Analysis of random walks using tabu lists

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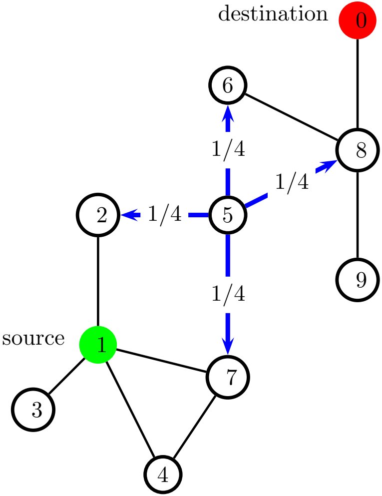
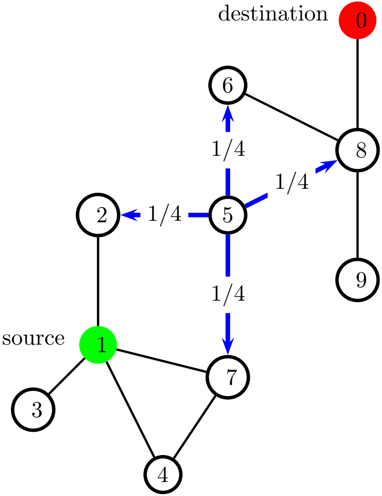
1. Random walk with tabu list
2. Termination
3. Optimal update rules for m-free graphs

Simple random walk routing



All graphs are finite, simple, connected, with at least two vertices.

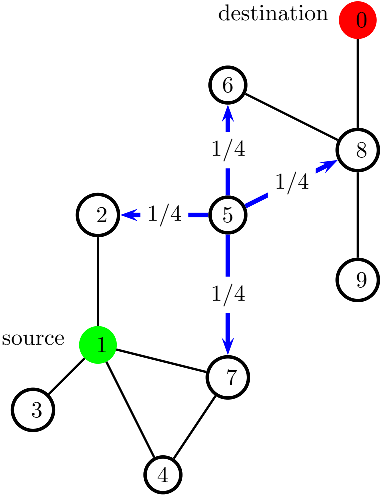
Simple random walk routing



At each step, the walker moves to a neighbor chosen uniformly at random.

The walker represents a packet forwarded from node to node.

Simple random walk routing



Advantages:

- ✓ no routing table;
- ✓ local computations;
- ✓ scalable;
- ✓ robust; and
- ✓ load-balanced.

Drawback : slow.

Mean hitting times

Hitting time: number of steps to deliver the packet to its final destination.

Motivation: reducing the mean hitting times.

Mean hitting times

Let $(X_n)_{n \geqslant 0}$ be a random walk on a graph G. For every two vertices x and y of G:

- The hitting time of y is the random variable

$$H_y(G) = \inf\{n \geqslant 0 \,:\, X_n = y\} \,.$$

- The mean hitting time of y starting from x is the mean value $E_x H_y(G)$.

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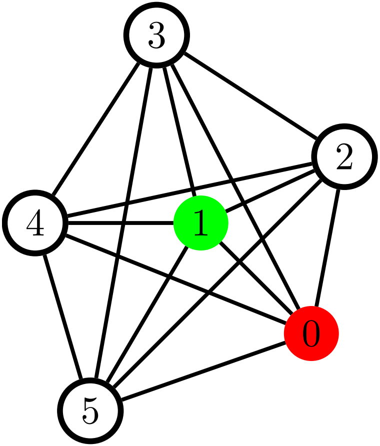

$E_1 H_0(K_6) = 5$

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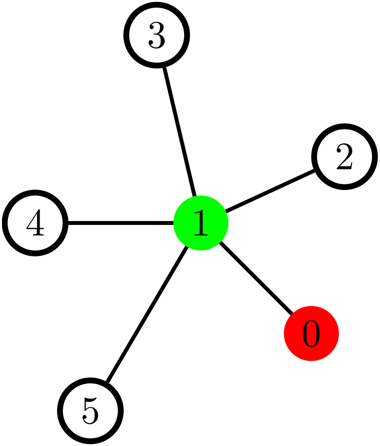

$E_1 H_0(S_6) = 2 \times 5$

Mean hitting times

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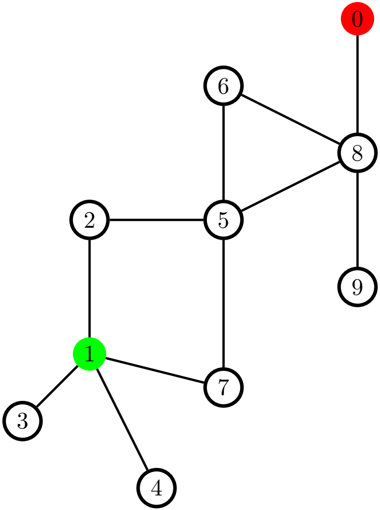
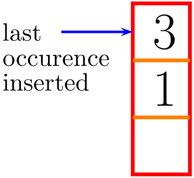


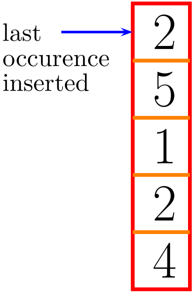
$E_1 H_0(G) = 39$

Random walk with tabu list

The walker has a finite memory in which it stores occurrences of vertices already visited, called tabu list. The occurrences are sorted in chronological order. The number of blocks is the length of the tabu list. The tabu list is full if all blocks are allocated. The policy to insert or remove occurrences of vertices in the tabu list is called the update rule.



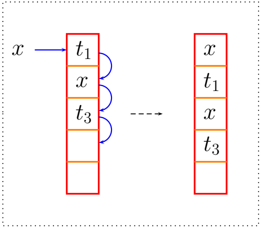
(a) The tabu list has length 3 with 2 allocated blocks.



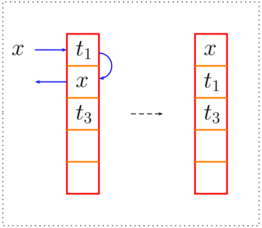
(b) The tabu list has length 5 and is full.

Random walk with tabu list

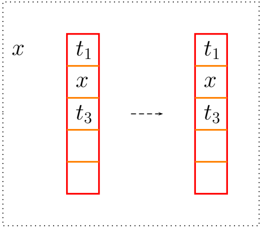
If the tabu list is not full, then three cases may occur:



(c) The current vertex is inserted.



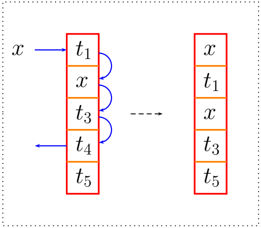
(d) The current vertex is inserted and an older occurrence of the current vertex is discarded.



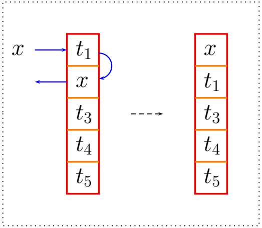
(e) The current vertex is not inserted.

Random walk with tabu list

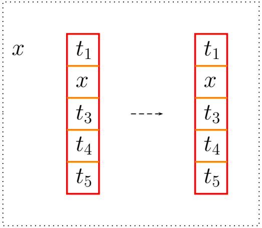
If the tabu list is full, then three cases may occur:



(f) The current vertex is inserted and an occurrence of another vertex is discarded.



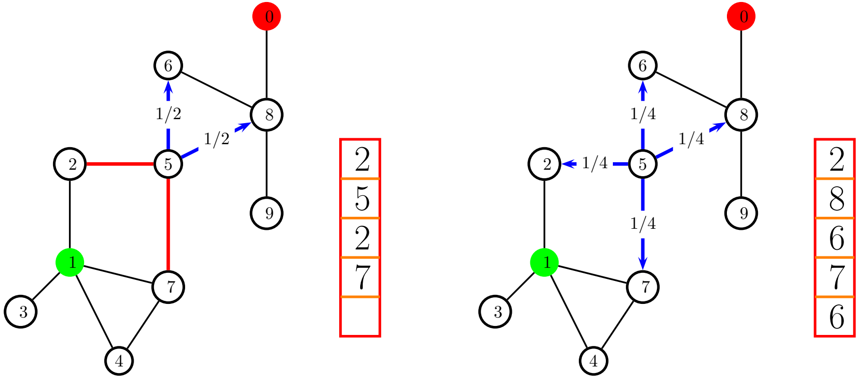
(g) The current vertex is inserted and an older occurrence of the current vertex is discarded.



(h) The current vertex is not inserted.

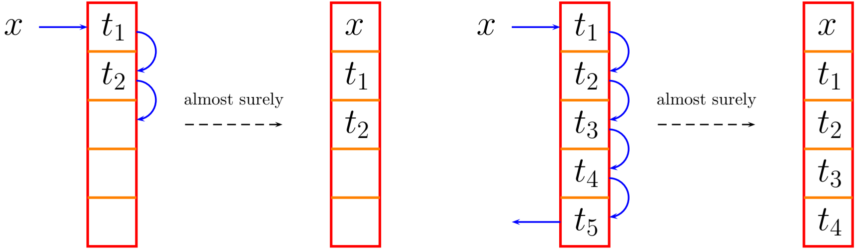
Random walk with tabu list

At each step, the walker moves to a neighbor chosen uniformly at random among those without occurrence in the tabu list. If all neighbors have an occurrence in the tabu list, the walker moves to a neighbor chosen uniformly at random.

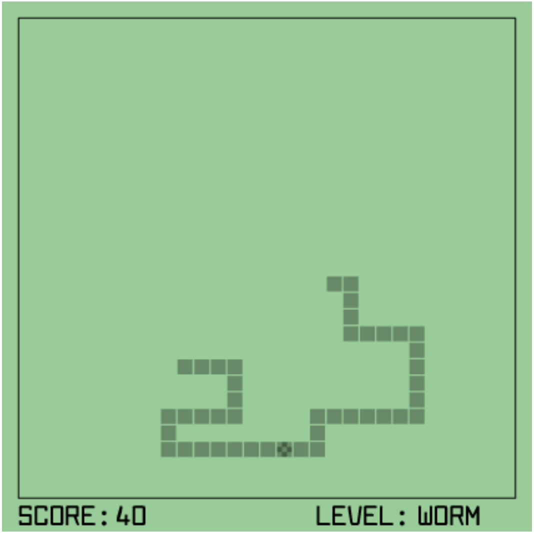


First In First Out

The $FIFO_m$ update rules, $m \geqslant 0$.



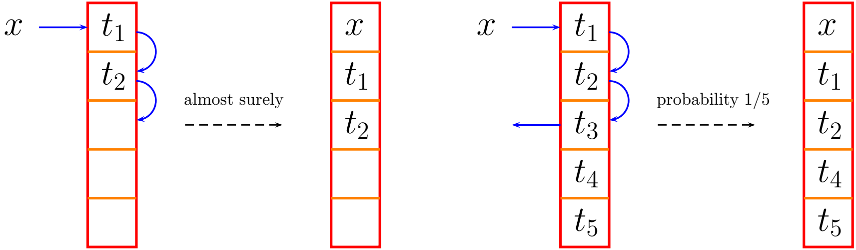
Here, $m = 5$. The current vertex is inserted almost surely. If the tabu list is full, then the occurence of the vertex stored in the last block is discarded, almost surely.



The $FIFO_{40}$ update rule.

Contrarily to the snake, the walker never dies.

A non deterministic update rule



The current vertex is inserted almost surely. If the tabu list is full, then an occurence in the tabu list, chosen uniformly at random, is discarded. Here, t_3 is discarded.

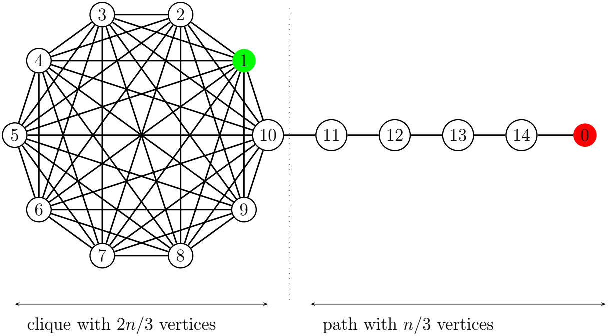
Hitting time in lollilop graph

For the simple random walk, G. Brightwell and P. Winkler showed in 1990:

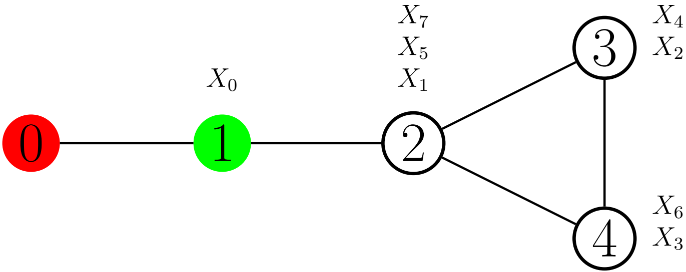
$$\max_{G:|\mathcal{V}|=n} \max_{x,y\in\mathcal{V}} E_x H_y(G) = E_1 H_0(Lollilop_n) = \frac{4n^3}{27}(1+o(1)) .$$

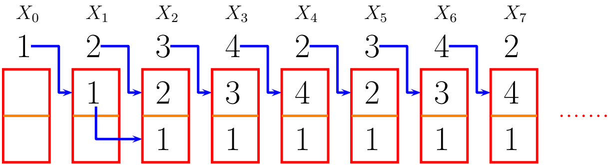
For the random walk with tabu list and update rule $FIFO_1$,

$$E_1 H_0(Lollilop_n) = \frac{4n^2}{9}(1+o(1)) .$$



Trap





In some cases, there is a positive probability that a vertex is never reached.

Finite mean hitting times

An update rule is called trivial if the current vertex is never inserted in the tabu list when the tabu list contains no element.

The unique update rule with zero length is trivial.

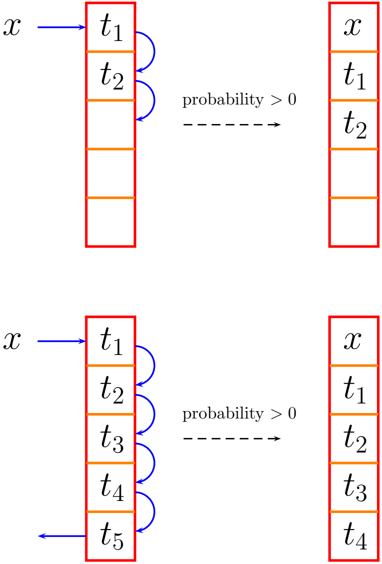
Consider a trivial update rule. If the tabu list is initially empty, then it remains empty forever and the walker performs a simple random walk. Consequently, all mean hitting times are finite for all graphs.

Finite mean hitting times

Theorem

Consider a non trivial update rule. All mean hitting times are finite for all connected graphs, if and only if, when the current vertex has no occurrence in the tabu list:

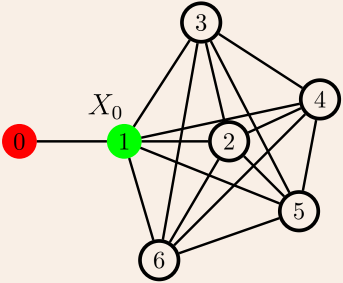
- *If the tabu list is not full, then the probability to insert the current vertex is positive.*
- *If the tabu list is full, then the probability to remove the last element is positive.*

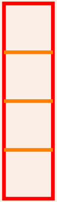


Finite mean hitting times

Sketch of proof.

If one of the two conditions is not fullfilled, then we exhibit a graph such that, with positive probability, the walker never reaches a given vertex. For example, in the following situation, the vertex 1 is never removed from the tabu list. Thus, the vertex 0 is never reached.





□

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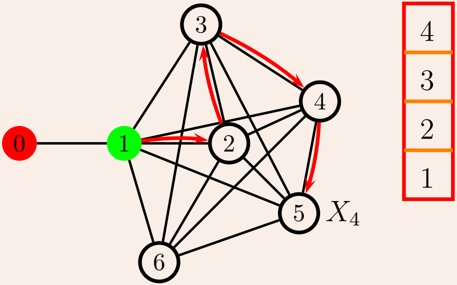

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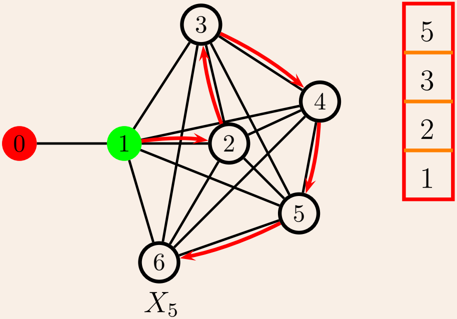

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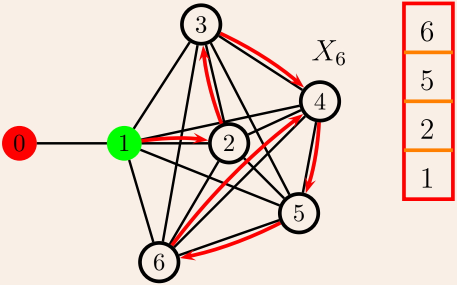

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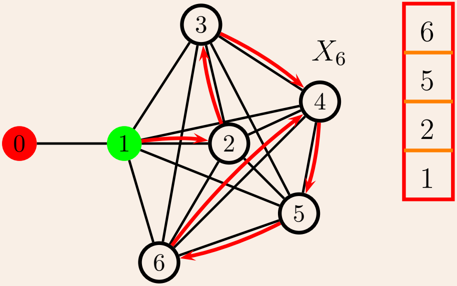

The reciprocal assertion follows from the theory of Markov chains on finite space.

□

1. Random walk with tabu list

2. Termination

3. Optimal update rules for *m*-free graphs

Optimal update rules for m-free graphs

A graph is m-free if a tabu random walk with $FIFO_m$ update rule cannot visit a vertex in its current tabu list, unless the current vertex has degree one.

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- All graphs are 1-free.
- A graph is 2-free if and only if it does not contain any triangle with a degree two vertex.
- Every $(m+1)$-regular graph is m-free.
- A graph with girth greater than or equal to $m+2$ is m-free.
- If $1 \leqslant k < m$, then all m-free graphs are k-free.

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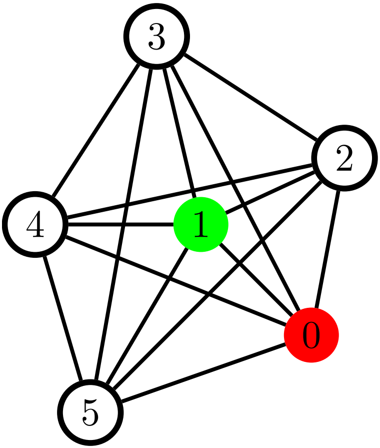

The clique with 6 elements is 5-regular, hence 4-free.

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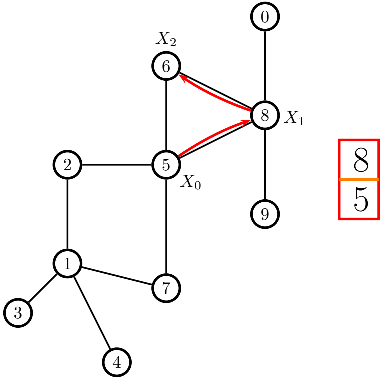

That graph is not 2-free because X_3 belongs to $\{5, 8\}$ almost surely.

Optimal update rules for m-free graphs

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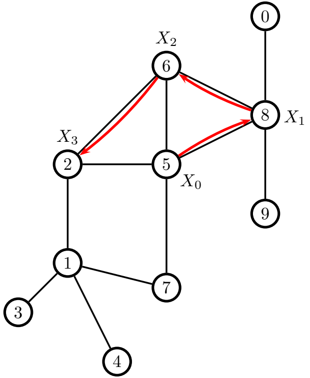

That graph is 2-free because it does not contain any triangle with a degree two vertex.

Theorem

For every nonnegative integer m and for every m-free graph, $FIFO_m$ yields minimal mean hitting times among all update rules with length less than or equal to m.

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Sketch of proof.

Let m be a positive integer. A walker $(X_n)_{n \geqslant 0}$ backtracks at step n if X_n belongs to $\{X_{\max\{n-2,0\}}, \dots, X_{\max\{n-m-1,0\}}\}$ while there exists a neighbor of X_{n-1} which does not have any occurrence in the tabu list.

On one hand, we show that any tabu random walk with $FIFO_m$ update rule does not backtrack on any m-free graph.

On the other hand, starting from any tabu random walk, we remove all backtracks to get a stochastic process that follows the same law than a tabu random walk with $FIFO_m$ update rule. □

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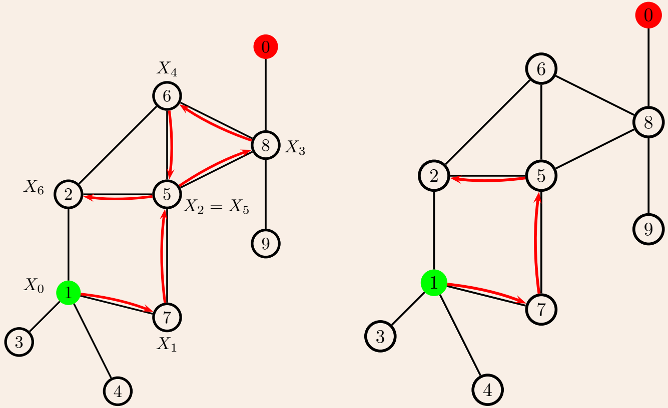
Sketch of proof.



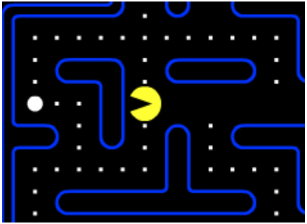
(a) The walker backtracks at step 5.

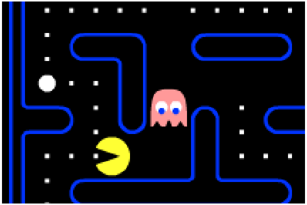
(b) We remove the backtracks.

Since all graphs are 1-free:

Corollary

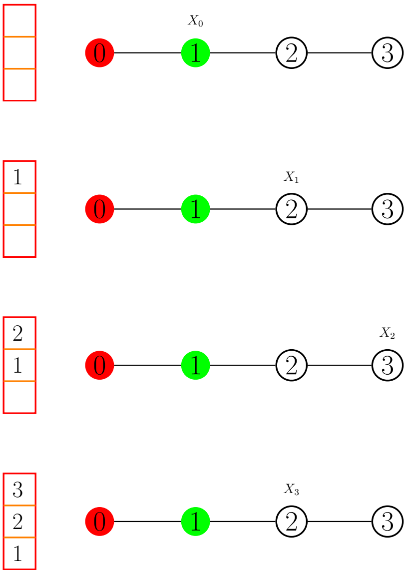
For every update rule R of length 0 or 1, the mean hitting times associated to R are greater than or equal to the corresponding mean hitting times associated to $FIFO_1$.





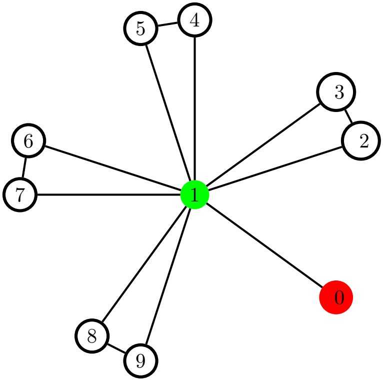
Your mean hitting times are always lower if someone follows you. It does not hold on all graphs for two people or more.

Better sometimes to forget

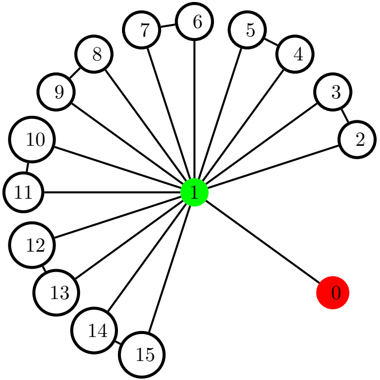


Comparisons between $FIFO_m$ update rules, $m \geqslant 0$

- K_n: clique with n vertices.
- L_n: path with n vertices.
- F_n: flower graph with n petals.



(a) F_4



(b) F_7

Comparisons between $FIFO_m$ update rules, $m \geqslant 0$

- K_n: clique with n vertices.
- L_n: path with n vertices.
- F_n: flower graph with n petals.

	0	1	2	3	4	$m \geqslant 5$
0	$\times$	K_3	K_3	K_3	K_3	K_3
1	$\times$	$\times$	K_3	K_3	K_3	K_3
2	F_7	F_4	$\times$	K_4	K_4	K_4
3	L_4	L_4	L_4	$\times$	K_5	K_5
4	L_4	L_4	L_4	L_5	$\times$	K_6
$k \geqslant 5$	L_4	L_4	L_4	L_5	L_6	$\begin{cases} L_{m+2} & \text{if } m < k \,, \\ \times & \text{if } m = k \,, \\ K_{k+2} & \text{if } m > k \,. \end{cases}$