Small inversions for smaller inversions

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Inversion, simple example

Even natural numbers

```
Inductive even : ∀ n, Prop :=
  | Ev0 :               even 0
  | Ev2 n : even n → even (S (S n)).
```

Basic usage

```
Lemma even_plus_left n m : even n → even (n + m) → even m.

IHen : even (n + m) → even m
enm : even (S (S (n + m)))
==============================
even m
```

Inversion

Purpose

Extract the information contained in a hypothesis H of type T

- where T is an inductive relation
- with some arguments having an inductive type

Expectations

- For each case (constructor), decompose H into ALL its components
- In particular, remove irrelevant cases

Essentially : (subtle) case analysis on H

- Simultaneous case analysis on H and its arguments
- game on dependent pattern-matching

Smaller inversion (part of the Braga method)

Joint work with Dominique Larchey Wendling [TYPES'18], [Proof&Computation II 2021]

Half of even numbers

```
Fixpoint half n (e: even n) {struct e} : nat :=
  match n return even n → nat with
  | 0       => λ _, 0
  | 1       => λ e, match even_inv e with end
  | S (S n) => λ e, S (half n (πeven e))
  end e.
```

Projection: getting ONE STRUCTURALLY SMALLER component

```
Definition πeven n (e: even (S (S n))) : even n :=
  match e in even m return
     let n := match m with S (S n) => n | _ => n end in
     let G := match m with S (S n) => True | _ => False end in G → even n
  with
  | Ev2 n e => λ _, e
  |  _       => λ fa, match fa with end
 end I.
```

Reasoning on half

Easy (induction on e)

```
Lemma double_half : ∀ n e, half n e + half n e = n.
```

Less easy: induction on e and inversion on e'

```
Lemma half_pirr : ∀ n (e e' : even n), half n e = half n e'.

e : even n
e' : even (S (S n))
================================
S (half n e) = half (S (S n)) e'
```

Unicity of e

Again: induction on e and inversion on e'

```
Lemma even_unique : ∀n (e e' : even n), e = e'.
```

But proof unicity should not be overrated here

- The returned result (sort Set/Type)
  cannot depend on an argument of sort Prop
- Simple example: unbounded linear search algorithm
  (see ConstructiveEpsilon.v in the std lib)

More sophisticated inversions

- Even bounded natural numbers
- Half of even bounded natural numbers
- Proof unicity for $=$ and $\leq$ in nat

Bounded natural numbers

```
Inductive t : nat → Set :=
  | F0 {n} : t (S n)
  | FS {n} : t n → t (S n).
```

Failures for standard inversion.

Inversion technologies

Standard tactic of Coq: fully automated [Cornes & Terrasse, 1995 ; Murthy?]

- Improved over the years, very impressive black box
- lack of control
- big underlying terms
- failures with dependent inductive types

Small inversions: handcrafted [Monin 2010, Monin & Shi 2013]

- Flexible approach with several variants
- Developed for a big experiment with CompCert
- Attempts towards automation (Braibant, Boutillier)

TYPE'2022

- Made clearer with auxiliary inductive types
- Improvement needed for dependent types

Small inversions with auxiliary inductive types

Receipe

Given an inductive relation `rel : Tx → Ty1 → ... Prop`
with "input" argument `x : Tx`, define:

- For each input case (constructor `C`) in `Tx`,
  an *auxiliary inductive relation* of type `Ty1 → ... Prop`
  by copy and paste of relevant telescopes of `rel`
  No recursion
- A *dispatch function* `rel_disp` from `x : Tx` to `Ty1 → ... Prop`
  by pattern matching on `x`
- Inversion lemma `rel_inv : rel → rel_disp` *(easy proof)*

Usage

- Given a hypothesis `R : rel (C...) expr_1...`
  perform `match rel_inv R with...`
- Boils down to the relevant *aux. inductive relation* corresponding to `(C...)`

Small inversions with auxiliary inductive types

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Usage

- Given a hypothesis `R : rel (C...) expr_1...`
  perform `match rel_inv R with...`
- Boils down to the relevant *aux. inductive relation* corresponding to `(C...)`

Small inversion for dependent (data) types

Explicit injectivity

When `R` occurs as an argument in the goal we need also the left inverse `rel_back` of `rel_inv` (trivial as well), and a proof of
`R = rel_back (rel_inv R)`.

Then rewrite the occurrences of `R` with `rel_back (rel_inv R)` before the pattern-matching on `rel_inv R`.

Improvement: built-in injectivity

- In the previous receipe, *add a last argument of shape* `C...`
- *Same code* for `rel_disp` and `rel_inv`
- Bonus: inline `rel_disp` in the statement of `rel_inv`

Basic small inversion on even [2021 talks]

```
Inductive even : ∀ n, Prop :=
  | Ev0 :            even 0
  | Ev2 n : even n → even (S (S n)).

Inductive even0 : Prop := even0_Ev0 : even0.
Inductive even1 : Prop :=.
Inductive even2 n : Prop := even2_Ev2 : even n → even2 n.

Definition even_inv {n} (e : even n) :
    match n return Prop with
    | 0       => even0
    | 1       => even1
    | S (S n) => even2 n
    end.
Proof. destruct e; constructor; assumption. Defined.

Definition even_back {n} (e : match n return Prop with...) : even n.
Proof... Defined.

Lemma even_inv_mono {n} (e : even n) : e = even_back (even_inv e).
Proof. destruct e; reflexivity. Qed.
```

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Definition even_back {n} (e : match n return Prop with...) : even n.
Proof... Defined.

Lemma even_inv_mono {n} (e : even n) : e = even_back (even_inv e).
Proof. destruct e; reflexivity. Qed.
```

Improved small inversion on even with built-in injectivity

```
Inductive even : ∀ n, Prop :=
  | Ev0 :              even 0
  | Ev2 n : even n → even (S (S n)).

Inductive is_Ev0 : even 0 → Prop := is_Ev0_intro : is_Ev0 Ev0.
Inductive no_Ev1 : even 1 → Prop :=.
Inductive is_Ev2 n : even (S (S n)) → Prop :=
  is_Ev2_intro : ∀ (e : even n), is_Ev2 n (Ev2 n e).

Definition even_inv {n} (e :  even n) :
    match n return even n → Prop with
    | 0       => is_Ev0
    | 1       => no_Ev1
    | S (S n) => is_Ev2 n
    end e.
Proof. destruct e; constructor. Defined.

(* Basic version *)
Inductive even0 : Prop := even0_Ev0 : even0.
Inductive even1 : Prop :=.
Inductive even2 n : Prop := even2_Ev2 : even n → even2 n.
```

Exercise: equality in nat with obvious UP

```
Inductive diag : nat → nat → Prop :=
| dia0 : diag 0 0
| diaS x y : diag x y → diag (S x) (S y).

(* small inversion : standard receipe with built-in injectivity *)
Inductive is_dia0 : diag 0 0 → Prop := ii00 : is_dia0 dia0.
Inductive is_diaS x y : diag (S x) (S y) → Prop :=
  iiSS : ∀ (d : diag x y), is_diaS x y (diaS x y d).
Inductive no_diag x y : diag x y → Prop := .

Definition diag_inv {x y} (d : diag x y) :
  match x, y return diag x y → Prop with
  | 0, 0     => is_dia0
  | S x, S y => is_diaS x y
  | x, y     => no_diag x y
  end d.
Proof. destruct d; constructor. Qed.
```

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  | S x, S y => is_diaS x y
  | x, y     => no_diag x y
  end d.
Proof. destruct d; constructor. Qed.
```

Simple explicit UIP in nat

```
Definition diag_refl {x} : diag x x.
 Proof.  induction x as [ | x IHx]; constructor. apply IHx. Defined.
Definition eq_diag {x y} (e : x = y) : diag x y.
 Proof. case e. apply diag_refl. Defined.
Definition diag_back {x} : ∀ y, diag x y → x = y.
Proof. induction x; destruct y; intro d; destruct (diag_inv d);
   [reflexivity | apply f_equal, (IHx _ d)]. Defined.
Lemma diag_back_isrefl {x} : ∀ (d : diag x x), eq_refl = diag_back d.
Proof. induction x  as [ | x IHx]; simpl; intro d; destruct (diag_inv d);
 [reflexivity | case (IHx d). cbn. reflexivity]. Qed.
Lemma diag_mono {x y} (e : x = y) : e = diag_back (eq_diag e).
 Proof. destruct e; destruct x as [ | x]; simpl.
  + destruct (diag_inv dia0); reflexivity.
  + destruct (diag_inv (diaS x x diag_refl)) as [d]. case (diag_back_isrefl d); reflexivity.
Qed.
Corollary UIP_nat (x: nat) (e : x = x) : eq_refl = e.
Proof. rewrite (diag_mono e). apply diag_back_isrefl. Qed.
```

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Lemma diag_mono {x y} (e : x = y) : e = diag_back (eq_diag e).
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Corollary UIP_nat (x: nat) (e : x = x) : eq_refl = e.
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Definition eq_diag {x y} (e : x = y) : diag x y.
 Proof. case e. apply diag_refl. Defined.

Definition diag_back {x} : ∀ y, diag x y → x = y.
Proof. induction x; destruct y; intro d; destruct (diag_inv d);
   [reflexivity | apply f_equal, (IHx _ d)]. Defined.

Lemma diag_back_isrefl {x} : ∀ (d : diag x x), eq_refl = diag_back d.
Proof. induction x  as [ | x IHx]; simpl; intro d; destruct (diag_inv d);
 [reflexivity | case (IHx d). cbn. reflexivity]. Qed.

Lemma diag_mono {x y} (e : x = y) : e = diag_back (eq_diag e).
 Proof. destruct e; destruct x as [ | x]; simpl.
  + destruct (diag_inv dia0); reflexivity.
  + destruct (diag_inv (diaS x x diag_refl)) as [d]. case (diag_back_isrefl d); reflexivity.
Qed.

Corollary UIP_nat (x: nat) (e : x = x) : eq_refl = e.
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```

Simple explicit UIP in nat

```
Definition diag_refl {x} : diag x x.
 Proof.  induction x as [ | x IHx]; constructor. apply IHx. Defined.
Definition eq_diag {x y} (e : x = y) : diag x y.
 Proof. case e. apply diag_refl. Defined.
Definition diag_back {x} : ∀ y, diag x y → x = y.
Proof. induction x; destruct y; intro d; destruct (diag_inv d);
    [reflexivity | apply f_equal, (IHx _ d)]. Defined.
Lemma diag_back_isrefl {x} : ∀ (d : diag x x), eq_refl = diag_back d.
Proof. induction x  as [ | x IHx]; simpl; intro d; destruct (diag_inv d);
 [reflexivity | case (IHx d). cbn. reflexivity]. Qed.
Lemma diag_mono {x y} (e : x = y) : e = diag_back (eq_diag e).
 Proof. destruct e; destruct x as [ | x]; simpl.
  + destruct (diag_inv dia0); reflexivity.
  + destruct (diag_inv (diaS x x diag_refl)) as [d]. case (diag_back_isrefl d); reflexivity.
Qed.
Corollary UIP_nat (x: nat) (e : x = x) : eq_refl = e.
Proof. rewrite (diag_mono e). apply diag_back_isrefl. Qed.
```

Horribly simpler proof of UIP in nat along the same scheme...

```
Fixpoint diagTF (x y : nat) : Prop :=
  match x, y with
  | 0, 0     => True
  | S x, S y => diagTF x y
  | _, _     => False
  end.
Definition diagTF_refl x : diagTF x x :=...
Definition eq_diagTF {x y} (e : x = y) : diagTF x y :=...
Definition diagTF_back {x} : ∀ y, diagTF x y → x = y :=...
Lemma diagTF_back_isrefl {x} : ∀ (d : diagTF x x), eq_refl = diagTF_back d.
Lemma diagTF_mono {x y} (e : x = y) : e = diagTF_back (eq_diagTF e).
Corollary UIP_nat (x: nat) (e : x = x) : eq_refl = e.
Proof. rewrite (diagTF_mono e). apply diagTF_back_isrefl. Qed.
```

... without diag and its inversion :(

Equality is too easy, what about $\leq$?

Inversion performed “as if” $\leq$ was defined as

```
Inductive le n : nat → Prop :=
| le_e_0   : n = 0   → n ≤ 0
| le_e_S m : n = S m → n ≤ S m
| le_S m   : n ≤ m   → n ≤ S m.
```

```
Definition eq_le n m (e : n = m) : n ≤ m :=
  match e with eq_refl => le_n n end.
```

```
Inductive le_0 [n] : n ≤ 0 → Prop :=
| le_0_e : ∀ e, le_0 (eq_le e).
```

```
Inductive le_Sm [m n] : n ≤ S m → Prop :=
| le_Sm_e : ∀ e, le_Sm (eq_le e)
| le_Sm_S : ∀ l, le_Sm (le_S n m l).
```

```
Lemma le_inv {n m} (l : n ≤ m) :
  match m with
  | 0 => le_0
  | S m => @le_Sm m
  end n l.
```

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| le_e_0   : n = 0   → n ≤ 0
| le_e_S m : n = S m → n ≤ S m
| le_S m   : n ≤ m   → n ≤ S m.
```

```
Definition eq_le n m (e : n = m) : n ≤ m :=
  match e with eq_refl => le_n n end.

Inductive le_0 [n] : n ≤ 0 → Prop :=
| le_0_e : ∀ e, le_0 (eq_le e).

Inductive le_Sm [m n] : n ≤ S m → Prop :=
| le_Sm_e : ∀ e, le_Sm (eq_le e)
| le_Sm_S : ∀ l, le_Sm (le_S n m l).

Lemma le_inv {n m} (l : n ≤ m) :
  match m with
  | O => le_0
  | S m => @le_Sm m
  end n l.
```

Unicity of proofs of $\leq$

```
Lemma eq_is_le_n {n} (e : n = n) : le_n n = eq_le e.
Proof. rewrite (UIP_refl_nat n e). reflexivity. Qed.

Lemma lenn_unique {n} (l : n ≤ n) : le_n n = l.
Proof.
  destruct n; destruct (le_inv l);  try apply eq_is_le_n. case (lt_irrefl _ l).
Qed.

Inductive is_le_S {n m} : n ≤ S m → Prop :=
| is_le_S_intro : ∀ l, is_le_S (le_S n m l).

Lemma leS_is_le_S n m (lS : n ≤ S m) : n ≤ m → is_le_S lS.
Proof.
  destruct (le_inv lS) as [ e | ll ];  intro l; try constructor.
  exfalso; rewrite e in l; apply (lt_irrefl _ l).
Qed.

Fixpoint le_unique {n m} (p : n ≤ m) : ∀ q, p = q.
 Proof.
  destruct p as [ | m p]; intro q; cbn.
  - destruct (lenn_unique q); reflexivity.
  - destruct (leS_is_le_S q p). apply f_equal, le_unique.
Qed.
```

Material

The Braga method

https://github.com/DmxLarchey/The-Braga-Method

Dominique Larchey-Wendling and Jean-François Monin.
The Braga Method: Extracting Certified Algorithms from Complex Recursive Schemes in Coq, chapter 8, pages 305–386.
In Klaus Mainzer, Peter Schuster, and Helmut Schwichtenberg, editors.
Proof and Computation II: From Proof Theory and Univalent Mathematics to Program Extraction and Verification.
World Scientific, September 2021.

Small inversions

http://home/jf/www/Proof/Small_inversions/2022/