Feedback Control I - PID Control

Course Feedback Control and Real-time SystemsI

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Overview

Recall

- Autonomous Linear Dynamical Systems
- Some Linear Algebra

Feedback Control

Autonomous Linear Dynamical Systems

Linear ordinary differential equation (ODE)

$$\dot{x} = Ax$$

with **initial state** $x(0) = x_{\text{ini}}$.

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \qquad A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix}$$

Solution (general form): Recall

Linear ordinary differential equation (ODE)

$$\dot{x} = Ax$$

with **initial state** $x(0) = x_{\text{ini}}$ has the solution

$$x(t) = e^{At} x_{\text{ini}},$$

$$e^{At} = \sum_{k=0}^{\infty} A^k \frac{t^k}{k!} = I + At + A^2 \frac{t^2}{2} + \cdot$$

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Determinant of a Matrix

matrix determinant

$$|M| = \sum_{(i_1,\ldots,i_n)} \pm a_{i_1} a_{i_2} \cdot a_{i_n}$$

where $(i_1, \ldots, i_n)$ are permutations of $1, \cdots, n$ and $\pm$ is $+$ $(-)$ for even (odd) permutations

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

- $|M| = |M^T|$
- if M (upper or lower) triangular, $|M| = m_{11} m_{22} \cdots m_{nn}$

Eigenvalues of a Matrix

characteristic polynomial of A:

$$|A - \lambda I| = 0$$

characteristic polynomial has n roots, called **Eigenvalues**

- EV of A and of A^T are identical
- if A is (upper or lower) triangular, its EV are $a_{11}, \ldots, a_{nn}$

Exercise: Stability

Compute the Eigenvalues to assess the stability of

1. coasting car
$$\begin{bmatrix} 0 & 1 \\ 0 & -\frac{\beta}{m} \end{bmatrix}$$
2. accelerating car
$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$
3. ideal pendulum
$$\begin{bmatrix} 0 & 1 \\ -\frac{g}{L} & 0 \end{bmatrix}$$
4. pendulum with air frinction
$$\begin{bmatrix} 0 & 1 \\ -\frac{g}{L} & -\frac{\beta}{m} \end{bmatrix}$$

Exercise: Stability

What happens to an accelerating car with air friction?

$$m\ddot{x}_1 = -\beta\dot{x}_1 + x_3$$

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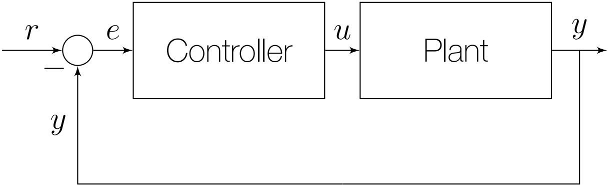
Feedback Control

- Linear Time-Invariant (LTI) Systems
- Proportional Feedback
- PD- and PI-Control
- PID Control

Feedback Control



- reference input r, here **for simplicity** $r = 0$
- reference error $e = r - y$
- control input u
- measurement y

Linear Control Systems

Linear time-invariant (LTI) system with inputs $u = \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}$,

outputs $y = \begin{bmatrix} y_1 \\ \vdots \\ y_l \end{bmatrix}$

$$\dot{x} = \underbrace{A}_{n \times n} x + \underbrace{B}_{n \times m} u$$
$$y = \underbrace{C}_{l \times n} x$$

typically $m \ll n$, $l \ll n$

Solution

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

solution from $x(0) = x_{\text{ini}}$ for given input function $u(t)$:

$$x(t) = e^{At}x_{\text{ini}} + e^{At}\int_{\tau=0}^{t} e^{-A\tau}Bu(\tau)d\tau$$
$$y(t) = Cx(t)$$

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Proportional Feedback

$$\dot{x} = Ax + Bu \qquad \boldsymbol{u = K_p e}$$

$$y = Cx \qquad e = r - y$$

closed-loop LTI system $\dot{x} = \tilde{A}x$ for $r = 0$:

$$u = K_p e = K_p(-y) = K_p(-Cx) = -\underbrace{K_p C}_{\tilde{K}} x$$

$$\begin{aligned} \dot{x} &= Ax + B(-\tilde{K}x) \\ &= (\underbrace{A - B\tilde{K}}_{\tilde{A}})x \end{aligned}$$

Pole Placement

Choose K_p to get suitable Eigenvalues $\lambda_i = a_i \pm \mathrm{i} b_i$ of

$$\tilde{A} = A - BK_pC$$

- stable: $a_i < 0$
- fast convergence: $a_i \ll 0$
- damping: $b_i < |a_i|$

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PD-Feedback

$$\dot{x} = Ax + Bu \qquad \boldsymbol{u = K_p e + K_d \dot{e}}$$

$$y = Cx \qquad e = r - y$$

closed-loop LTI system $\dot{x} = \tilde{A}x$ for $r = 0$:

$$\dot{e} = -\dot{y} = -C\dot{x} = -CAx - CBu$$

$$\begin{aligned} u &= -K_p Cx + K_d(-CAx - CBu) \\ (I + K_d CB)u &= -(K_p + K_d CA)x \\ u &= -\underbrace{(I + K_d CB)^{-1}(K_p C + K_d CA)}_{\tilde{K}}x \end{aligned}$$

$$\dot{x} = (\underbrace{A - B\tilde{K}}_{\tilde{A}})x$$

PI-Feedback

$$\dot{x} = Ax + Bu \qquad \boldsymbol{u = K_p e + K_i \int_0^t e(\tau) d\tau}$$

$$y = Cx \qquad e = r - y$$

closed-loop LTI system with variable $u_i = K_i \int_0^t e(\tau) d\tau$

$$\dot{u}_i = K_i e = -K_i y = -K_i C x$$

$$\dot{x} = Ax + B(K_p e + u_i) = (A - BK_p C)x + Bu_i$$

$$\begin{bmatrix} \dot{x} \\ \dot{u}_i \end{bmatrix} = \begin{bmatrix} A - BK_p C & B \\ -K_i C & 0 \end{bmatrix} \begin{bmatrix} x \\ u_i \end{bmatrix}$$

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PID-Feedback

$$\dot{x} = Ax + Bu \qquad \boldsymbol{u = K_p e + K_i \int_0^t e(\tau) d\tau + K_d \dot{e}}$$

$$y = Cx \qquad e = r - y$$

closed-loop LTI system with $\dot{u}_i = -K_i C x$

$$u = -K_p C x + u_i - K_d C A x - K_d C B u$$

$$(I + K_d C B) u = -(K_p C + K_d C A) x + u_i$$

$$u = -\underbrace{(I + K_d C B)^{-1} (K_p C + K_d C A)}_{\tilde{K}_x} x + \underbrace{(I + K_d C B)^{-1}}_{\tilde{K}_i} u_i$$

$$\begin{bmatrix} \dot{x} \\ \dot{u}_i \end{bmatrix} = \begin{bmatrix} A - B\tilde{K}_x & B\tilde{K}_i \\ -K_i C & 0 \end{bmatrix} \begin{bmatrix} x \\ u_i \end{bmatrix}$$

PID-Control

PID controllers:

- most widespread control type in industry
- typically scalar feedback ($m = l = 1$)

tuned manually (difficult) or automatically

Ziegler-Nichols Tuning:

1. $K_i = K_d = 0$, increase K_p until oscillation, record $K_u := K_p$ period T_u
2. Set $K_p = 0.6 K_u$, $K_i = 1.2 K_u / T_u$, $K_d = 3 K_u T_u / 40$