First-order logic
Part one :
Language and Truth Value of Formulae

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Homework : solution using ND

$$(p \Rightarrow \neg j) \wedge (\neg p \Rightarrow j) \wedge (j \Rightarrow m) \Rightarrow m \vee p$$

Structure of first-order logic

A non-empty domain (more than two elements)

Three categories :

- Terms representing the elements of the domain or functions on these elements
- Relations
- Formulae describing the interactions between the relations thanks to connectives and quantifiers

Remark :

Two particular symbols (quantifiers) : $\forall$ (universal quantification) and $\exists$ (existential quantification).

Structure of first-order logic

Examples :

- the term *parent*(x) intended to mean the parent of x,
- the formula $\forall x \exists y\ parent(y, x)$ indicates that every individual has a parent.

Syllogism

A cheap horse is rare.
Everything that is rare is expensive.
Hence a cheap horse is expensive.

$$\forall x(horse(x) \land cheap(x) \Rightarrow rare(x))$$
$$\forall x(rare(x) \Rightarrow expensive(x))$$
$$\forall x(horse(x) \land cheap(x) \Rightarrow expensive(x))$$

Syllogism

A cheap horse is rare.
Everything that is rare is expensive.
Hence a cheap horse is expensive.

Nothing bothers you ?

Everything that is expensive is not cheap and vice versa.

$$\forall x(\mathit{cheap}(x) \Leftrightarrow \neg \mathit{expensive}(x))$$

Now we have a contradiction.

Usage

First-order logic allows us to model :

- a **single** non-empty domain,
- functions over the domain, and
- relations over the domain.

Overview

Vocabulary

- Two propositional constants : $\bot$ and $\top$
- Variables : sequence of letters and digits starting with one of the following lower case letters : u,v,w,x,y,z.
- Connectives : $\neg, \wedge, \vee, \Rightarrow, \Leftrightarrow$
- Quantifiers : $\forall$ the **universal** quantification and $\exists$ the **existential** quantification
- Punctuation : the comma « , » and the open « (» and closing «) »parenthesis.
- Ordinary and special symbols :
  - **an ordinary symbol** is a sequence of letters and digits not starting by one of the following lower case letters : u,v,w,x,y,z.
  - **a special symbol** is $+, -, *, /, =, \neq, <, \leq, >, \geq, \ldots$

Example 4.1.1

- $x, x1, x2, y$ are variables,
- *man*, *parent*, *succ*, 12, 24, $f1$ are ordinary symbols, the ordinary symbols will represent :
  - **functions** (numerical constants or multiple argument functions) or
  - **relations** (propositional variables or multiple argument relations).
- $x = y$, $z > 3$ are examples for special symbols

Term

Definition 4.1.2

- an ordinary symbol is a term,
- a variable is a term,
- if $t_1, \ldots, t_n$ are terms and if s is a (ordinary or special) symbol then $s(t_1, \ldots, t_n)$ is a term.

Example 4.1.3

$x, a, f(x1, x2, g(y)), +(x, *(y, z)), +(5, 42)$ are terms

On the contrary, $f(\bot, 2, y)$ is not a term.

Note that $42(1, y, 3)$ is also a term, but by convention function and relation names are ordinary symbols starting with letters.

Atomic formula

Definition 4.1.4 atomic formulae

- $\top$ and $\bot$ are atomic formulae
- an ordinary symbol is an atomic formula
- if $t_1, \ldots, t_n$ are terms and if s is a (ordinary or special) symbol then $s(t_1, \ldots, t_n)$ is an atomic formula.

Example 4.1.5 :

- $f(1, +(5, 42), g(z))$, a, and $+(x, *(y, z))$ are atomic formulae
- x and $A \vee f(4, 2, 6)$ are not atomic formulae

Syntax *v.s.* Semantics

The set of **terms** and the set of **atomic formulae** are not disjoint.

For example $p(x)$ is a term and an atomic formula.

When t is a term and an atomic formula simultaneously, we distinguish $[[t]]$, the value of t seen as a term of $[t]$, value of t seen as a formula.

(Strict) Formula

Definition 4.1.6

- an atomic formula is a formula,
- if A is a formula then $\neg A$ is a formula,
- if A and B are formulae and if $\circ$ one of the following operations $\vee, \wedge, \Rightarrow, \Leftrightarrow$ then $(A \circ B)$ is a formula ,
- if A is a formula and if x is any variable then $\forall x\ A$ and $\exists x\ A$ are formulae.

Example 4.1.7

- $man(x)$, $parents(son(y), mother(Alice))$, $=(x, +(f(x), g(y)))$ are atomic formulae, hence formulae.
- On the contrary

is a non-atomic formula.

(Strict) Formula : Examples

Among these expressions, which ones are strict formulae :

- x
- a
- $(a(x) \Rightarrow b) \wedge a(x) \Rightarrow b$
- $\exists x((\bot \Rightarrow a(x)) \wedge b(x))$
- $\exists x \exists y < (-(x, y), +(a, y))$
- $((a < b) \Rightarrow ((2 * b) > (2 * a)))$

Infix notations

Prioritized formulae : the symbols of the functions $+, -, *, /$ and the symbols of the relations $=, \neq, <, >, \leq, \geq$ are written in the usual manner.

Example 4.1.8

- $\leq (*(3, x), +(y, 5))$ is abbreviated as
- $+(x, *(y, z))$ is abbreviated as

Inverse transformation

Prioritize

- connectives have a lower priority than the relations
- quantifiers have the same priority as negation.
- $=, \neq, <, \leq, >, \geq$ have a lower priority than $+, -, *, /$

Table 4.1 summary of priorities

Priorities decreasing from top to bottom.

OPERATIONS	
$-+$ unary	
$*, /$ binary	left associative
$+, -$ binary	left associative
RELATIONS	
$=, \neq, <, \leq, >, \geq$	
NEGATION, QUANTIFIERS	
$\neg, \forall, \exists$	
BINARY CONNECTIVES	
$\wedge$	left associative
$\vee$	left associative
$\Rightarrow$	**right** associative
$\Leftrightarrow$	left associative

Prioritized formulae

Definition 4.1.9 prioritized formulae

A prioritized formula is inductively defined as follows :

- An atomic formula is a prioritized formula.
- If A is a prioritized formula then $\neg A$ is a prioritized formula.
- If A and B are prioritized formulae then $A \circ B$ is a prioritized formula.
- If A is a prioritized formula and if x is *any* variable then $\forall x\ A$ and $\exists x\ A$ are prioritized formulae.
- If A is a prioritized formula (A) is a prioritized formula.

Examples

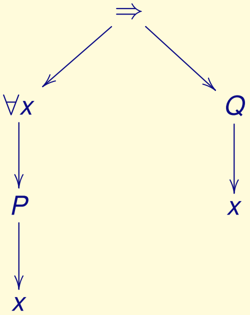
Example 4.1.10

- $\forall x P(x) \wedge \forall x Q(x) \Leftrightarrow \forall x (P(x) \wedge Q(x))$ is an abbreviation of
- $\forall x \forall y \forall z (x \leq y \wedge y \leq z \Rightarrow x \leq z)$ is an abbreviation of ?

Tree representation

Example 4.1.11 $\forall x P(x) \Rightarrow Q(x)$

the left-hand side operand of the implication is $\forall x P(x)$.



Idea

- The meaning of the formula $x + 2 = 4$ depends on x
  The meaning of the formula $x = x$ depends on x as well
  x is free in the previous formulae
- The meaning of $\forall x(x + 2 = y)$ does not depend on x
  The meaning of $\forall x(x + 0 = x)$ does not depend on x
  x is not free in these two formulae

Free and bound occurrences

Definition 4.2.1

- In $\forall x\ A$ or $\exists x\ A$, the scope of the binding of x is A.
- An occurrence of x in A is bound if it is in the scope of a binding of x, otherwise it is said to be free

If we represent a formula by a tree :

- A bound occurrence of x is
- An occurrence of x is free if

Example 4.2.2

$\forall x P(\mathbf{x}, y) \wedge \exists z R(\underline{x}, z)$

Free, bound variables

Definition 4.2.3

- The variable x is a free variable of a formula if and only if there is a free occurrence of x in the formula.
- A variable x is a bound variable of a formula if and only if there is abound occurence of x in the formula.
- A formula without free variable is also called a closed formula.

Remark 4.2.4

A variable can be simultaneously free and bound. For example, in the formula $\forall x P(x) \vee Q(x)$, x is both free and bound.

Remark 4.2.5

By definition, a variable which does not appear in a formula (0 occurrence) is **NOT** free in this formula.

Example 4.2.6

The free variables of the formula of example 4.2.2 are x and y.

Declaring a symbol

Definition 4.3.1

A symbol declaration is a triple denoted by s^{gn} where :

- s is a symbol
- g one of the letters f (a function) or r (a relation)
- n is a natural number.

Remark 4.3.3

If the context gives an implicit declaration of a symbol, we omit the exponent.

Example : **equal** is always a 2 arguments relation, we abbreviate the declaration $=^{r2}$ as $=$.

Symbol declaration : Example

Example 4.3.2

- $parent^{r2}$ is a **relation** (**r**) with **2** arguments
- $*^{f2}$ is **function** (**f**) with **2** arguments
- man^{r1} a unary **relation**

Signature

Definition 4.3.4

A signature is a set of symbol declarations.

Let $n > 0$ and Σ a signature, the symbol s is :

1. a constant of the signature if and only if $s^{f0} \in \Sigma$
2. a symbol of the function of n arguments of the signature, if and only if $s^{fn} \in \Sigma$
3. a propositional variable of the signature if and only if $s^{r0} \in \Sigma$
4. a symbol of the relation of n arguments of the signature, if and only if $s^{rn} \in \Sigma$

Examples in mathematics (1/2)

$0^{f0}, 1^{f0}, +^{f2}, -^{f2}, *^{f2}, =^{r2}$ is a signature for arithmetics.

Remark :

- We write : 0, 1, $+$ and $-$ (with two arguments), $*$ and $=$.
- Note that plus and minus have two arguments, (the symbol will not be used with only one argument).

Examples in mathematics (2/2)

Example 4.3.5 (Set theory)

A possible signature is $\in, =$

All other operations can be defined from these relations.

Overloading

Definition 4.3.6

A symbol is overloaded in a signature, when this signature has two distinct declarations of the same symbol.

Example 4.3.7 : the minus sign is often overloaded.

- the opposite of a number
- the subtraction of two numbers

In what follows, in this course, we prohibit the use of overloaded symbols in signatures.

Term over a signature

Definition 4.3.8

Let Σ be a signature, a term over Σ is :

- either a variable,
- or a constant s where $s^{f0} \in \Sigma$,
- or a term of the form $s(t_1, \ldots, t_n)$, where $n \geq 1$, $s^{fn} \in \Sigma$ and $t_1, \ldots, t_n$ are terms over Σ.

The set of terms over the signature Σ is denoted by T_Σ.

Atomic formula over a signature

Definition 4.3.9

Let Σ a signature, an atomic formula over Σ is :

- either one of the constants $\top, \bot$,
- or a propositional variable s where $s^{r0} \in \Sigma$,
- or an expression $s(t_1, \ldots, t_n)$ where $n \geq 1$, $s^{rn} \in \Sigma$ and $t_1, \ldots, t_n$ are terms over Σ.

Formula over a signature

Definition 4.3.10

A formula over a signature Σ is a formula, whose atomic sub-formulae are atomic formulae over Σ (according to definition 4.3.9).

F_Σ denotes the set of formulae over the signature Σ.

Example 4.3.11

$\forall x\ (p(x) \Rightarrow \exists y\ q(x, y))$ is a formula over signature $\Sigma = \{p^{r1}, q^{r2}, h^{f1}, c^{f0}\}$.

But it is also a formula over the signature $\Sigma' = \{p^{r1}, q^{r2}\}$, since the symbols h and c are not in the formula.

Associated signature

Definition 4.3.12

The signature associated to a formula is the smallest signature Σ such that the formula is a member of F_{Σ}, it is the smallest signature allowing to write the formula.

Example 4.3.13

The associated signature of formula $\forall x\ (p(x) \Rightarrow \exists y\ q(x, y))$ is

Associated signature

Definition 4.3.14

The associated signature to a set of formulae is the union of the associated signatures of all formulae of the set.

Example 4.3.15

The associated signature of a set of two formulae
$\forall x(p(x) \Rightarrow \exists y\ q(x,y)), \forall u\ \forall v\ (u + s(v) = s(u) + v)$ is

Interpretation

Definition 4.3.16

An interpretation I over a signature Σ is defined by a non-empty domain D and an application which maps every symbol $s^{gn} \in \Sigma$ to its value s_I^{gn} as follows :

1. s_I^{f0} is an element of D.
2. s_I^{fn} where $n \geq 1$ is a function from D^n to D, in other words, a function of n arguments.
3. s_I^{r0} is 0 or 1.
4. s_I^{rn} where $n \geq 1$, is a subset of D^n, in other words, a relation having n arguments.

Example 4.3.17

Let I be the interpretation of domain $D = \{1, 2, 3\}$ where the binary relation *friend* is true for pairs $(1, 2)$, $(1, 3)$ and $(2, 3)$, i.e., $friend_I^{r2} = \{(1, 2), (1, 3), (2, 3)\}$.

$friend(2, 3)$ is true in interpretation I. On the other hand, $friend(2, 1)$ is false in interpretation I.

Remark 4.3.18

In all interpretations, the symbol $=$ maps to the set $\{(d, d) \mid d \in D\}$.

Example 4.3.19

Let us consider the following signature.

- $Anne^{f0}$, $Bernard^{f0}$ and $Claude^{f0}$: the first names Anne, Bernard, and Claude denote constants,
- a^{r2} : the letter a denotes a two-argument relation (we read $a(x,y)$ as x likes y) and
- c^{f1} : the letter c denotes a single argument function (we read $c(x)$ as the friend of x).

A possible interpretation over this signature is the interpretation I of domain $D = \{0,1,2\}$ where :

- $Anne_I^{f0} = 0$, $Bernard_I^{f0} = 1$, and $Claude_I^{f0} = 2$.
- $a_I^{r2} = \{(0,1),(1,0),(2,0)\}$.
- $c_I^{f1}(0) = 1, c_I^{f1}(1) = 0, c_I^{f1}(2) = 2$. Note that the domain of any function is D. In particular, function c_I^{f1} is defined everywhere, which makes it necessary to artificially define $c_I^{f1}(2)$ even if Claude, denoted by 2, has no friend.

Interpretation of a set of formulae

Definition 4.3.20

The interpretation of a set of formulae is an interpretation for the signature associated to this set of formulae.

State, assignment

Definition 4.3.21

A state e of an interpretation is an application from the set of variables to the interpretation domain.

Definition 4.3.22

An assignment is a pair (I, e) composed of an interpretation I and a state e.

Example 4.3.23

Let the domain $D = \{1,2,3\}$ and the interpretation I where the binary relation *friend* is true only for the pairs $(1,2)$, $(1,3)$ and $(2,3)$, i.e., $friend_I^{r2} = \{(1,2),(1,3),(2,3)\}$.

Let e the state which maps x to 2 and y to 1.

The assignment (I,e) makes the relation $friend(x,y)$ false.

Remark 4.3.24

The truth value of a formula depends only on its free variables and on its symbols. In order to evaluate a formula without free variables, the state is useless.

- For a formula with no free variables, simply give an interpretation I of the symbols of the formula. For any state e, we will identify (I, e) and I. Depending on the context, I will be considered either as an interpretation or as an assignment of an arbitrary state.
- For a formula with free variables, we therefore need an assignment.

Terms

Definition 4.3.25 Evaluation

The evaluation of a term t is inductively defined as :

1. if t is a variable, then $[\![t]\!]_{(I,e)} = e(t)$,
2. if t is a constant, then $[\![t]\!]_{(I,e)} = t_I^{f0}$,
3. if $t = s(t_1, \ldots, t_n)$ where s is a symbol and $t_1, \ldots, t_n$ are terms, then $[\![t]\!]_{(I,e)} = s_I^{fn}([\![t_1]\!]_{(I,e)}, \ldots, [\![t_n]\!]_{(I,e)})$.

Example 4.3.26

Let I the interpretation of domain $\mathbb{N}$ which maps the symbols $1^{f0}, *^{f2}, +^{f2}$ to their usual values.
Let e the state such that $x = 2, y = 3$.
Compute $[\![x * (y + 1)]\!]_{(I,e)}$.

Formulae

Definition 4.3.27 Truth value of an atomic formula

The truth value of an atomic formula is given by the following inductive rules :

1. $[\top]_{(I,e)} = 1, [\bot]_{(I,e)} = 0$. In the example, we allow the replacement of $\top$ by its value 1 and $\bot$ by its value 0.
2. Let s a propositional variable, $[s]_{(I,e)} = s_I^{r0}$.
3. Let $A = s(t_1, \ldots, t_n)$ where s is a symbol and $t_1, \ldots, t_n$ are terms. If $([\![t_1]\!]_{(I,e)}, \ldots, [\![t_n]\!]_{(I,e)}) \in s_I^{rn}$ then $[A]_{(I,e)} = 1$ else $[A]_{(I,e)} = 0$.

Example 4.3.31

Let I be the interpretation of domain $D = \{1, 2, 3\}$ where the binary relation *friend* is true for the pairs $(1, 2)$, $(1, 3)$ and $(2, 3)$, i.e., $friend_I^{r2} = \{(1, 2), (1, 3), (2, 3)\}$.

The formula $friend(1, 2) \land friend(2, 3) \Rightarrow friend(1, 3)$ is true in the interpretation I, *i.e.*, $[friend(1, 2) \land friend(2, 3) \Rightarrow friend(1, 3)]_I = 1$.

Example 4.3.29

Let us consider the following signature.

- $Anne^{f0}$, $Bernard^{f0}$ and $Claude^{f0}$: the first names Anne, Bernard, and Claude denote constants,
- a^{r2} : letter a denotes a two-argument relation (we read $a(x,y)$ as x likes y) and
- c^{f1} : the letter c denotes a one-argument function (we read $c(x)$ as the friend of x).

Let I the interpretation of domain $D = \{0,1,2\}$ over this signature where :

- $Anne_I^{f0} = 0$, $Bernard_I^{f0} = 1$, and $Claude_I^{f0} = 2$.
- $a_I^{r2} = \{(0,1),(1,0),(2,0)\}$.
- $c_I^{f1}(0) = 1, c_I^{f1}(1) = 0, c_I^{f1}(2) = 2$. Note that the domain of any function is D. In particular, function c_I^{f1} is defined everywhere, which makes it necessary to artificially define $c_I^{f1}(2)$ even if Claude, denoted by 2, has no friend.

Example 4.3.29

We obtain :

- $[a(Anne, Bernard)]_I =$

- $[a(Anne, Claude)]_I =$

Example 4.3.29

Let e the state $x = 0, y = 2$. We have :

- $[a(x, c(x))]_{(I,e)} =$

- $[a(y, c(y))]_{(I,e)} =$

Make sure to distinguish (depending on the context), the elements of the domain $0, 1$ and the truth values $0, 1$.

Example 4.3.29

We have :

- $[(Anne = Bernard)]_I =$
- $[(c(Anne) = Anne)]_I =$
- $[(c(c(Anne)) = Anne)]_I =$

Truth value of a formula 4.3.30

1. Propositional connectives have the same meaning as in propositional logic.
2. Let x a variable and B a formula. $[\forall x B]_{(I,e)} = 1$ if and only if $[B]_{(I,f)} = 1$ for all state f identical to e, except for x. Let $d \in D$. Let us denote $e[x = d]$ the state identical to the e, except for the variable x, whose state $e[x = d]$ associates the value d. The above definition can be written as :

   $$[\forall x B]_{(I,e)} = min_{d \in D}[B]_{(I,e[x=d])} = \prod_{d \in D} [B]_{(I,e[x=d])},$$

   where the product is the boolean product.
3. $[\exists x B]_{(I,e)} = 1$ if and only if there is a state f identical to e, except for x, such that $[B]_{(I,f)} = 1$. The above definition can be written as :

   $$[\exists x B]_{(I,e)} = max_{d \in D}[B]_{(I,e[x=d])} = \sum_{d \in D} [B]_{(I,e[x=d])},$$

   where the sum is the boolean sum.

Example 4.3.32

Let us use the interpretation I given in example 4.3.19.
(Reminder $D = \{0, 1, 2\}$)

- $[\exists x\ a(x, x)]_I =$

- $[\forall x \exists y\ a(x, y)]_I =$

Example 4.3.32

- $[\exists y \forall x \; a(x, y)]_I =$

Remark 4.3.33

The formulae $\forall x \exists y \; a(x, y)$ and $\exists y \forall x \; a(x, y)$ do not have the same value. Exchanging an existential quantification and an universal quantification does not preserve the truth value of a formula.

Conclusion : Next course

- Interpret a first order formula (contd.)
- Important equivalences

Conclusion

Thank you for your attention.

Questions ?