Leader Election in Asymmetric Labeled Unidirectional Rings

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Leader election
Unidirectional rings
Homonym processes
Deterministic algorithm
Asynchronous message-passing
Leader Election in Rings

Deterministic solution

Anonymous processes

Impossible
[Angluin, 80]
[Lynch, 96]
State of the Art

Leader Election in Rings

**Anonymous processes**

- Deterministic solution: Impossible
  - [Angluin, 80]
  - [Lynch, 96]

- Probabilistic solution: Possible
  - [Xu and Srimani, 06]
  - [Kutten et al., 13]
Leader Election in Rings

**Anonymous processes**
- Impossible
  - [Angluin, 80]
  - [Lynch, 96]

**Identified processes**
- Possible
  - [LeLann, 77]
  - [Chang and Roberts, 79]
  - [Peterson, 82]

**Deterministic solution**

**Probabilistic solution**
- Possible
  - [Xu and Srimani, 06]
  - [Kutten et al., 13]
State of the Art

Leader Election in Rings

**Anonymous processes**
- Deterministic solution: Impossible
  - [Angluin, 80]
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**Homonym processes**
- Deterministic solution: Possible
  - [LeLann, 77]
  - [Chang and Roberts, 79]
  - [Peterson, 82]
- Probabilistic solution: Possible
  - [Xu and Srimani, 06]
  - [Kutten et al., 13]

**Identified processes**
- Probabilistic solution: Possible
  - [Yamashita and Kameda, 89]
# Leader Election in Rings of Homonym Processes

<table>
<thead>
<tr>
<th></th>
<th>PT/MT</th>
<th>Asynch.</th>
<th>Uni./Bi.</th>
<th>Known</th>
<th>Ring Class</th>
<th># Msg</th>
<th>Time</th>
<th>Memory</th>
</tr>
</thead>
<tbody>
<tr>
<td>[Delporte et al., 14]</td>
<td>MT</td>
<td>✔️</td>
<td>Bi.</td>
<td></td>
<td></td>
<td>?</td>
<td>?</td>
<td>?</td>
</tr>
<tr>
<td></td>
<td>PT</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$O(n \log n)$</td>
<td>?</td>
<td>?</td>
</tr>
<tr>
<td>[Dobrev, Pelc, 04]</td>
<td>PT</td>
<td></td>
<td>Bi. + uni.</td>
<td>$m \leq n$</td>
<td>Decided if inputs are unambiguous</td>
<td>$O(n \log n)$</td>
<td>$O(M)$</td>
<td>$O(nb)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>✔️</td>
<td></td>
<td></td>
<td></td>
<td>$O(nM)$</td>
<td>?</td>
<td>$O(Mb)$</td>
</tr>
<tr>
<td>[SSS 2016]</td>
<td>PT</td>
<td>✔️</td>
<td>Uni.</td>
<td>$k$</td>
<td>$\exists$ unique label and # proc with same label $\leq k$</td>
<td>$O(kn)$</td>
<td>$O(kn)$</td>
<td>$O(\log k + b)$</td>
</tr>
<tr>
<td>[IPDPS 2017]</td>
<td>PT</td>
<td>✔️</td>
<td>Uni.</td>
<td>$k$</td>
<td>Asymmetric labelling and # proc with same label $\leq k$</td>
<td>$O(n^2 + kn)$</td>
<td>$O(kn)$</td>
<td>$O(kn b)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$O(k^2 n^2)$</td>
<td>$O(k^2 n^2)$</td>
<td>$O(\log k + b)$</td>
</tr>
</tbody>
</table>

- **Uni**: Unidirectional / **Bi**: Bidirectional
- **MT** = Message-terminating: Processes do not explicitly terminate but only a finite number of messages are exchanged.
- **PT** = Process-terminating: Every process eventually halts.
Contributions

- **MT-LE**: Message-Terminating Leader Election
- **PT-LE**: Process-Terminating Leader Election
- $A$: Rings with asymmetric labelling
- $\overline{A}$: Rings with symmetric labelling
- $U^*$: Rings with at least one unique label
- $\mathcal{K}_k$: Rings with no more than $k$ processes with the same label

MT-LE Impossible [Angluin, 80]
Contributions

- MT-LE: Message-Terminating Leader Election
- PT-LE: Process-Terminating Leader Election

\[ \mathcal{A} \]: Rings with asymmetric labelling
\[ \overline{\mathcal{A}} \]: Rings with symmetric labelling
\[ \mathcal{U}^* \]: Rings with at least one unique label
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PT-LE Impossible
Contributions

- MT-LE: Message-Terminating Leader Election
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- \( A \): Rings with asymmetric labelling
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\[ \text{PT-LE Impossible} \Rightarrow \text{PT-LE Impossible} \]
Contributions

- MT-LE: Message-Terminating Leader Election
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MT-LE Impossible
Contributions

- MT-LE: Message-Terminating Leader Election
- PT-LE: Process-Terminating Leader Election

PT-LE Algorithm for $\mathcal{U}^* \cap \mathcal{K}_k$ [SSS 2016]

- $\mathcal{A}$: Rings with asymmetric labelling
- $\overline{\mathcal{A}}$: Rings with symmetric labelling
- $\mathcal{U}^*$: Rings with at least one unique label
- $\mathcal{K}_k$: Rings with no more than $k$ processes with the same label
First PT-LE Algorithm for $A \cap K_k$

- **Chosen Leader:**
  process whose LabelSequence = LyndonWord(LabelSequence)
  Lyndon Word = smallest rotation in lexicographic order

- **Label Sequence at $p_1$:**
  $LS_{p_1} = 12212$
  Rotations:
  1. $12212$ ( = $LS_{p_1}$ )
  2. $21221$ ( = $LS_{p_2}$ )
  3. $12122$ ( = $LS_{p_3}$ )
  4. $21212$ ( = $LS_{p_4}$ )
  5. $22121$ ( = $LS_{p_5}$ )

  Lyndon Word (LW) does not equal $LS_{p_1}$.
First PT-LE Algorithm for $\mathcal{A} \cap \mathcal{K}_k$

- **Chosen Leader:**
  process whose $\text{LabelSequence} = \text{LyndonWord}(\text{LabelSequence})$
  Lyndon Word = smallest rotation in lexicographic order

- **Local label aggregation**

[Diagram of a labeled unidirectional ring with nodes labeled 1, 2, and 2, arrows indicating label aggregation.]
First PT-LE Algorithm for $\mathcal{A} \cap \mathcal{K}_k$

- **Chosen Leader:**
  process whose LabelSequence = LyndonWord(LabelSequence)
  Lyndon Word = smallest rotation in lexicographic order

- **Local label aggregation**

During the election process, each node compares its own label with the labels of its neighbors and chooses a leader based on the Lyndon Word condition. The diagram illustrates the sequence of labels and how they are aggregated locally.

At the end of the election, if no node has detected a leader, it means that no one has agreed on a leader, which is indicated by the termination detection condition.
- **Chosen Leader:**
  process whose LabelSequence = LyndonWord(LabelSequence)
  Lyndon Word = smallest rotation in lexicographic order

- Local label aggregation

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Anaïs Durand
Leader Election in Asymmetric Labeled Unidirectional Rings
First PT-LE Algorithm for $\mathcal{A} \cap \mathcal{K}_k$

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  Lyndon Word = smallest rotation in lexicographic order

- **Local label aggregation**

- **⚠️ Do not know $n$**
  
  $\Rightarrow$ Leader cannot detect its election
First PT-LE Algorithm for $A \cap K_k$

- **Chosen Leader:**
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  Lyndon Word = smallest rotation in lexicographic order

- Local label aggregation

- Do not know $n$ ⇒ Leader cannot detect its election

- Termination detection = $(2k + 1) \times$ the same label ⇒ at least 2 times the sequence of labels

$k = 3$

Smallest repeating prefix = $\text{LabelSequence}$
  $= \text{LyndonWord}(\text{Smallest repeating prefix})$
Time complexity: at most \((2k + 2)n\) time units

Message complexity: at most \(n^2(2k + 1)\) messages

Memory: \((2k + 1)nb + 2b + 3\) bits, where \(b = \) number of bits to store an ID

Asymptotically optimal time complexity but

Large memory requirement
Second PT-LE Algorithm for $\mathcal{A} \cap \mathcal{K}_k$

- **Decrease memory usage** $\Rightarrow$ Peterson principle with radix sort

**Phase 1**

Known

---

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Leader Election in Asymmetric Labeled Unidirectional Rings

9/15
Second PT-LE Algorithm for $\mathcal{A} \cap \mathcal{K}_k$

- **Decrease memory usage** ⇒ Peterson principle with radix sort

**Phase 1**

Diagram shows a labeled unidirectional ring with nodes labeled 1, 2, and 3, and arrows indicating communication and state transitions. Nodes are marked with labels indicating their state and position in the algorithm.
Decrease memory usage ⇒ Peterson principle with radix sort
Second PT-LE Algorithm for $\mathcal{A} \cap \mathcal{K}_k$

- **Decrease memory usage** $\Rightarrow$ Peterson principle with radix sort

### Phase 3

Known

12212

21221

12122

22121

\[ \begin{array}{c}
 p_1 \\
 p_2 \\
 p_3 \\
 p_4 \\
 p_5 \\
\end{array} \]

\[ \begin{array}{c}
 1 \\
 2 \\
 2 \\
 2 \\
 1 \\
\end{array} \]
Second PT-LE Algorithm for $A \cap K_k$

- **Decrease memory usage** $\Rightarrow$ Peterson principle with radix sort

Known Phase 3
Second PT-LE Algorithm for $\mathcal{A} \cap \mathcal{K}_k$

- **Phase Shift**

**Phase 1**

**Known**

12212

**Phase 2**

Known

12212

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Leader Election in Asymmetric Labeled Unidirectional Rings
Second PT-LE Algorithm for $\mathcal{A} \cap \mathcal{K}_k$

**Execution**

- **Phase 1**
- **Phase 2**
- **Phase 3**

**Shift**

**Synchronization**

...
Second PT-LE Algorithm for $A \cap K_k$

**Termination Detection:** $\text{count} = k+1$

$\text{count} = \# \text{ phases where } \text{Known} = \text{Label}$

**Phase 1**

Diagram:

- Node 1
- Node 2
- Node 1
- Node 2
- Node 1
- Node 2
- Node 1

Arrows:

- $p_1$
- $p_2$
- $p_3$
- $p_4$
- $p_5$

Count:

- $\text{count} = 1$

Known:

- Node 1
- Node 2
- Node 1
- Node 2
- Node 1
- Node 2
- Node 1
Termination Detection: \( \text{count} = k+1 \)

\( \text{count} = \# \text{ phases where Known = Label} \)

Phase 2
Termination Detection: count = k+1
count = # phases where Known = Label

Phase 3
Second PT-LE Algorithm for $A \cap K_k$

- **Termination Detection:** count = $k+1$
  
  count = \# phases where Known = Label

**Phase 4**

```
12122
```

Known
Second PT-LE Algorithm for $A \cap K_k$

**Termination Detection:** count = $k+1$

$\text{count} = \# \text{ phases where Known} = \text{Label}$

---

**Phase 5**

![Diagram showing phase 5 with counts and labels](image-url)
Termination Detection: count = k+1
count = \# phases where Known = Label

Phase 6
Termination Detection: count = k+1

\[
\text{count} = \# \text{ phases where } \text{Known} = \text{Label}
\]

Phase 7

\[
\begin{align*}
\text{p}_1 & \quad \text{p}_2 & \quad \text{p}_3 & \quad \text{p}_4 & \quad \text{p}_5 \\
1 & \quad 2 & \quad 1 & \quad 2 & \quad 1
\end{align*}
\]

\[
\text{count} = 3
\]

\[
\text{Known} = 12122
\]
Second PT-LE Algorithm for \( A \cap \mathcal{K}_k \)

**Termination Detection:**
\[ \text{count} = k+1 \]
\[ \text{count} = \# \text{ phases where Known = Label} \]

---

**Phase 8**

\[ k = 3 \]
\[ \text{count} = 4 = k+1 \]
Second PT-LE Algorithm for $\mathcal{A} \cap \mathcal{K}_k$

- **Memory:** $2 \lceil \log k \rceil + 3b + 5$ bits, where $b = \text{number of bits to store an ID}$

- **Time complexity:** $O(k^2 n^2)$ time units

- **Message complexity:** $O(k^2 n^2)$ messages

Asymptotically optimal memory requirement but Large time complexity
# Conclusion

<table>
<thead>
<tr>
<th>Class</th>
<th>Message-terminating leader election impossible</th>
<th>Process-terminating leader election impossible</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\overline{A})</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(K_k)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(U^*)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(A)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Class</th>
<th>Lower Bound on Time</th>
<th>Time</th>
<th>Nbr of Msgs</th>
<th>Memory</th>
</tr>
</thead>
<tbody>
<tr>
<td>(U^* \cap K_k) [SSS 2016]</td>
<td>(\Omega(kn))</td>
<td>(O(kn))</td>
<td>(O(n^2 + kn))</td>
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</tr>
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<td>(A \cap K_k)</td>
<td>(\Omega(kn))</td>
<td>(O(kn))</td>
<td>(O(n^2k))</td>
<td>(O(knb))</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(O(k^2n^2))</td>
<td>(O(k^2n^2))</td>
<td>(O(\log k + b))</td>
</tr>
</tbody>
</table>

\(b = \#\) bits to store a label

- \(A\): Rings with asymmetric labelling
- \(\overline{A}\): Rings with symmetric labelling
- \(U^*\): Rings with at least one unique label
- \(K_k\): Rings with no more than \(k\) processes with the same label
Thank you for your attention.

Do you have any questions?