

First-order logic

Part one :

Language and Truth Value of Formulae

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Homework : solution using ND

$$(p \Rightarrow \neg j) \wedge (\neg p \Rightarrow j) \wedge (j \Rightarrow m) \Rightarrow m \vee p$$

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context	number	proof	justification
1	1	suppose $(p \Rightarrow \neg j) \wedge (\neg p \Rightarrow j) \wedge (j \Rightarrow m)$	

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$$(p \Rightarrow \neg j) \wedge (\neg p \Rightarrow j) \wedge (j \Rightarrow m) \Rightarrow m \vee p$$

context	number	proof	justification
1	1	suppose $(p \Rightarrow \neg j) \wedge (\neg p \Rightarrow j) \wedge (j \Rightarrow m)$	
1	2	$(p \Rightarrow \neg j) \wedge (\neg p \Rightarrow j)$	$\wedge E1$ 1

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$$(p \Rightarrow \neg j) \wedge (\neg p \Rightarrow j) \wedge (j \Rightarrow m) \Rightarrow m \vee p$$

context	number	proof	justification
1	1	suppose $(p \Rightarrow \neg j) \wedge (\neg p \Rightarrow j) \wedge (j \Rightarrow m)$	
1	2	$(p \Rightarrow \neg j) \wedge (\neg p \Rightarrow j)$	$\wedge E1$ 1
1	3	$(j \Rightarrow m)$	$\wedge E2$ 1

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$$(p \Rightarrow \neg j) \wedge (\neg p \Rightarrow j) \wedge (j \Rightarrow m) \Rightarrow m \vee p$$

context	number	proof	justification
1	1	suppose $(p \Rightarrow \neg j) \wedge (\neg p \Rightarrow j) \wedge (j \Rightarrow m)$	
1	2	$(p \Rightarrow \neg j) \wedge (\neg p \Rightarrow j)$	$\wedge E1$ 1
1	3	$(j \Rightarrow m)$	$\wedge E2$ 1
1	4	$(\neg p \Rightarrow j)$	$\wedge E2$ 2

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$$(p \Rightarrow \neg j) \wedge (\neg p \Rightarrow j) \wedge (j \Rightarrow m) \Rightarrow m \vee p$$

context	number	proof	justification
1	1	suppose $(p \Rightarrow \neg j) \wedge (\neg p \Rightarrow j) \wedge (j \Rightarrow m)$	
1	2	$(p \Rightarrow \neg j) \wedge (\neg p \Rightarrow j)$	$\wedge E1$ 1
1	3	$(j \Rightarrow m)$	$\wedge E2$ 1
1	4	$(\neg p \Rightarrow j)$	$\wedge E2$ 2
1, 5	5	suppose $(\neg p \vee p) \Rightarrow \perp$	

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$$(p \Rightarrow \neg j) \wedge (\neg p \Rightarrow j) \wedge (j \Rightarrow m) \Rightarrow m \vee p$$

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1	1	suppose $(p \Rightarrow \neg j) \wedge (\neg p \Rightarrow j) \wedge (j \Rightarrow m)$	
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1	3	$(j \Rightarrow m)$	$\wedge E2$ 1
1	4	$(\neg p \Rightarrow j)$	$\wedge E2$ 2
1, 5	5	suppose $(\neg p \vee p) \Rightarrow \perp$	
1, 5, 6	6	suppose p	

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$$(p \Rightarrow \neg j) \wedge (\neg p \Rightarrow j) \wedge (j \Rightarrow m) \Rightarrow m \vee p$$

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1	1	suppose $(p \Rightarrow \neg j) \wedge (\neg p \Rightarrow j) \wedge (j \Rightarrow m)$	
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1	3	$(j \Rightarrow m)$	$\wedge E2$ 1
1	4	$(\neg p \Rightarrow j)$	$\wedge E2$ 2
1, 5	5	suppose $(\neg p \vee p) \Rightarrow \perp$	
1, 5, 6	6	suppose p	
1, 5, 6	7	$\neg p \vee p$	$\vee I$ 6

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$$(p \Rightarrow \neg j) \wedge (\neg p \Rightarrow j) \wedge (j \Rightarrow m) \Rightarrow m \vee p$$

context	number	proof	justification
1	1	suppose $(p \Rightarrow \neg j) \wedge (\neg p \Rightarrow j) \wedge (j \Rightarrow m)$	
1	2	$(p \Rightarrow \neg j) \wedge (\neg p \Rightarrow j)$	$\wedge E1$ 1
1	3	$(j \Rightarrow m)$	$\wedge E2$ 1
1	4	$(\neg p \Rightarrow j)$	$\wedge E2$ 2
1, 5	5	suppose $(\neg p \vee p) \Rightarrow \perp$	
1, 5, 6	6	suppose p	
1, 5, 6	7	$\neg p \vee p$	$\vee I1$ 6
1, 5, 6	8	\perp	$\Rightarrow E$ 5, 7

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1	1	suppose $(p \Rightarrow \neg j) \wedge (\neg p \Rightarrow j) \wedge (j \Rightarrow m)$	
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1	3	$(j \Rightarrow m)$	$\wedge E2$ 1
1	4	$(\neg p \Rightarrow j)$	$\wedge E2$ 2
1, 5	5	suppose $(\neg p \vee p) \Rightarrow \perp$	
1, 5, 6	6	suppose p	
1, 5, 6	7	$\neg p \vee p$	$\vee I1$ 6
1, 5, 6	8	\perp	$\Rightarrow E$ 5, 7
1, 5	9	hence $\neg p$	$\Rightarrow I$ 6, 8

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1, 5	5	suppose $(\neg p \vee p) \Rightarrow \perp$	
1, 5, 6	6	suppose p	
1, 5, 6	7	$\neg p \vee p$	$\vee I1$ 6
1, 5, 6	8	\perp	$\Rightarrow E$ 5, 7
1, 5	9	hence $\neg p$	$\Rightarrow I$ 6, 8
1, 5	10	$\neg p \vee p$	$\vee I1$ 9

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1, 5, 6	7	$\neg p \vee p$	$\vee I1$ 6
1, 5, 6	8	\perp	$\Rightarrow E$ 5, 7
1, 5	9	hence $\neg p$	$\Rightarrow I$ 6, 8
1, 5	10	$\neg p \vee p$	$\vee I1$ 9
1, 5	11	\perp	$\Rightarrow E$ 5, 10

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1	1	suppose $(p \Rightarrow \neg j) \wedge (\neg p \Rightarrow j) \wedge (j \Rightarrow m)$	
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1, 5	5	suppose $(\neg p \vee p) \Rightarrow \perp$	
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1, 5, 6	8	\perp	$\Rightarrow E$ 5, 7
1, 5	9	hence $\neg p$	$\Rightarrow I$ 6, 8
1, 5	10	$\neg p \vee p$	$\vee I1$ 9
1, 5	11	\perp	$\Rightarrow E$ 5, 10
1	12	hence $\neg \neg(\neg p \vee p)$	$\Rightarrow I$ 5, 11

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1, 5	5	suppose $(\neg p \vee p) \Rightarrow \perp$	
1, 5, 6	6	suppose p	
1, 5, 6	7	$\neg p \vee p$	$\vee I1$ 6
1, 5, 6	8	\perp	$\Rightarrow E$ 5, 7
1, 5	9	hence $\neg p$	$\Rightarrow I$ 6, 8
1, 5	10	$\neg p \vee p$	$\vee I1$ 9
1, 5	11	\perp	$\Rightarrow E$ 5, 10
1	12	hence $\neg\neg(\neg p \vee p)$	$\Rightarrow I$ 5, 11
1	13	$\neg p \vee p$	RAA, 12

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1	1	suppose $(p \Rightarrow \neg j) \wedge (\neg p \Rightarrow j) \wedge (j \Rightarrow m)$	
1	2	$(p \Rightarrow \neg j) \wedge (\neg p \Rightarrow j)$	$\wedge E1$ 1
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1	4	$(\neg p \Rightarrow j)$	$\wedge E2$ 2
1, 5	5	suppose $(\neg p \vee p) \Rightarrow \perp$	
1, 5, 6	6	suppose p	
1, 5, 6	7	$\neg p \vee p$	$\vee I1$ 6
1, 5, 6	8	\perp	$\Rightarrow E$ 5, 7
1, 5	9	hence $\neg p$	$\Rightarrow I$ 6, 8
1, 5	10	$\neg p \vee p$	$\vee I1$ 9
1, 5	11	\perp	$\Rightarrow E$ 5, 10
1	12	hence $\neg\neg(\neg p \vee p)$	$\Rightarrow I$ 5, 11
1	13	$\neg p \vee p$	RAA, 12
1, 14	14	suppose $\neg p$	

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1	2	$(p \Rightarrow \neg j) \wedge (\neg p \Rightarrow j)$	$\wedge E1$ 1
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1, 5	5	suppose $(\neg p \vee p) \Rightarrow \perp$	
1, 5, 6	6	suppose p	
1, 5, 6	7	$\neg p \vee p$	$\vee I1$ 6
1, 5, 6	8	\perp	$\Rightarrow E$ 5, 7
1, 5	9	hence $\neg p$	$\Rightarrow I$ 6, 8
1, 5	10	$\neg p \vee p$	$\vee I1$ 9
1, 5	11	\perp	$\Rightarrow E$ 5, 10
1	12	hence $\neg\neg(\neg p \vee p)$	$\Rightarrow I$ 5, 11
1	13	$\neg p \vee p$	RAA, 12
1, 14	14	suppose $\neg p$	
1, 14	15	j	$\Rightarrow E$ 4, 14

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1	2	$(p \Rightarrow \neg j) \wedge (\neg p \Rightarrow j)$	$\wedge E1$ 1
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1	4	$(\neg p \Rightarrow j)$	$\wedge E2$ 2
1, 5	5	suppose $(\neg p \vee p) \Rightarrow \perp$	
1, 5, 6	6	suppose p	
1, 5, 6	7	$\neg p \vee p$	$\vee I1$ 6
1, 5, 6	8	\perp	$\Rightarrow E$ 5, 7
1, 5	9	hence $\neg p$	$\Rightarrow I$ 6, 8
1, 5	10	$\neg p \vee p$	$\vee I1$ 9
1, 5	11	\perp	$\Rightarrow E$ 5, 10
1	12	hence $\neg\neg(\neg p \vee p)$	$\Rightarrow I$ 5, 11
1	13	$\neg p \vee p$	RAA, 12
1, 14	14	suppose $\neg p$	
1, 14	15	j	$\Rightarrow E$ 4, 14
1, 14	16	m	$\Rightarrow E$ 3, 15

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1	1	suppose $(p \Rightarrow \neg j) \wedge (\neg p \Rightarrow j) \wedge (j \Rightarrow m)$	
1	2	$(p \Rightarrow \neg j) \wedge (\neg p \Rightarrow j)$	$\wedge E1$ 1
1	3	$(j \Rightarrow m)$	$\wedge E2$ 1
1	4	$(\neg p \Rightarrow j)$	$\wedge E2$ 2
1, 5	5	suppose $(\neg p \vee p) \Rightarrow \perp$	
1, 5, 6	6	suppose p	
1, 5, 6	7	$\neg p \vee p$	$\vee I1$ 6
1, 5, 6	8	\perp	$\Rightarrow E$ 5, 7
1, 5	9	hence $\neg p$	$\Rightarrow I$ 6, 8
1, 5	10	$\neg p \vee p$	$\vee I1$ 9
1, 5	11	\perp	$\Rightarrow E$ 5, 10
1	12	hence $\neg\neg(\neg p \vee p)$	$\Rightarrow I$ 5, 11
1	13	$\neg p \vee p$	RAA, 12
1, 14	14	suppose $\neg p$	
1, 14	15	j	$\Rightarrow E$ 4, 14
1, 14	16	m	$\Rightarrow E$ 3, 15
1, 14	17	$m \vee p$	$\vee I1$ 16

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1	2	$(p \Rightarrow \neg j) \wedge (\neg p \Rightarrow j)$	$\wedge E1$ 1
1	3	$(j \Rightarrow m)$	$\wedge E2$ 1
1	4	$(\neg p \Rightarrow j)$	$\wedge E2$ 2
1, 5	5	suppose $(\neg p \vee p) \Rightarrow \perp$	
1, 5, 6	6	suppose p	
1, 5, 6	7	$\neg p \vee p$	$\vee I1$ 6
1, 5, 6	8	\perp	$\Rightarrow E$ 5, 7
1, 5	9	hence $\neg p$	$\Rightarrow I$ 6, 8
1, 5	10	$\neg p \vee p$	$\vee I1$ 9
1, 5	11	\perp	$\Rightarrow E$ 5, 10
1	12	hence $\neg\neg(\neg p \vee p)$	$\Rightarrow I$ 5, 11
1	13	$\neg p \vee p$	RAA, 12
1, 14	14	suppose $\neg p$	
1, 14	15	j	$\Rightarrow E$ 4, 14
1, 14	16	m	$\Rightarrow E$ 3, 15
1, 14	17	$m \vee p$	$\vee I1$ 16
1	18	hence $\neg p \Rightarrow p \vee m$	$\Rightarrow I$ 17, 14

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	1	suppose $(p \Rightarrow \neg j) \wedge (\neg p \Rightarrow j) \wedge (j \Rightarrow m)$	
1	2	$(p \Rightarrow \neg j) \wedge (\neg p \Rightarrow j)$	$\wedge E1$ 1
1	3	$(j \Rightarrow m)$	$\wedge E2$ 1
1	4	$(\neg p \Rightarrow j)$	$\wedge E2$ 2
1, 5	5	suppose $(\neg p \vee p) \Rightarrow \perp$	
1, 5, 6	6	suppose p	
1, 5, 6	7	$\neg p \vee p$	$\vee I1$ 6
1, 5, 6	8	\perp	$\Rightarrow E$ 5, 7
1, 5	9	hence $\neg p$	$\Rightarrow I$ 6, 8
1, 5	10	$\neg p \vee p$	$\vee I1$ 9
1, 5	11	\perp	$\Rightarrow E$ 5, 10
1	12	hence $\neg\neg(\neg p \vee p)$	$\Rightarrow I$ 5, 11
1	13	$\neg p \vee p$	RAA, 12
1, 14	14	suppose $\neg p$	
1, 14	15	j	$\Rightarrow E$ 4, 14
1, 14	16	m	$\Rightarrow E$ 3, 15
1, 14	17	$m \vee p$	$\vee I1$ 16
1	18	hence $\neg p \Rightarrow p \vee m$	$\Rightarrow I$ 17, 14
1, 19	19	suppose p	

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1	4	$(\neg p \Rightarrow j)$	$\wedge E2$ 2
1, 5	5	suppose $(\neg p \vee p) \Rightarrow \perp$	
1, 5, 6	6	suppose p	
1, 5, 6	7	$\neg p \vee p$	$\vee I1$ 6
1, 5, 6	8	\perp	$\Rightarrow E$ 5, 7
1, 5	9	hence $\neg p$	$\Rightarrow I$ 6, 8
1, 5	10	$\neg p \vee p$	$\vee I1$ 9
1, 5	11	\perp	$\Rightarrow E$ 5, 10
1	12	hence $\neg\neg(\neg p \vee p)$	$\Rightarrow I$ 5, 11
1	13	$\neg p \vee p$	RAA, 12
1, 14	14	suppose $\neg p$	
1, 14	15	j	$\Rightarrow E$ 4, 14
1, 14	16	m	$\Rightarrow E$ 3, 15
1, 14	17	$m \vee p$	$\vee I1$ 16
1	18	hence $\neg p \Rightarrow p \vee m$	$\Rightarrow I$ 17, 14
1, 19	19	suppose p	
1, 19	20	$p \vee m$	$\vee I1$ 19

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$$(p \Rightarrow \neg j) \wedge (\neg p \Rightarrow j) \wedge (j \Rightarrow m) \Rightarrow m \vee p$$

context	number	proof	justification
1	1	suppose $(p \Rightarrow \neg j) \wedge (\neg p \Rightarrow j) \wedge (j \Rightarrow m)$	
1	2	$(p \Rightarrow \neg j) \wedge (\neg p \Rightarrow j)$	$\wedge E$ 1
1	3	$(j \Rightarrow m)$	$\wedge E$ 1
1	4	$(\neg p \Rightarrow j)$	$\wedge E$ 2
1, 5	5	suppose $(\neg p \vee p) \Rightarrow \perp$	
1, 5, 6	6	suppose p	
1, 5, 6	7	$\neg p \vee p$	$\vee I$ 6
1, 5, 6	8	\perp	$\Rightarrow E$ 5, 7
1, 5	9	hence $\neg p$	$\Rightarrow I$ 6, 8
1, 5	10	$\neg p \vee p$	$\vee I$ 9
1, 5	11	\perp	$\Rightarrow E$ 5, 10
1	12	hence $\neg\neg(\neg p \vee p)$	$\Rightarrow I$ 5, 11
1	13	$\neg p \vee p$	RAA, 12
1, 14	14	suppose $\neg p$	
1, 14	15	j	$\Rightarrow E$ 4, 14
1, 14	16	m	$\Rightarrow E$ 3, 15
1, 14	17	$m \vee p$	$\vee I$ 16
1	18	hence $\neg p \Rightarrow p \vee m$	$\Rightarrow I$ 17, 14
1, 19	19	suppose p	
1, 19	20	$p \vee m$	$\vee I$ 19
1	21	hence $p \Rightarrow p \vee m$	$\Rightarrow I$ 20, 19

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$$(p \Rightarrow \neg j) \wedge (\neg p \Rightarrow j) \wedge (j \Rightarrow m) \Rightarrow m \vee p$$

context	number	proof	justification
1	1	suppose $(p \Rightarrow \neg j) \wedge (\neg p \Rightarrow j) \wedge (j \Rightarrow m)$	
1	2	$(p \Rightarrow \neg j) \wedge (\neg p \Rightarrow j)$	$\wedge E$ 1
1	3	$(j \Rightarrow m)$	$\wedge E$ 1
1	4	$(\neg p \Rightarrow j)$	$\wedge E$ 2
1, 5	5	suppose $(\neg p \vee p) \Rightarrow \perp$	
1, 5, 6	6	suppose p	
1, 5, 6	7	$\neg p \vee p$	$\vee I$ 6
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1, 5	9	hence $\neg p$	$\Rightarrow I$ 6, 8
1, 5	10	$\neg p \vee p$	$\vee I$ 9
1, 5	11	\perp	$\Rightarrow E$ 5, 10
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1	13	$\neg p \vee p$	RAA, 12
1, 14	14	suppose $\neg p$	
1, 14	15	j	$\Rightarrow E$ 4, 14
1, 14	16	m	$\Rightarrow E$ 3, 15
1, 14	17	$m \vee p$	$\vee I$ 16
1	18	hence $\neg p \Rightarrow p \vee m$	$\Rightarrow I$ 17, 14
1, 19	19	suppose p	
1, 19	20	$p \vee m$	$\vee I$ 19
1	21	hence $p \Rightarrow p \vee m$	$\Rightarrow I$ 20, 19
1	22	$m \vee p$	$\vee E$ 21, 18, 13

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$$(p \Rightarrow \neg j) \wedge (\neg p \Rightarrow j) \wedge (j \Rightarrow m) \Rightarrow m \vee p$$

context	number	proof	justification
1	1	suppose $(p \Rightarrow \neg j) \wedge (\neg p \Rightarrow j) \wedge (j \Rightarrow m)$	
1	2	$(p \Rightarrow \neg j) \wedge (\neg p \Rightarrow j)$	$\wedge E1$ 1
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1	4	$(\neg p \Rightarrow j)$	$\wedge E2$ 2
1, 5	5	suppose $(\neg p \vee p) \Rightarrow \perp$	
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1, 14	14	suppose $\neg p$	
1, 14	15	j	$\Rightarrow E$ 4, 14
1, 14	16	m	$\Rightarrow E$ 3, 15
1, 14	17	$m \vee p$	$\vee I1$ 16
1	18	hence $\neg p \Rightarrow p \vee m$	$\Rightarrow I$ 17, 14
1, 19	19	suppose p	
1, 19	20	$p \vee m$	$\vee I1$ 19
1	21	hence $p \Rightarrow p \vee m$	$\Rightarrow I$ 20, 19
1	22	$m \vee p$	$\vee E$ 21, 18, 13
	23	hence $(p \Rightarrow \neg j) \wedge (\neg p \Rightarrow j) \wedge (j \Rightarrow m) \Rightarrow m \vee p$	$\Rightarrow I$ 22, 1

Structure of first-order logic

A non-empty domain (more than two elements)

Structure of first-order logic

A non-empty domain (more than two elements)

Three categories :

- ▶ **Terms** representing the elements of the domain or functions on these elements
- ▶ **Relations**
- ▶ **Formulae** describing the interactions between the relations thanks to connectives and quantifiers

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A non-empty domain (more than two elements)

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Remark :

Two particular symbols (quantifiers) : \forall (universal quantification) and \exists (existential quantification).

Structure of first-order logic

Examples :

- ▶ the term $parent(x)$ intended to mean the parent of x ,
- ▶ the formula $\forall x \exists y parent(y, x)$ indicates that every individual has a parent.

Syllogism

A cheap horse is rare.
Everything that is rare is expensive.
Hence a cheap horse is expensive.

Syllogism

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Hence a cheap horse is expensive.

$$\begin{aligned}\forall x(\textit{horse}(x) \wedge \textit{cheap}(x) \Rightarrow \textit{rare}(x)) \\ \forall x(\textit{rare}(x) \Rightarrow \textit{expensive}(x)) \\ \forall x(\textit{horse}(x) \wedge \textit{cheap}(x) \Rightarrow \textit{expensive}(x))\end{aligned}$$

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Nothing bothers you ?

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Everything that is expensive is not cheap and vice versa.

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Nothing bothers you ?

Everything that is expensive is not cheap and vice versa.

$$\forall x(\text{cheap}(x) \Leftrightarrow \neg \text{expensive}(x))$$

Syllogism

A cheap horse is rare.
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Nothing bothers you ?

Everything that is expensive is not cheap and vice versa.

$$\forall x(cheap(x) \Leftrightarrow \neg expensive(x))$$

Now we have a contradiction.

Usage

First-order logic allows us to model :

- ▶ a **single** non-empty **domain**,
- ▶ **functions** over the domain, and
- ▶ **relations** over the domain.

Overview

Introduction

Language

(Strict) Formulae

Prioritized formulae

Free vs. bound

Truth value of formulae

Declaring a symbol

Signature

Interpretation

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Vocabulary

- ▶ **Two propositional constants** : \perp and \top
- ▶ **Variables** : sequence of letters and digits starting with one of the following lower case letters : u,v,w,x,y,z.
- ▶ **Connectives** : $\neg, \wedge, \vee, \Rightarrow, \Leftrightarrow$
- ▶ **Quantifiers** : \forall the **universal** quantification and \exists the **existential** quantification
- ▶ **Punctuation** : the comma \langle , \rangle and the open $\langle (\rangle$ and closing $\langle \rangle \rangle$ parenthesis.
- ▶ **Ordinary and special symbols** :
 - ▶ **an ordinary symbol** is a sequence of letters and digits not starting by one of the following lower case letters : u,v,w,x,y,z.
 - ▶ **a special symbol** is $+, -, *, /, =, \neq, <, \leq, >, \geq, \dots$

Example 4.1.1

- ▶ x, x_1, x_2, y are **variables**,
- ▶ $man, parent, succ, 12, 24, f_1$ are **ordinary symbols**, the ordinary symbols will represent :
 - ▶ **functions** (numerical constants or multiple argument functions) or
 - ▶ **relations** (propositional variables or multiple argument relations).
- ▶ $x = y, z > 3$ are examples for **special symbols**

Term

Definition 4.1.2

- ▶ an ordinary symbol is a term,
- ▶ a variable is a term,
- ▶ if t_1, \dots, t_n are terms and if s is a (ordinary or special) symbol then $s(t_1, \dots, t_n)$ is a term.

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Example 4.1.3

$x, a, f(x_1, x_2, g(y)), +(x, *(y, z)), +(5, 42)$ are terms

On the contrary, $f(\perp, 2, y)$ is not a term.

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$x, a, f(x_1, x_2, g(y)), +(x, *(y, z)), +(5, 42)$ are terms

On the contrary, $f(\perp, 2, y)$ is not a term.

Note that $42(1, y, 3)$ is also a term, but by convention function and relation names are ordinary symbols starting with letters.

Atomic formula

Definition 4.1.4 atomic formulae

- ▶ \top and \perp are atomic formulae
- ▶ an ordinary symbol is an atomic formula
- ▶ if t_1, \dots, t_n are terms and if s is a (ordinary or special) symbol then $s(t_1, \dots, t_n)$ is an atomic formula.

Atomic formula

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Example 4.1.5 :

- ▶ $f(1, +(5, 42), g(z))$, a , and $+(x, *(y, z))$ are atomic formulae
- ▶ x and $A \vee f(4, 2, 6)$ are not atomic formulae

Syntax v.s. Semantics

The set of **terms** and the set of **atomic formulae** are not disjoint.

For example $p(x)$ is a term and an atomic formula.

When t is a term and an atomic formula simultaneously, we distinguish $[[t]]$, the value of t seen as a term of $[t]$, value of t seen as a formula.

(Strict) Formula

Definition 4.1.6

- ▶ an atomic formula is a formula,
- ▶ if A is a formula then $\neg A$ is a formula,
- ▶ if A and B are formulae and if \circ one of the following operations $\vee, \wedge, \Rightarrow, \Leftrightarrow$ then $(A \circ B)$ is a formula ,
- ▶ if A is a formula and if x is **any** variable then $\forall x A$ and $\exists x A$ are formulae.

Example 4.1.7

- ▶ $man(x)$, $parents(son(y), mother(Alice))$, $= (x, +(f(x), g(y)))$ are **atomic formulae**, hence formulae.
- ▶ On the contrary

$$\forall x (man(x) \Rightarrow man(Socrate))$$

is a **non-atomic formula**.

(Strict) Formula : Examples

Among these expressions, which ones are strict formulae :

- ▶ x

(Strict) Formula : Examples

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- ▶ $((a < b) \Rightarrow ((2 * b) > (2 * a)))$

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Infix notations

Prioritized formulae : the symbols of the functions $+$, $-$, $*$, $/$ and the symbols of the relations $=$, \neq , $<$, $>$, \leq , \geq are written in the usual manner.

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$$3 * x \leq y + 5$$

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Example 4.1.8

- ▶ $\leq (* (3, x), + (y, 5))$ is abbreviated as

$$3 * x \leq y + 5$$

- ▶ $+(x, *(y, z))$ is abbreviated as

$$x + y * z$$

Inverse transformation

Prioritize

- ▶ **connectives** have a lower priority than the relations
- ▶ quantifiers have the same priority as negation.
- ▶ $=, \neq, <, \leq, >, \geq$ have a lower priority than $+, -, *, /$

Table 4.1 summary of priorities

Priorities decreasing from top to bottom.

OPERATIONS	
$-$, $+$ unary	
$*$, $/$ binary	left associative
$+$, $-$ binary	left associative
RELATIONS	
$=, \neq, <, \leq, >, \geq$	
NEGATION, QUANTIFIERS	
\neg, \forall, \exists	
BINARY CONNECTIVES	
\wedge	left associative
\vee	left associative
\Rightarrow	right associative
\Leftrightarrow	left associative

Prioritized formulae

Definition 4.1.9 prioritized formulae

A **prioritized formula** is inductively defined as follows :

- ▶ An atomic formula is a prioritized formula.
- ▶ If A is a prioritized formula then $\neg A$ is a prioritized formula.
- ▶ If A and B are prioritized formulae then $A \circ B$ is a prioritized formula.
- ▶ If A is a prioritized formula and if x is *any* variable then $\forall x A$ and $\exists x A$ are prioritized formulae.
- ▶ If A is a prioritized formula (A) is a prioritized formula.

Examples

Example 4.1.10

- ▶ $\forall xP(x) \wedge \forall xQ(x) \Leftrightarrow \forall x(P(x) \wedge Q(x))$ is an **abbreviation** of

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$$((\forall xP(x) \wedge \forall xQ(x)) \Leftrightarrow \forall x(P(x) \wedge Q(x)))$$

- ▶ $\forall x\forall y\forall z(x \leq y \wedge y \leq z \Rightarrow x \leq z)$ is an **abbreviation** of ?

Examples

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- ▶ $\forall xP(x) \wedge \forall xQ(x) \Leftrightarrow \forall x(P(x) \wedge Q(x))$ is an **abbreviation** of

$$((\forall xP(x) \wedge \forall xQ(x)) \Leftrightarrow \forall x(P(x) \wedge Q(x)))$$

- ▶ $\forall x\forall y\forall z(x \leq y \wedge y \leq z \Rightarrow x \leq z)$ is an **abbreviation** of

$$\forall x\forall y\forall z((\leq(x, y) \wedge \leq(y, z)) \Rightarrow \leq(x, z))$$

Tree representation

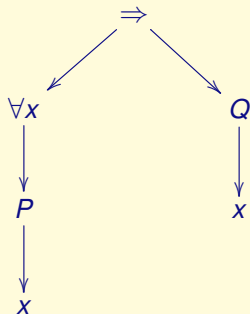
Example 4.1.11 $\forall xP(x) \Rightarrow Q(x)$

the left-hand side operand of the implication is $\forall xP(x)$.

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Idea

- ▶ The **meaning** of the formula $x + 2 = 4$ depends on x
The **meaning** of the formula $x = x$ depends on x as well
 x is free in the previous formulae

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- ▶ The **meaning** of the formula $x + 2 = 4$ depends on x
The **meaning** of the formula $x = x$ depends on x as well
 x is free in the previous formulae
- ▶ The **meaning** of $\forall x(x + 2 = y)$ does not depend on x
The **meaning** of $\forall x(x + 0 = x)$ does not depend on x
 x is not free in these two formulae

Free and bound occurrences

Definition 4.2.1

- ▶ In $\forall x A$ or $\exists x A$, the **scope of the binding** of x is A .

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- ▶ A bound occurrence of x is

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below a node $\exists x$ or $\forall x$.

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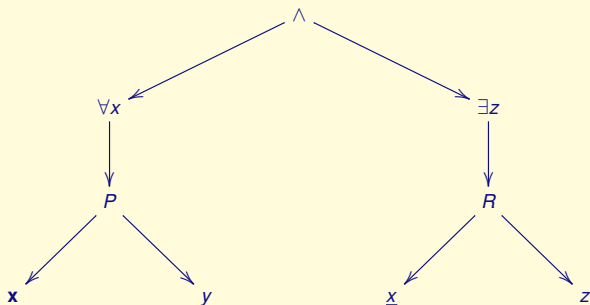
is not below such a node.

Example 4.2.2

$$\forall x P(\mathbf{x}, y) \wedge \exists z R(\underline{x}, z)$$

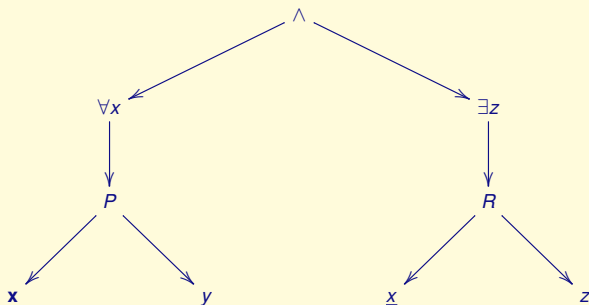
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Example 4.2.2

$$\forall x P(\mathbf{x}, y) \wedge \exists z R(\underline{x}, z)$$



- ▶ The bold occurrence of x is bound.
- ▶ The underlined occurrence of x is free.
- ▶ The occurrence of z is bound.

Free, bound variables

Definition 4.2.3

- ▶ The variable x is a **free variable** of a formula if and only if there is a free occurrence of x in the formula.
- ▶ A variable x is a **bound variable** of a formula if and only if there is a bound occurrence of x in the formula.
- ▶ A formula without free variable is also called a **closed formula**.

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Remark 4.2.4

A variable can be simultaneously free and bound. For example, in the formula $\forall xP(x) \vee Q(x)$, x is both free and bound.

Remark 4.2.5

By definition, a variable which does not appear in a formula (0 occurrence) is **NOT** free in this formula.

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Example 4.2.6

The free variables of the formula of example 4.2.2 are x and y .

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Declaring a symbol

Definition 4.3.1

A **symbol declaration** is a triple denoted by s^{gn} where :

- ▶ s is a symbol
- ▶ g one of the letters f (a function) or r (a relation)
- ▶ n is a natural number.

Declaring a symbol

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- ▶ s is a symbol
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- ▶ n is a natural number.

Remark 4.3.3

If the context gives an implicit declaration of a symbol, we omit the exponent.

Example : **equal** is always a 2 arguments relation, we abbreviate the declaration $=^{r2}$ as $=$.

Symbol declaration : Example

Example 4.3.2

- ▶ $parent^{r2}$ is a **relation (r)** with **2** arguments
- ▶ $*^{f2}$ is **function (f)** with **2** arguments
- ▶ man^{r1} a unary **relation**

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Signature

Definition 4.3.4

A **signature** is a set of symbol declarations.

Let $n > 0$ and Σ a signature, the symbol s is :

1. a **constant** of the signature if and only if $s^{f0} \in \Sigma$
2. a **symbol of the function of n arguments** of the signature, if and only if $s^{fn} \in \Sigma$
3. a **propositional variable** of the signature if and only if $s^{r0} \in \Sigma$
4. a **symbol of the relation of n arguments** of the signature, if and only if $s^{rn} \in \Sigma$

Examples in mathematics (1/2)

$0^{f0}, 1^{f0}, +^{f2}, -^{f2}, *^{f2}, =^{r2}$ is a signature for arithmetics.

Examples in mathematics (1/2)

$0^{f0}, 1^{f0}, +^{f2}, -^{f2}, *^{f2}, =^{r2}$ is a signature for arithmetics.

Remark :

- ▶ We write : 0, 1, + and – (with two arguments), * and =.
- ▶ Note that plus and minus have two arguments, (the symbol will not be used with only one argument).

Examples in mathematics (2/2)

Example 4.3.5 (Set theory)

A possible signature is $\in, =$

Examples in mathematics (2/2)

Example 4.3.5 (Set theory)

A possible signature is $\in, =$

All other operations can be defined from these relations.

Overloading

Definition 4.3.6

A symbol is **overloaded** in a signature, when this signature has two distinct declarations of the same symbol.

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Example 4.3.7 : the minus sign is often overloaded.

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- ▶ the subtraction of two numbers

Overloading

Definition 4.3.6

A symbol is **overloaded** in a signature, when this signature has two distinct declarations of the same symbol.

Example 4.3.7 : the minus sign is often overloaded.

- ▶ the opposite of a number
- ▶ the subtraction of two numbers

In what follows, in this course, we prohibit the use of overloaded symbols in signatures.

Term over a signature

Definition 4.3.8

Let Σ be a signature, a **term** over Σ is :

- ▶ either a variable,
- ▶ or a constant s where $s^{f_0} \in \Sigma$,
- ▶ or a term of the form $s(t_1, \dots, t_n)$, where $n \geq 1$, $s^{f_n} \in \Sigma$ and t_1, \dots, t_n are terms over Σ .

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The set of terms over the signature Σ is denoted by T_Σ .

Atomic formula over a signature

Definition 4.3.9

Let Σ a signature, an **atomic formula** over Σ is :

- ▶ either one of the constants \top, \perp ,
- ▶ or a propositional variable s where $s^{r_0} \in \Sigma$,
- ▶ or an expression $s(t_1, \dots, t_n)$ where $n \geq 1$, $s^{r_n} \in \Sigma$ and t_1, \dots, t_n are terms over Σ .

Formula over a signature

Definition 4.3.10

A **formula** over a signature Σ is a formula, whose atomic sub-formulae are atomic formulae over Σ (according to definition 4.3.9).

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A **formula** over a signature Σ is a formula, whose atomic sub-formulae are atomic formulae over Σ (according to definition 4.3.9).

F_{Σ} denotes the set of formulae over the signature Σ .

Example 4.3.11

$\forall x (p(x) \Rightarrow \exists y q(x, y))$ is a formula over signature
 $\Sigma = \{p^{r1}, q^{r2}, h^{f1}, c^{f0}\}$.

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$\forall x (p(x) \Rightarrow \exists y q(x, y))$ is a formula over signature
 $\Sigma = \{p^{r1}, q^{r2}, h^{f1}, c^{f0}\}$.

But it is also a formula over the signature $\Sigma' = \{p^{r1}, q^{r2}\}$, since the symbols h and c are not in the formula.

Associated signature

Definition 4.3.12

The **signature associated** to a formula is the smallest signature Σ such that the formula is a member of F_Σ , it is the smallest signature allowing to write the formula.

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Example 4.3.13

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Associated signature

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Example 4.3.13

The associated signature of formula $\forall x (p(x) \Rightarrow \exists y q(x, y))$ is

p^{r1}, q^{r2} .

Associated signature

Definition 4.3.14

The **associated signature** to a set of formulae is the union of the associated signatures of all formulae of the set.

Associated signature

Definition 4.3.14

The **associated signature** to a set of formulae is the union of the associated signatures of all formulae of the set.

Example 4.3.15

The associated signature of a set of two formulae

$\forall x(p(x) \Rightarrow \exists y q(x, y)), \forall u \forall v (u + s(v) = s(u) + v)$ is

$$\Sigma = \{p^{r1}, q^{r2}, +^{f2}, s^{f1}, =^{r2}\}.$$

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Definition 4.3.16

An **interpretation** I over a signature Σ is defined by a non-empty domain D and an application which maps every symbol $s^{gn} \in \Sigma$ to its value s_I^{gn} as follows :

1. s_I^{f0} is an element of D .
2. s_I^{fn} where $n \geq 1$ is a function from D^n to D , in other words, a function of n arguments.
3. s_I^{r0} is 0 or 1.
4. s_I^{rn} where $n \geq 1$, is a subset of D^n , in other words, a relation having n arguments.

Example 4.3.17

Let I be the interpretation of domain $D = \{1, 2, 3\}$ where the binary relation *friend* is true for pairs (1, 2), (1, 3) and (2, 3), i.e.,
 $friend_I^{r^2} = \{(1, 2), (1, 3), (2, 3)\}$.

friend(2, 3) is true in interpretation I . On the other hand, *friend*(2, 1) is false in interpretation I .

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Remark 4.3.18

In all interpretations, the symbol $=$ maps to the set $\{(d, d) \mid d \in D\}$.

Example 4.3.19

Let us consider the following signature.

- ▶ $Anne^{f_0}$, $Bernard^{f_0}$ and $Claude^{f_0}$: the first names Anne, Bernard, and Claude denote constants,
- ▶ a^{r_2} : the letter a denotes a two-argument relation (we read $a(x, y)$ as x likes y) and
- ▶ c^{f_1} : the letter c denotes a single argument function (we read $c(x)$ as the friend of x).

Example 4.3.19

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- ▶ $Anne^{f_0}$, $Bernard^{f_0}$ and $Claude^{f_0}$: the first names Anne, Bernard, and Claude denote constants,
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- ▶ c^{f_1} : the letter c denotes a single argument function (we read $c(x)$ as the friend of x).

A possible interpretation over this signature is the interpretation I of domain $D = \{0, 1, 2\}$ where :

- ▶ $Anne_I^{f_0} = 0$, $Bernard_I^{f_0} = 1$, and $Claude_I^{f_0} = 2$.
- ▶ $a_I^{f_2} = \{(0, 1), (1, 0), (2, 0)\}$.
- ▶ $c_I^{f_1}(0) = 1$, $c_I^{f_1}(1) = 0$, $c_I^{f_1}(2) = 2$. Note that the domain of any function is D . In particular, function $c_I^{f_1}$ is defined everywhere, which makes it necessary to artificially define $c_I^{f_1}(2)$ even if Claude, denoted by 2, has no friend.

Interpretation of a set of formulae

Definition 4.3.20

The **interpretation of a set of formulae** is an interpretation for the signature associated to this set of formulae.

State, assignment

Definition 4.3.21

A **state** e of an interpretation is an application from the set of variables to the interpretation domain.

State, assignment

Definition 4.3.21

A **state** e of an interpretation is an application from the set of variables to the interpretation domain.

Definition 4.3.22

An **assignment** is a pair (I, e) composed of an interpretation I and a state e .

Example 4.3.23

Let the domain $D = \{1, 2, 3\}$ and the interpretation I where the binary relation *friend* is true only for the pairs $(1, 2)$, $(1, 3)$ and $(2, 3)$, i.e., $friend_I^{r2} = \{(1, 2), (1, 3), (2, 3)\}$.

Let e the state which maps x to 2 and y to 1.

The assignment (I, e) makes the relation $friend(x, y)$ false.

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Remark 4.3.24

The truth value of a formula depends only on its free variables and on its symbols. In order to evaluate a formula without free variables, the state is useless.

- ▶ For a formula **with no free variables**, simply give an interpretation I of the symbols of the formula. For any state e , we will identify (I, e) and I . Depending on the context, I will be considered either as an interpretation or as an assignment of an arbitrary state.
- ▶ For a formula **with free variables**, we therefore need an assignment.

Terms

Definition 4.3.25 Evaluation

The evaluation of a term t is inductively defined as :

1. if t is a variable, then $\llbracket t \rrbracket_{(I,e)} = e(t)$,
2. if t is a constant, then $\llbracket t \rrbracket_{(I,e)} = t_I^{f_0}$,
3. if $t = s(t_1, \dots, t_n)$ where s is a symbol and t_1, \dots, t_n are terms, then $\llbracket t \rrbracket_{(I,e)} = s_I^{f_n}(\llbracket t_1 \rrbracket_{(I,e)}, \dots, \llbracket t_n \rrbracket_{(I,e)})$.

Example 4.3.26

Let I the interpretation of domain \mathbb{N} which maps the symbols $1^{f0}, *^{f2}, +^{f2}$ to their usual values.

Let e the state such that $x = 2, y = 3$.

Compute $\llbracket x * (y + 1) \rrbracket_{(I,e)}$.

Example 4.3.26

Let I the interpretation of domain \mathbb{N} which maps the symbols $1^{f0}, *^{f2}, +^{f2}$ to their usual values.

Let e the state such that $x = 2, y = 3$.

Compute $\llbracket x * (y + 1) \rrbracket_{(I,e)}$.

$$\begin{aligned}\llbracket x * (y + 1) \rrbracket_{(I,e)} &= \llbracket x \rrbracket_{(I,e)} * \llbracket (y + 1) \rrbracket_{(I,e)} = \\ \llbracket x \rrbracket_{(I,e)} * (\llbracket y \rrbracket_{(I,e)} + \llbracket 1 \rrbracket_{(I,e)}) &= e(x) * (e(y) + 1) = 2 * (3 + 1) = 8.\end{aligned}$$

Formulae

Definition 4.3.27 Truth value of an atomic formula

The truth value of an atomic formula is given by the following inductive rules :

1. $[\top]_{(I,e)} = 1, [\perp]_{(I,e)} = 0$. In the example, we allow the replacement of \top by its value 1 and \perp by its value 0.
2. Let s a propositional variable, $[s]_{(I,e)} = s_I^{r_0}$.
3. Let $A = s(t_1, \dots, t_n)$ where s is a symbol and t_1, \dots, t_n are terms. If $([t_1]_{(I,e)}, \dots, [t_n]_{(I,e)}) \in s_I^{r_n}$ then $[A]_{(I,e)} = 1$ else $[A]_{(I,e)} = 0$.

Example 4.3.31

Let I be the interpretation of domain $D = \{1, 2, 3\}$ where the binary relation *friend* is true for the pairs (1, 2), (1, 3) and (2, 3), i.e., $friend_I^2 = \{(1, 2), (1, 3), (2, 3)\}$.

Example 4.3.31

Let I be the interpretation of domain $D = \{1, 2, 3\}$ where the binary relation *friend* is true for the pairs (1, 2), (1, 3) and (2, 3), i.e., $friend_I^2 = \{(1, 2), (1, 3), (2, 3)\}$.

The formula $friend(1, 2) \wedge friend(2, 3) \Rightarrow friend(1, 3)$ is true in the interpretation I , i.e., $[friend(1, 2) \wedge friend(2, 3) \Rightarrow friend(1, 3)]_I = 1$.

Example 4.3.29

Let us consider the following signature.

- ▶ $Anne^{f0}$, $Bernard^{f0}$ and $Claude^{f0}$: the first names Anne, Bernard, and Claude denote constants,
- ▶ a^{r2} : letter a denotes a two-argument relation (we read $a(x, y)$ as x likes y) and
- ▶ c^{f1} : the letter c denotes a one-argument function (we read $c(x)$ as the friend of x).

Let I the interpretation of domain $D = \{0, 1, 2\}$ over this signature where :

- ▶ $Anne_I^{f0} = 0$, $Bernard_I^{f0} = 1$, and $Claude_I^{f0} = 2$.
- ▶ $a_I^{r2} = \{(0, 1), (1, 0), (2, 0)\}$.
- ▶ $c_I^{f1}(0) = 1$, $c_I^{f1}(1) = 0$, $c_I^{f1}(2) = 2$. Note that the domain of any function is D . In particular, function c_I^{f1} is defined everywhere, which makes it necessary to artificially define $c_I^{f1}(2)$ even if Claude, denoted by 2, has no friend.

Example 4.3.29

We obtain :

$$\blacktriangleright [a(\textit{Anne}, \textit{Bernard})]_I =$$

Example 4.3.29

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▶ $[a(\textit{Anne}, \textit{Bernard})]_I =$

1 since $(\llbracket \textit{Anne} \rrbracket_I, \llbracket \textit{Bernard} \rrbracket_I) = (0, 1) \in a_I^{r^2}$.
--

▶ $[a(\textit{Anne}, \textit{Claude})]_I =$

Example 4.3.29

We obtain :

▶ $[a(\textit{Anne}, \textit{Bernard})]_I =$

1 since $(\llbracket \textit{Anne} \rrbracket_I, \llbracket \textit{Bernard} \rrbracket_I) = (0, 1) \in a_I^{r2}$.

▶ $[a(\textit{Anne}, \textit{Claude})]_I =$

0 since $(\llbracket \textit{Anne} \rrbracket_I, \llbracket \textit{Claude} \rrbracket_I) = (0, 2) \notin a_I^{r2}$.

Example 4.3.29

Let e the state $x = 0, y = 2$. We have :

$$\blacktriangleright [a(x, c(x))]_{(I, e)} =$$

Example 4.3.29

Let e the state $x = 0, y = 2$. We have :

► $[a(x, c(x))]_{(l, e)} =$

1 since

$$([\![x]\!]_{(l, e)}, [\![c(x)]\!]_{(l, e)}) = (0, c_l^{f1}([\![x]\!]_{(l, e)})) = (0, c_l^{f1}(0)) = (0, 1) \in a_l^{r2}.$$

► $[a(y, c(y))]_{(l, e)} =$

Example 4.3.29

Let e the state $x = 0, y = 2$. We have :

► $[a(x, c(x))]_{(l, e)} =$

1 since
 $(\llbracket x \rrbracket_{(l, e)}, \llbracket c(x) \rrbracket_{(l, e)}) = (0, c_l^{f1}(\llbracket x \rrbracket_{(l, e)})) = (0, c_l^{f1}(0)) = (0, 1) \in a_l^{r2}.$

► $[a(y, c(y))]_{(l, e)} =$

0 since
 $(\llbracket y \rrbracket_{(l, e)}, \llbracket c(y) \rrbracket_{(l, e)}) = (2, c_l^{f1}(\llbracket y \rrbracket_{(l, e)})) = (2, c_l^{f1}(2)) = (2, 2) \notin a_l^{r2}.$

Make sure to distinguish (depending on the context), the elements of the domain 0, 1 and the truth values 0, 1.

Example 4.3.29

We have :

$$\blacktriangleright [(Anne = Bernard)]_I =$$

Example 4.3.29

We have :

$$\blacktriangleright \llbracket (Anne = Bernard) \rrbracket_I =$$

0, since $(\llbracket Anne \rrbracket_I, \llbracket Bernard \rrbracket_I) = (0, 1) \notin =_I^{r^2}$.

$$\blacktriangleright \llbracket (c(Anne) = Anne) \rrbracket_I =$$

Example 4.3.29

We have :

▶ $[(Anne = Bernard)]_I =$

0, since $(\llbracket Anne \rrbracket_I, \llbracket Bernard \rrbracket_I) = (0, 1) \notin =_I^{r^2}$.

▶ $[(c(Anne) = Anne)]_I =$

0, since

$(\llbracket c(Anne) \rrbracket_I, \llbracket Anne \rrbracket_I) = (c_I^{f_1}(\llbracket Anne \rrbracket_I), 0) = (c_I^{f_1}(0), 0) = (1, 0) \notin =_I^{r^2}$.

▶ $[(c(c(Anne)) = Anne)]_I =$

Example 4.3.29

We have :

- ▶ $[(Anne = Bernard)]_I =$

0, since $(\llbracket Anne \rrbracket_I, \llbracket Bernard \rrbracket_I) = (0, 1) \notin =_I^{r^2}$.

- ▶ $[(c(Anne) = Anne)]_I =$

0, since

$(\llbracket c(Anne) \rrbracket_I, \llbracket Anne \rrbracket_I) = (c_I^{f^1}(\llbracket Anne \rrbracket_I), 0) = (c_I^{f^1}(0), 0) = (1, 0) \notin =_I^{r^2}$.

- ▶ $[(c(c(Anne)) = Anne)]_I =$

1, since $(\llbracket c(c(Anne)) \rrbracket_I, \llbracket Anne \rrbracket_I) = (c_I^{f^1}(\llbracket c(Anne) \rrbracket_I), 0) = (c_I^{f^1}(c_I^{f^1}(0)), 0) = (c_I^{f^1}(1), 0) = (0, 0) \in =_I^{r^2}$.

Truth value of a formula 4.3.30

1. Propositional connectives have the same meaning as in propositional logic.
2. Let x a variable and B a formula. $[\forall x B]_{(l,e)} = 1$ if and only if $[B]_{(l,f)} = 1$ for all state f identical to e , except for x . Let $d \in D$. Let us denote $e[x = d]$ the state identical to the e , except for the variable x , whose state $e[x = d]$ associates the value d . The above definition can be written as :

$$[\forall x B]_{(l,e)} = \min_{d \in D} [B]_{(l,e[x=d])} = \prod_{d \in D} [B]_{(l,e[x=d])},$$

where the product is the boolean product.

3. $[\exists x B]_{(l,e)} = 1$ if and only if there is a state f identical to e , except for x , such that $[B]_{(l,f)} = 1$. The above definition can be written as :

$$[\exists x B]_{(l,e)} = \max_{d \in D} [B]_{(l,e[x=d])} = \sum_{d \in D} [B]_{(l,e[x=d])},$$

where the sum is the boolean sum.

Example 4.3.32

Let us use the interpretation I given in example 4.3.19.

(Reminder $D = \{0, 1, 2\}$)

▶ $[\exists x a(x, x)]_I =$

Example 4.3.32

Let us use the interpretation I given in example 4.3.19.

(Reminder $D = \{0, 1, 2\}$)

▶ $[\exists x a(x, x)]_I =$

$\max\{[a(0, 0)]_I, [a(1, 1)]_I, [a(2, 2)]_I\} = 0$ since $(0, 0), (1, 1), (2, 2) \notin a_I^2$.

According to the definition, we have : $[\exists x a(x, x)]_I =$

$[a(0, 0)]_I + [a(1, 1)]_I + [a(2, 2)]_I = 0 + 0 + 0 = 0.$

▶ $[\forall x \exists y a(x, y)]_I =$

Example 4.3.32

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According to the definition, we have : $[\exists x a(x, x)]_I =$

$[a(0, 0)]_I + [a(1, 1)]_I + [a(2, 2)]_I = 0 + 0 + 0 = 0.$

► $[\forall x \exists y a(x, y)]_I =$

$\min\{\max\{[a(0, 0)]_I, [a(0, 1)]_I, [a(0, 2)]_I\}, \max\{[a(1, 0)]_I, [a(1, 1)]_I, [a(1, 2)]_I\}, \max\{[a(2, 0)]_I, [a(2, 1)]_I, [a(2, 2)]_I\}\} = \min\{\max\{0, 1, 0\}, \max\{1, 0, 0\}, \max\{1, 0, 0\}\} = \min\{1, 1, 1\} = 1.$

According to the definition, we have : $[\forall x \exists y a(x, y)]_I =$

$([a(0, 0)]_I + [a(0, 1)]_I + [a(0, 2)]_I) \cdot ([a(1, 0)]_I + [a(1, 1)]_I + [a(1, 2)]_I) \cdot$

$([a(2, 0)]_I + [a(2, 1)]_I + [a(2, 2)]_I) =$

$(0 + 1 + 0) \cdot (1 + 0 + 0) \cdot (1 + 0 + 0) = 1 \cdot 1 \cdot 1 = 1.$

Example 4.3.32

► $[\exists y \forall x a(x, y)]_I =$

Example 4.3.32

► $[\exists y \forall x a(x, y)]_I =$

$$\max\{\min\{[a(0,0)]_I, [a(1,0)]_I, [a(2,0)]_I\}, \min\{[a(0,1)]_I, [a(1,1)]_I, [a(2,1)]_I\}, \min\{[a(0,2)]_I, [a(1,2)]_I, [a(2,2)]_I\}\} = \max\{\min\{0, 1, 1\}, \min\{1, 0, 0\}, \min\{0, 0, 0\}\} = \max\{0, 0, 0\} = 0.$$

According to the definition, we have : $[\exists y \forall x a(x, y)]_I =$
 $[a(0,0)]_I \cdot [a(1,0)]_I \cdot [a(2,0)]_I + [a(0,1)]_I \cdot [a(1,1)]_I \cdot [a(2,1)]_I + [a(0,2)]_I \cdot [a(1,2)]_I \cdot [a(2,2)]_I = 0 \cdot 1 \cdot 1 + 1 \cdot 0 \cdot 0 + 0 \cdot 0 \cdot 0 = 0 + 0 + 0 = 0.$

Example 4.3.32

► $[\exists y \forall x a(x, y)]_I =$

$$\max\{\min\{[a(0,0)]_I, [a(1,0)]_I, [a(2,0)]_I\}, \min\{[a(0,1)]_I, [a(1,1)]_I, [a(2,1)]_I\}, \min\{[a(0,2)]_I, [a(1,2)]_I, [a(2,2)]_I\}\} = \max\{\min\{0, 1, 1\}, \min\{1, 0, 0\}, \min\{0, 0, 0\}\} = \max\{0, 0, 0\} = 0.$$

According to the definition, we have : $[\exists y \forall x a(x, y)]_I =$
 $[a(0,0)]_I \cdot [a(1,0)]_I \cdot [a(2,0)]_I + [a(0,1)]_I \cdot [a(1,1)]_I \cdot [a(2,1)]_I + [a(0,2)]_I \cdot [a(1,2)]_I \cdot [a(2,2)]_I = 0 \cdot 1 \cdot 1 + 1 \cdot 0 \cdot 0 + 0 \cdot 0 \cdot 0 = 0 + 0 + 0 = 0.$

Remark 4.3.33

The formulae $\forall x \exists y a(x, y)$ and $\exists y \forall x a(x, y)$ do not have the same value. Exchanging an existential quantification and an universal quantification does not preserve the truth value of a formula.

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Conclusion : Next course

- ▶ Interpret a first order formula (contd.)
- ▶ Important equivalences

Conclusion

Thank you for your attention.

Questions ?