prove $\neg A \vee B$ in the environment $A \Rightarrow B$, *i.e.*, $A \Rightarrow B \models \neg A \vee B$.

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reference		formula	
(*i*)		$A \Rightarrow B$	
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1,2	4	$\neg A \vee B$	$\vee I2\ 3$
1,2	5	$\perp$	$\Rightarrow E\ 1, 4$
1	6	`Therefore` $\neg A$	$\Rightarrow I\ 2, 5$
1	7	$\neg A \vee B$	$\vee I1\ 6$
1	8	$\perp$	$\Rightarrow E\ 1, 7$
	9	`Therefore` $\neg\neg(\neg A \vee B)$	$\Rightarrow I\ 1, 8$
	10	$\neg A \vee B$	$RAA\ 9$

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Theorem 2

If a formula A is deduced from an environment Γ ($\Gamma \vdash A$) then A is a consequence of Γ ($\Gamma \models A$).

Every proof written in an environment Γ is correct!

In particular, if $\Gamma = \emptyset$, then $\vdash A$ implies $\models A$.

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Every proof written in an environment Γ is correct!

In particular, if $\Gamma = \emptyset$, then $\vdash A$ implies $\models A$.

Let $\Gamma \vdash P : A$. Proof by induction on the number of lines i in P:

- Let H_i be the context and C_i the conclusion of the i^{th} line in P. (We let $H_0 = \emptyset$. If P is empty, we let $C_0 = A$.)
- We show that for every k we have $\Gamma, H_k \models C_k$.

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- We show that for every k we have $\Gamma, H_k \models C_k$.

Hence, for the last line (n) of the proof, we have $\Gamma \models A$

(Remember that H_n is empty and $C_n = A$.)

Base case

Assume that A is derived from Γ by an empty proof.

That is, A is a member of Γ.

Hence $\Gamma \models A$. Since $H_0 = \emptyset$, we can conclude that $\Gamma, H_0 \models A$, so $\Gamma, H_0 \models C_0$.

Induction hypothesis

Assume that for every line $i < k$ of the proof P we have $\Gamma,\ H_i \models C_i$.

Let us show that $\Gamma,\ H_k \models C_k$.

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Assume that for every line $i < k$ of the proof P we have $\Gamma,\ H_i \models C_i$.

Let us show that $\Gamma,\ H_k \models C_k$.

Three possible cases:

- Line k is "`Assume` C_k".
- Line k is "`Therefore` C_k".
- Line k is "C_k".

Line k is "Assume C_k"

The formula C_k is the last formula of H_k.

Then $H_k \models C_k$.

Then $\Gamma, H_k \models C_k$.

The line k is "`Therefore` C_k"

C_k is the formula $B \Rightarrow D$ where:

- B is the last formula of H_{k-1}: $H_{k-1} = H_k, B$
- D is either a formula in Γ or is usable on the previous line $k-1$.

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(2) If D is usable on the previous line.

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(1) If D is a formula of Γ.
$\Gamma \models D$
$\Gamma,\ H_k \models D$.
Since $D \models B \Rightarrow D$, we conclude that $\Gamma,\ H_k \models B \Rightarrow D$, *i.e.*,
$\Gamma,\ H_k \models C_k$.

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(2) If D is usable on the previous line.
Hence $\exists i < k$ such that $D = C_i$ and H_i is a prefix of H_{k-1}.
By induction hypothesis, $\Gamma,\ H_i \models D$.

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C_k is the formula $B \Rightarrow D$ where:

- B is the last formula of H_{k-1}: $H_{k-1} = H_k, B$
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(1) If D is a formula of Γ.
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(2) If D is usable on the previous line.
Hence $\exists i < k$ such that $D = C_i$ and H_i is a prefix of H_{k-1}.
By induction hypothesis, $\Gamma,\ H_i \models D$.
Since H_i is a prefix of H_{k-1}, we have $\Gamma,\ H_{k-1} \models D$ which can also be written $\Gamma,\ H_k,\ B \models D$.
Therefore $\Gamma,\ H_k \models B \Rightarrow D$, *i.e.*, $\Gamma,\ H_k \models C_k$.

Line k is "C_k"

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By induction hypothesis (and similarly to the previous case), we have:
$\Gamma, H_{k-1} \models D$ and $\Gamma, H_{k-1} \models E$.

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Finally $D,\ E \models D \wedge E$. Transitively, $\Gamma,\ H_k \models D \wedge E$, *i.e.*, $\Gamma,\ H_k \models C_k$.

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Finally $D,\ E \models D \wedge E$. Transitively, $\Gamma,\ H_k \models D \wedge E$, *i.e.*, $\Gamma,\ H_k \models C_k$.

For the other rules, it is the same proof, you just have to prove that the conclusion is a consequence of the premises.

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We prove the completeness of the rules only for formulas containing the following logic symbols: $\bot$, $\wedge$, $\vee$, $\Rightarrow$.

This is enough because additional symbols $\top$, $\neg$ and $\Leftrightarrow$ are abbreviations.

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Theorem 3

Let Γ be a finite set of formulae and A a formula.
If $\Gamma \models A$ then $\Gamma \vdash A$.

Definitions

A literal is either a variable x or an implication $x \Rightarrow \bot$.
x and $x \Rightarrow \bot$ (abbreviated as $\neg x$) are complementary literals.

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We define a measure m of formulae and of lists of formulae as:

- $m(\bot) = 0$
- $m(x) = 1$
- $m(A \Rightarrow B) = 1 + m(A) + m(B)$
- $m(A \wedge B) = 1 + m(A) + m(B)$
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A literal is either a variable x or an implication $x \Rightarrow \perp$.
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- $m(x) = 1$
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- $m(A \wedge B) = 1 + m(A) + m(B)$
- $m(A \vee B) = 2 + m(A) + m(B)$
- $m(\Gamma) = \sum_{A \in \Gamma} m(A)$

For example, let $A = (a \vee \neg a)$.
$m(\neg a) = 2$, $m(A) = 5$ and $m(A,\ (b \wedge b),\ A) = 13$.

Induction

We define $P(n)$ to be the following property:

If $m(\Gamma,\ A) = n$, then if $\Gamma \models A$ then $\Gamma \vdash A$.

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$$\text{If } m(\Gamma,\ A) = n, \text{ then if } \Gamma \models A \text{ then } \Gamma \vdash A.$$

To show that $P(n)$ holds for every integer n, we use "strong" induction:

Induction

We define $P(n)$ to be the following property:

$$\text{If } m(\Gamma, A) = n\text{, then if } \Gamma \models A \text{ then } \Gamma \vdash A.$$

To show that $P(n)$ holds for every integer n, we use "strong" induction:

Assume that for every $i < k$, property $P(i)$ holds.

Assume that $m(\Gamma, A) = k$ and $\Gamma \models A$.

Let us show that $\Gamma \vdash A$.

Decomposition

Idea: we decompose Γ, A in order to apply the induction hypothesis.

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- A is **undecomposable** if A is $\bot$ or a variable
- Γ is **undecomposable** if Γ is a list of literals or contains $\bot$.

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We study three cases:

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Case 1: Neither A, nor Γ is decomposable.

Case 2: A is decomposable.

We decompose A in two sub-formulae B and C.

We obtain $m(\Gamma, B) < m(\Gamma, A)$ and $m(\Gamma, C) < m(\Gamma, A)$ and so we can apply the induction hypothesis.

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We obtain $m(\Gamma, B) < m(\Gamma, A)$ and $m(\Gamma, C) < m(\Gamma, A)$ and so we can apply the induction hypothesis.

Case 3: Γ is decomposable: we choose in Γ a decomposable formula.

(*i.e.*, other than $\bot$, x, and $x \Rightarrow \bot$ where x is a variable).

We decompose it.

The new set Γ' satisfies $m(\Gamma', A) < m(\Gamma, A)$, and so we can apply the induction hypothesis.

Case 1 : neither A, nor Γ are decomposable

Then:

- Γ is a list of literals or contains the formula $\perp$.
- A is $\perp$ or a variable.

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Then:

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(a) If $\bot \in \Gamma$ then A can be derived from $\bot$ by the rule *Efq*.

Case 1 : neither A, nor Γ are decomposable

Then:

- Γ is a list of literals or contains the formula $\bot$.
- A is $\bot$ or a variable.

(a) If $\bot \in \Gamma$ then A can be derived from $\bot$ by the rule *Efq*.
(b) If $\bot \notin \Gamma$ and Γ is a list of literals, then we have two cases:

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(b) If $\bot \notin \Gamma$ and Γ is a list of literals, then we have two cases:

- $A = \bot$.
  Since $\Gamma \models A$, there are two complementary literals in Γ.
  Therefore A can be derived from Γ by the rule $\Rightarrow$E.

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- $A = \bot$.
  Since $\Gamma \models A$, there are two complementary literals in Γ.
  Therefore A can be derived from Γ by the rule $\Rightarrow$E.
- A is a variable.
  Since $\Gamma \models A$:

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Then:

- Γ is a list of literals or contains the formula $\perp$.
- A is $\perp$ or a variable.

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(b) If $\perp \notin \Gamma$ and Γ is a list of literals, then we have two cases:

- $A = \perp$.
  Since $\Gamma \models A$, there are two complementary literals in Γ.
  Therefore A can be derived from Γ by the rule $\Rightarrow$E.
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  Since $\Gamma \models A$, there are two complementary literals in Γ.
  Therefore A can be derived from Γ by the rule $\Rightarrow$E.
- A is a variable.
  Since $\Gamma \models A$:
  - either Γ contains two complementary literals, and similarly $\Gamma \vdash A$ by $\Rightarrow$E and then *Efq*
  - or $A \in \Gamma$ and in this case $\Gamma \vdash A$ by an empty proof.

Case 2: A is decomposable into B and C

A is decomposed into $B \wedge C$, $B \vee C$, or $B \Rightarrow C$.

We only study the case $A = B \wedge C$, the other cases are similar.

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Since $\Gamma \models A$ and $A = B \wedge C$, we have $\Gamma \models B$ and $\Gamma \models C$.

Case 2: A is decomposable into B and C

A is decomposed into $B \wedge C$, $B \vee C$, or $B \Rightarrow C$.

We only study the case $A = B \wedge C$, the other cases are similar.

Since $\Gamma \models A$ and $A = B \wedge C$, we have $\Gamma \models B$ and $\Gamma \models C$.

Now $m(B) < m(A)$ and $m(C) < m(A)$, hence $m(\Gamma, B) < k$ and $m(\Gamma, C) < k$.

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Now $m(B) < m(A)$ and $m(C) < m(A)$, hence $m(\Gamma, B) < k$ and $m(\Gamma, C) < k$.

By induction hypothesis, there exist two proofs P and Q such that $\Gamma \vdash P : B$ and $\Gamma \vdash Q : C$.

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We only study the case $A = B \wedge C$, the other cases are similar.

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Now $m(B) < m(A)$ and $m(C) < m(A)$, hence $m(\Gamma, B) < k$ and $m(\Gamma, C) < k$.

By induction hypothesis, there exist two proofs P and Q such that $\Gamma \vdash P : B$ and $\Gamma \vdash Q : C$.

Hence the proof "$P,\ Q,\ A$" is a proof of A in the environment Γ.

Case 3: Γ is decomposable

There is a decomposable formula in Γ which is either:

- $B \wedge C$
- $B \vee C$
- $B \Rightarrow C$ where $C \neq \bot$
- $(B \wedge C) \Rightarrow \bot$
- $(B \vee C) \Rightarrow \bot$
- $(B \Rightarrow C) \Rightarrow \bot$

We only study the first case.

Remark: the four last cases are due to the fact that $x \Rightarrow \bot$ is undecomposable whenever x is a variable.

Γ is a permutation of the list $(B \wedge C), \Delta$

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Γ and $(B \wedge C), \Delta$ have the same measure.

Since $\Gamma \models A$, we have $(B \wedge C), \Delta \models A$ and so $B, C, \Delta \models A$.

$m(B) + m(C) < m(B \wedge C)$

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Since $\Gamma \models A$, we have $(B \wedge C), \Delta \models A$ and so $B, C, \Delta \models A$.

$m(B) + m(C) < m(B \wedge C)$

Hence $m(B, C, \Delta, A) < m((B \wedge C), \Delta, A) = m(\Gamma, A) = k$.

By induction hypothesis, there exist a proof P such that $B, C, \Delta \vdash P : A$.

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Γ and $(B \wedge C), \Delta$ have the same measure.

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Hence $m(B, C, \Delta, A) < m((B \wedge C), \Delta, A) = m(\Gamma, A) = k$.

By induction hypothesis, there exist a proof P such that $B, C, \Delta \vdash P : A$.

Since

- B can be derived from $(B \wedge C)$ by the rule $\wedge$E1 and
- C can be derived from $(B \wedge C)$ by the rule $\wedge$E2

We have "$B,\ C,\ P$" is a proof of A in the environment Γ. So $\Gamma \vdash A$.

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The proof of completeness is constructive, that is it provides a complete (recursive) algorithm, **or equivalently** a set of tactics to construct the proofs of a formula in an environment.

However, these tactics can lead to long proofs.

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It is better then to use “optimized” tactics.

For example, to prove $B \vee C$:

- First try to prove B
- If failure, then try to prove C
- Otherwise, use Tactic 10 (prove C under the hypothesis $\neg B$)

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For example, to prove $B \vee C$:

- First try to prove B
- If failure, then try to prove C
- Otherwise, use Tactic 10 (prove C under the hypothesis $\neg B$)

Before explaining these tactics ... A few number of small proofs are hard coded!

$P1 : \neg B \Rightarrow C \models B \vee C$

environment			
reference		formula	
i		$\neg B \Rightarrow C$	
context	number	line	rule
1	1	`Assume` $\neg(B \vee C)$	
1,2	2	`Assume` B	
1,2	3	$B \vee C$	$\vee I1$ 2
1,2	4	$\bot$	$\Rightarrow E$ 1,3
1	5	`Therefore` $\neg B$	$\Rightarrow I$ 2,4
1	6	C	$\Rightarrow E$ i,5
1	7	$B \vee C$	$\vee I2$ 6
1	8	$\bot$	$\Rightarrow E$ 1,7
	9	`Therefore` $\neg\neg(B \vee C)$	$\Rightarrow I$ 1,8
	10	$B \vee C$	RAA 9

$P2 : B \Rightarrow C \models \neg B \vee C$

environment			
reference		formula	
i		$B \Rightarrow C$	
context	number	line	rule
1	1	Assume $\neg(\neg B \vee C)$	
1,2	2	Assume $\neg B$	
1,2	3	$\neg B \vee C$	$\vee I1$ 2
1,2	4	$\perp$	$\Rightarrow E$ 3,1
1	5	Therefore $\neg\neg B$	$\Rightarrow E$ 2,4
1	6	B	RAA 5
1	7	C	$\Rightarrow E$ i,6
1	8	$\neg B \vee C$	$\vee I2$ 7
1	9	$\perp$	$\Rightarrow E$ 1,8
	10	Therefore $\neg\neg(\neg B \vee C)$	$\Rightarrow E$ 1,9
	11	$\neg B \vee C$	RAA 10

$P3 : \neg(B \wedge C) \models \neg B \vee \neg C$

environment			
reference		formula	
i		$\neg(B \wedge C)$	
context	number	line	rule
1	1	Assume $\neg(\neg B \vee \neg C)$	
1,2	2	Assume $\neg B$	
1,2	3	$\neg B \vee \neg C$	$\vee I1$ 2
1,2	4	$\perp$	$\Rightarrow E$ 1,3
1	5	Therefore $\neg\neg B$	$\Rightarrow I$ 2,4
1	6	B	RAA 5
1,7	7	Assume $\neg C$	
1,7	8	$\neg B \vee \neg C$	$\vee I2$ 7
1,7	9	$\perp$	$\Rightarrow E$ 8,1
1	10	Therefore $\neg\neg C$	$\Rightarrow I$ 7,9
1	11	C	RAA 10
1	12	$B \wedge C$	$\wedge I$ 6,11
1	13	$\perp$	$\Rightarrow E$ i,12
	14	Therefore $\neg\neg(\neg B \vee \neg C)$	$\Rightarrow I$ 1,13
	15	$\neg B \vee \neg C$	RAA 14

$P4 : \neg(B \vee C) \models \neg B$

environment			
reference		formula	
i		$\neg(B \vee C)$	
context	number	line	rule
1	1	Assume B	
1	2	$B \vee C$	$\vee I1$ 1
1	3	$\bot$	$\Rightarrow E\ i$,2
	4	Therefore $\neg B$	$\Rightarrow I$ 1,3

Similarly, $P5 : \neg(B \vee C) \models \neg C$

$P6 : \neg(B \Rightarrow C) \models B$

environment			
reference		formula	
i		$\neg(B \Rightarrow C)$	
context	number	line	rule
1	1	Assume $\neg B$	
1,2	2	Assume B	
1,2	3	$\bot$	$\Rightarrow E$ 2,1
1,2	4	C	*Efq* 3
1	5	Therefore $B \Rightarrow C$	$\Rightarrow I$ 2,4
1	6	$\bot$	$\Rightarrow E$ i,5
	7	Therefore $\neg\neg B$	$\Rightarrow I$ 1,6
	8	B	*RAA* 7

$P7 : \neg(B \Rightarrow C) \models \neg C$

environment			
reference		formula	
i		$\neg(B \Rightarrow C)$	
context	number	line	rule
1	1	`Assume` C	
1,2	2	`Assume` B	
1	3	`Therefore` $B \Rightarrow C$	$\Rightarrow I$ 1,2
1	4	$\perp$	$\Rightarrow E$ i,3
	5	`Therefore` $\neg C$	$\Rightarrow I$ 1,4

Proof Tactics

We wish to prove A in the environment Γ

The 13 following tactics must be used in the following order!

Tactic 1

If $A \in \Gamma$, then the proof is empty.

Tactic 2

If *A* is the conclusion of a rule *R* whose premises are in Γ, then the proof is

context	line	justification
Γ	*A*	*R*

Tactic 3

If Γ contains a contradiction, *i.e.* two formulae of the form B and $\neg B$, then proof is

context	line	justification
Γ	$\bot$	$\Rightarrow$E
Γ	A	*Efq*

Tactic 4

If $A = B \wedge C$, then

context	line	justification
Γ	...	
Γ	B	
Γ	...	
Γ	C	
Γ	$B \wedge C$	$\wedge I$

Tactic 4

If $A = B \wedge C$, then

context	line	justification
Γ	...	
Γ	B	
Γ	...	
Γ	C	
Γ	$B \wedge C$	$\wedge I$

The proofs can fail (if it is asked to prove a formula that is unprovable in the given environment).

if the proof of B or C fails, then A is not valid.

To simplify the remaining, we do not highlight the failure cases anymore, **unless they must be followed by another tactic.**

Tactic 5

If $A = B \Rightarrow C$, then prove C under hypothesis B, or equivalently prove C in the environment Γ, B, let P be the proof.

context	line	justification
Γ, B	`Assume` B	
Γ, B	$\cdots$	P
Γ, B	C	
Γ	`Therefore` $B \Rightarrow C$	$\Rightarrow I$

Tactic 6

If $A = B \vee C$, then prove B, let P be the proof.

context	line	justification
Γ	$\cdots$	P
Γ	B	
Γ	$B \vee C$	$\vee I1$

If the proof of B fails, then prove C, let Q be the proof.

context	line	justification
Γ	$\cdots$	Q
Γ	C	
Γ	$B \vee C$	$\vee I2$

If the proof of C fails, try the following tactics.

Tactic 7

If $\Gamma = \Gamma', B \wedge C$, then prove A in the environment Γ', B, C, let P the proof of A in Γ', B, C.

context	line	justification
$\Gamma', B \wedge C$	B	$\wedge E1$
$\Gamma', B \wedge C$	C	$\wedge E2$
$\Gamma', B \wedge C$	$\cdots$	P
$\Gamma', B \wedge C$	A	

Tactic 8

If $\Gamma = \Gamma', B \vee C$, then

- prove A in the environment Γ', B, let P be the proof
- prove A in the environment Γ', C, let Q be the proof

context	line	justification
$\Gamma', B \vee C, B$	Assume B	
$\Gamma', B \vee C, B$	$\cdots$	P
$\Gamma', B \vee C, B$	A	
$\Gamma', B \vee C$	Therefore $B \Rightarrow A$	$\Rightarrow I$
$\Gamma', B \vee C, C$	Assume C	
$\Gamma', B \vee C, C$	$\cdots$	Q
$\Gamma', B \vee C, C$	A	
$\Gamma', B \vee C$	Therefore $C \Rightarrow A$	$\Rightarrow I$
$\Gamma', B \vee C$	A	$\vee E$

Tactic 9

If $\Gamma = \Gamma', \neg(B \vee C)$, then

- prove $\neg B$ by $P4$,
- prove $\neg C$ by $P5$, and
- prove A in the environment $\Gamma', \neg B, \neg C$, let P be the proof.

context	line	justification
$\Gamma', \neg(B \vee C)$	$\cdots$	$P4$
$\Gamma', \neg(B \vee C)$	$\neg B$	
$\Gamma', \neg(B \vee C)$	$\cdots$	$P5$
$\Gamma', \neg(B \vee C)$	$\neg C$	
$\Gamma', \neg(B \vee C)$	$\cdots$	P
$\Gamma', \neg(B \vee C)$	A	

Tactic 10

If $A = B \vee C$, then prove C in the environment $\Gamma, \neg B$, let P be the proof.

context	line	justification
$\Gamma, \neg B$	Assume $\neg B$	
$\Gamma, \neg B$	$\cdots$	P
$\Gamma, \neg B$	C	
Γ	Therefore $\neg B \Rightarrow C$	
Γ	$\cdots$	$P1$
Γ	A	$A = B \vee C$

Tactic 11

If $\Gamma = \Gamma', \neg(B \wedge C)$, then prove $\neg B \vee \neg C$ by $P3$, and reason case by case as follows:

- prove A in the environment $\Gamma', \neg B$, let P the proof;
- prove A in the environment $\Gamma', \neg C$, let Q the proof.

context	line	justification
Γ	$\cdots$	$P3$
Γ	$\neg B \vee \neg C$	
$\Gamma, \neg B$	Assume $\neg B$	
$\Gamma, \neg B$	$\cdots$	P
$\Gamma, \neg B$	A	
Γ	Therefore $\neg B \Rightarrow A$	
$\Gamma, \neg C$	Assume $\neg C$	
$\Gamma, \neg C$	$\cdots$	Q
$\Gamma, \neg C$	A	
Γ	Therefore $\neg C \Rightarrow A$	
Γ	A	$\vee E$

Tactic 12

If $\Gamma = \Gamma', \neg(B \Rightarrow C)$, then

- prove B by $P6$,
- prove $\neg C$ by $P7$, and
- prove A in the environment $\Gamma', B, \neg C$, let P be the proof.

context	line	justification
$\Gamma', \neg(B \Rightarrow C)$	$\ldots$	$P6$
$\Gamma', \neg(B \Rightarrow C)$	B	
$\Gamma', \neg(B \Rightarrow C)$	$\cdots$	$P7$
$\Gamma', \neg(B \Rightarrow C)$	$\neg C$	
$\Gamma', \neg(B \Rightarrow C)$	$\cdots$	P
$\Gamma', \neg(B \Rightarrow C)$	A	

Tactic 13

If $\Gamma = \Gamma', B \Rightarrow C$ with $C \neq \bot$, *i.e.* if $B \Rightarrow C$ is not $\neg B$, then prove $\neg B \vee C$ in the environment $B \Rightarrow C$ by $P2$, and then we reason by cases:

- prove A in the environment $\Gamma', \neg B$, let P the proof;
- prove A in the environment Γ', C, let Q the proof.

context	line	justification
$\Gamma', B \Rightarrow C$	$\cdots$	$P2$
$\Gamma', B \Rightarrow C$	$\neg B \vee C$	
$\Gamma', B \Rightarrow C, \neg B$	Assume $\neg B$	
$\Gamma', B \Rightarrow C, \neg B$	$\cdots$	P
$\Gamma', B \Rightarrow C, \neg B$	A	
$\Gamma', B \Rightarrow C$	Therefore $\neg B \Rightarrow A$	
$\Gamma', B \Rightarrow C, C$	Assume C	
$\Gamma', B \Rightarrow C, C$	$\cdots$	Q
$\Gamma', B \Rightarrow C, C$	A	
$\Gamma', B \Rightarrow C$	Therefore $C \Rightarrow A$	
$\Gamma', B \Rightarrow C$	A	$\vee E$

Example

Proof of Peirce's law:

$$((p \Rightarrow q) \Rightarrow p) \Rightarrow p$$

$((p \Rightarrow q) \Rightarrow p) \Rightarrow p$: proof plan

Tactic 5 is mandatory!

Proof Q:
`Assume` $(p \Rightarrow q) \Rightarrow p$
Q_1 : proof of p in the environment $(p \Rightarrow q) \Rightarrow p$
`Therefore` $((p \Rightarrow q) \Rightarrow p) \Rightarrow p$

$((p \Rightarrow q) \Rightarrow p) \Rightarrow p$: proof plan

Tactic 5 is mandatory!

Proof Q:
`Assume` $(p \Rightarrow q) \Rightarrow p$
Q_1 : proof of p in the environment $(p \Rightarrow q) \Rightarrow p$
`Therefore` $((p \Rightarrow q) \Rightarrow p) \Rightarrow p$

Q_1 necessarily uses Tactic 13: indeed, Q_1 is written in the environment $B \Rightarrow C$ where $B = p \Rightarrow q,\ C = p$.

Proof Plan for Q_1

Proof of $A = p$ in the environment $B \Rightarrow C$ where $B = p \Rightarrow q,\ C = p$

Proof Q_1 :

$Q_{11} = P2$ where $P2$ is the proof of $\neg B \vee C$ in the environment $B \Rightarrow C$

`Assume` $\neg B$

Q_{12}: proof of $A = p$ in the environment $\neg B$

`Therefore` $\neg B \Rightarrow A$

`Assume` C

Q_{13}: proof of $A = p$ in the environment C

`Therefore` $C \Rightarrow A$

A

Proof of Q_1

Q_{13}, *i.e.*, proof of $A = p$ in the environment $C = p$: empty, since $A = C = p$.

Q_{12}: proof of $A = p$ in the environment $\neg B = \neg(p \Rightarrow q)$. The proof is actually $P6$.

By gluing pieces $Q_1,\ Q_{11},\ Q_{12},\ Q_{13}$, we obtain the proof Q.

Proof of Q_1

Q_{13}, *i.e.*, proof of $A = p$ in the environment $C = p$: empty, since $A = C = p$.

Q_{12}: proof of $A = p$ in the environment $\neg B = \neg(p \Rightarrow q)$. The proof is actually $P6$.

By gluing pieces $Q_1, Q_{11}, Q_{12}, Q_{13}$, we obtain the proof Q.

Below we show how to find the proof Q_{12} without using the tactics.

Proof of Q_{12}: $A = p$ in the environment $\neg(p \Rightarrow q)$

The only rule, which does not lead to a deadlock, is the reduction ad absurdum.

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The only rule, which does not lead to a deadlock, is the reduction ad absurdum.

Hence this proof is of the form:

```
Assume ¬p
  Q121: proof of ⊥ in the environment ¬(p ⇒ q), ¬p
Therefore ¬¬p
p
```

Proof of Q_{12}: $A = p$ in the environment $\neg(p \Rightarrow q)$

The only rule, which does not lead to a deadlock, is the reduction ad absurdum.

Hence this proof is of the form:

```
Assume ¬p
    Q121: proof of ⊥ in the environment ¬(p ⇒ q), ¬p
Therefore ¬¬p
p
```

To obtain $\perp$ in the environment $\neg(p \Rightarrow q), \neg p$,
$p \Rightarrow q$ must be derived. Hence, Q_{121} is:

```
Assume p
⊥
q
Therefore p ⇒ q
⊥
```

Plan

Introduction

Preliminaries

Specificities of Natural Deduction

Rules

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Examples

Correctness

Completeness

Algorithm

Conclusion

Automated proofs

To automatically obtain the proofs in the system, one recommends to use the following software (implementing the 13 previous tactics):

http://teachinglogic.liglab.fr/DN/

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Extension to First-Order formulae: complete, but undecidable.

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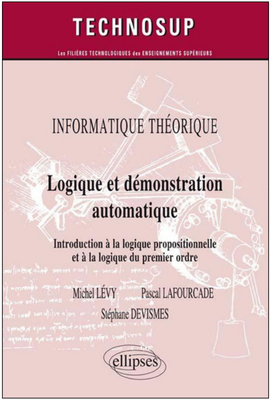
Extension to First-Order formulae: complete, but undecidable.

Several proof assistants (like `Coq`) are based on the (first-order) natural deduction.

For omitted details

See



In French, sorry!

Slides

Available on my Webpage:

`http://www-verimag.imag.fr/~devismes/`

Conclusion

Thank you for your attention.