Data Refinement: model-oriented proof methods and their comparison

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Overview

- Refinement
- Data refinement
- Simulation
- Equivalence between assertional and relational characterizations of downward simulation
- Sound and relatively complete proof system for a minimal Hoare logic
- Theorems: Reynolds' method, VDM reduced to downward simulation for total correctness

Questions answered in this talk

- What is a (data) refinement step?
- How to find and prove such a step?
- How to judge the solutions given by others?

Refinement (1)

> Given a pair of programs called concrete and abstract, the concrete program refines the abstract program correctly whenever the use of the concrete program does not lead to an observation which is not also an observation of the abstract program. [Gardiner & Morgan, 1993]

So, what is observable?

Refinement (2)

So, what is observable?

In our setting of sequential, imperative programs, only the binary relation between initial and final states is considered observable.

Given a class $Prog$ of programs and a function

$$\mathcal{P}[\![.]\!] : Prog \rightarrow 2^{\Sigma \times \Sigma}$$

that maps each program to its initial/final state relation, program $S \in Prog$ *refines* $T \in Prog$ is defined by

$$\mathcal{P}[\![S]\!] \subseteq \mathcal{P}[\![T]\!],$$

abbreviated to

$$S \subseteq T.$$

Refinement (3)

Example 1 Let S_1 and S_2 denote statements not involving variables s and l. Compare the following two programs; they refine each other.

```
begin
    var $s$ : finset of $\mathbb{N}$ ; $s := \emptyset$;
    $S_1$;
    $s := s \cup \{x\}$;
    $S_2$;
    $y :=$ a member of $s$
end
```

```
begin
    var $l : \mathbb{N}^*$ ; $l :=$ nil;
    $S_1$;
    $l :=$ append$(l, x)$;
    $S_2$;
    $y :=$ first$(l)$
end
```

This refinement step comprises of replacing the variable s (ranging over finite subsets of the natural numbers) and operations on it by the sequence-valued variable l and corresponding operations.

Refinement (4)

Initial/final state behaviour of S_1 and S_2 in terms of value-transformations of x, y are global w.r.t. S_1 and S_2:
x and y are called normal variables.

In contrast s, t are data-representation variables. Their values are only visible *inside* S_1 and S_2, because these variables vary according to the **abstraction** level.

Representation variables are not observable outside a program.

Data types

How to formalize the interesting part of two programs such as those in the example from the refinement point of view?

Definition 1 [data type] Given a finite set of variables $\bar{x}$, called normal variables, another (disjoint) finite set of variables $\bar{a}$, called representation variables, and a finite index set J, define state spaces Σ and Σ^{A} by $\Sigma \overset{\text{def}}{=} [\bar{x} \to \mathbb{V}]$ and $\Sigma^{\mathrm{A}} \overset{\text{def}}{=} [\bar{x} \cup \bar{a} \to \mathbb{V}]$. Let $A_j \subseteq \Sigma^{\mathrm{A}} \times \Sigma^{\mathrm{A}}$ for $j \in J$. Let initialization $AI \subseteq \Sigma \times \Sigma^{\mathrm{A}}$, and finalization $AF \subseteq \Sigma^{\mathrm{A}} \times \Sigma$. Then we call

$$\mathcal{A} = (AI, (A_j)_{j \in J}, AF)$$

a data type.

Note relational characterization of $\mathcal{A}$: $A_j \subseteq \Sigma^{\mathrm{A}} \times \Sigma^{\mathrm{A}}$.

Program Skeletons

A program skeleton maps each data type to a relation constructed from the operations A_j and operations on the normal variables using sequential composition, non-deterministic choice and recursion.

Example 2 $P(\mathcal{A}) = A_1 \,;\, A_2 \cup A_3$ and $P(\mathcal{C}) = C_1 \,;\, C_2 \cup C_3$.

Obviously, there are infinitely many program skeletons (unless $J = \emptyset$).

Data refinement (1)

Compare two levels of abstraction:

A ———————————————— of data type $\mathcal{A}$

C ———————————————— of data type $\mathcal{C}$

with $\mathcal{A}$ and $\mathcal{C}$ compatible (index sets J plus set $\bar{x}$ of normal variables the same).

$\mathcal{C}$ should refine/implement $\mathcal{A}$.

As mentioned before, the data type representation variables (e.g., s and l) themselves are NOT observable. $\Rightarrow$

When defining that $\mathcal{C}$ refines $\mathcal{A}$, the particular way a data type representation is defined should, therefore, not be observable:

Data refinement (2)

When defining that $\mathcal{C}$ refines $\mathcal{A}$, the particular way a data type representation is defined should, therefore, not be observable:

$$\underbrace{CI\ ;\ \ldots\ ;\ CF}_{\substack{CI,\ CF \text{ hide the transformation} \\ \text{of } \bar{c} \text{ by } \{C_j\}_{j \in J}}} \quad \subseteq \quad \underbrace{AI\ ;\ \ldots\ ;\ AF}_{\substack{AI,\ AF \text{ hide the transformation} \\ \text{of } \bar{a} \text{ by } \{A_j\}_{j \in J}}}$$

Moreover, the fact that one data type refines another should hold for all program skeletons using those data types:

$$CI\ ;\ P(\mathcal{C})\ ;\ CF \subseteq AI\ ;\ P(\mathcal{A})\ ;\ AF,$$

for all program skeletons P concerned. $\Rightarrow$
This involves proving infinitely many proof obligations.

Data refinement (3)

Definition 2 Data type $\mathcal{C} = (CI, (C_j)_{j \in J}, CF)$ *refines* data type $\mathcal{A} = (AI, (A_j)_{j \in J}, AF)$ iff, for all program skeletons P:

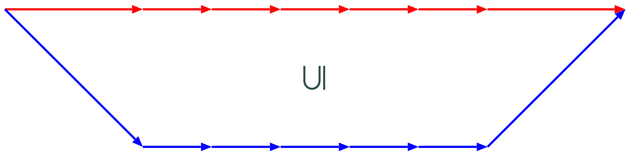
$$CI \; ; P(\mathcal{C}) \; ; CF \subseteq AI \; ; P(\mathcal{A}) \; ; AF$$

Technical note: $\mathcal{C}$ uses $\bar{c}$ (disjoint from $\bar{x}$ and $\bar{a}$) and $\Sigma^{\mathrm{C}} = [\bar{x} \cup \bar{c} \longrightarrow \mathbb{V}]$ instead of $\bar{a}$ and Σ^{A}. Moreover, $\mathcal{C}$ and $\mathcal{A}$ use the same index set J. I.e., $\mathcal{C}$ and $\mathcal{A}$ are *compatible*.

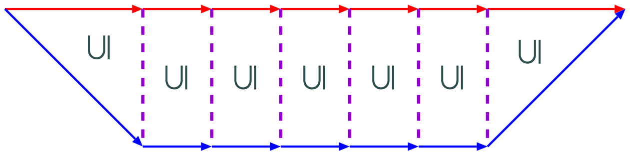
Hence, in order to prove data refinement, one has to prove **infinitely many proof obligations**.

Why simulation?

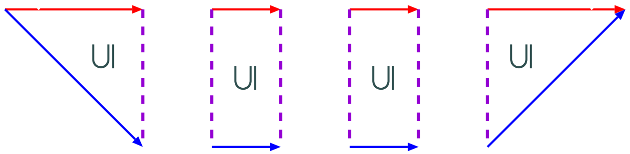
Instead of proving infinitely many proof obligations such as



directly, one would like to use induction. This requires invention of a relationship ρ between abstract and concrete level representation.



To focus on (the finite number of) base cases



one has to guarantee that *induction steps are for free*.

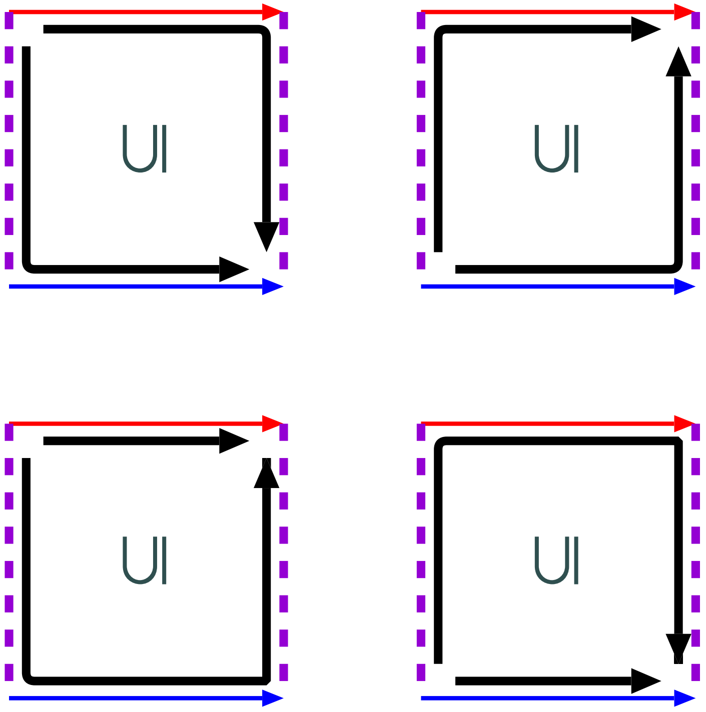
Local conditions for simulation (1)

Consider a relation $\rho \subseteq \Sigma^{\mathrm{A}} \times \Sigma^{\mathrm{C}}$ between abstract and concrete states. Then there are essentially four ways in which weak commutativity of diagram

$$\begin{tikzcd}
{} \arrow[r, "A_j"] \arrow[d, dash, dashed, "\rho"'] \arrow[dr, phantom, "\rotatebox[origin=c]{90}{$\subseteq$}"] & {} \arrow[d, dash, dashed, "\rho"] \\
{} \arrow[r, "C_j"'] & {}
\end{tikzcd}$$

can be defined, possibly using inverses of ρ.

Local conditions for simulation (2)

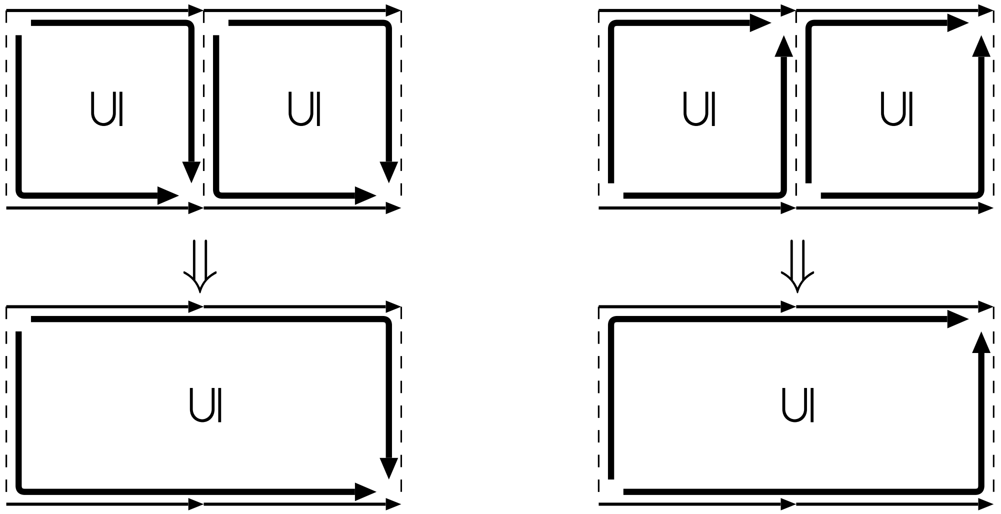


The induction step for sequential composition is free only for the first two, called downward and upward simulation, resp.

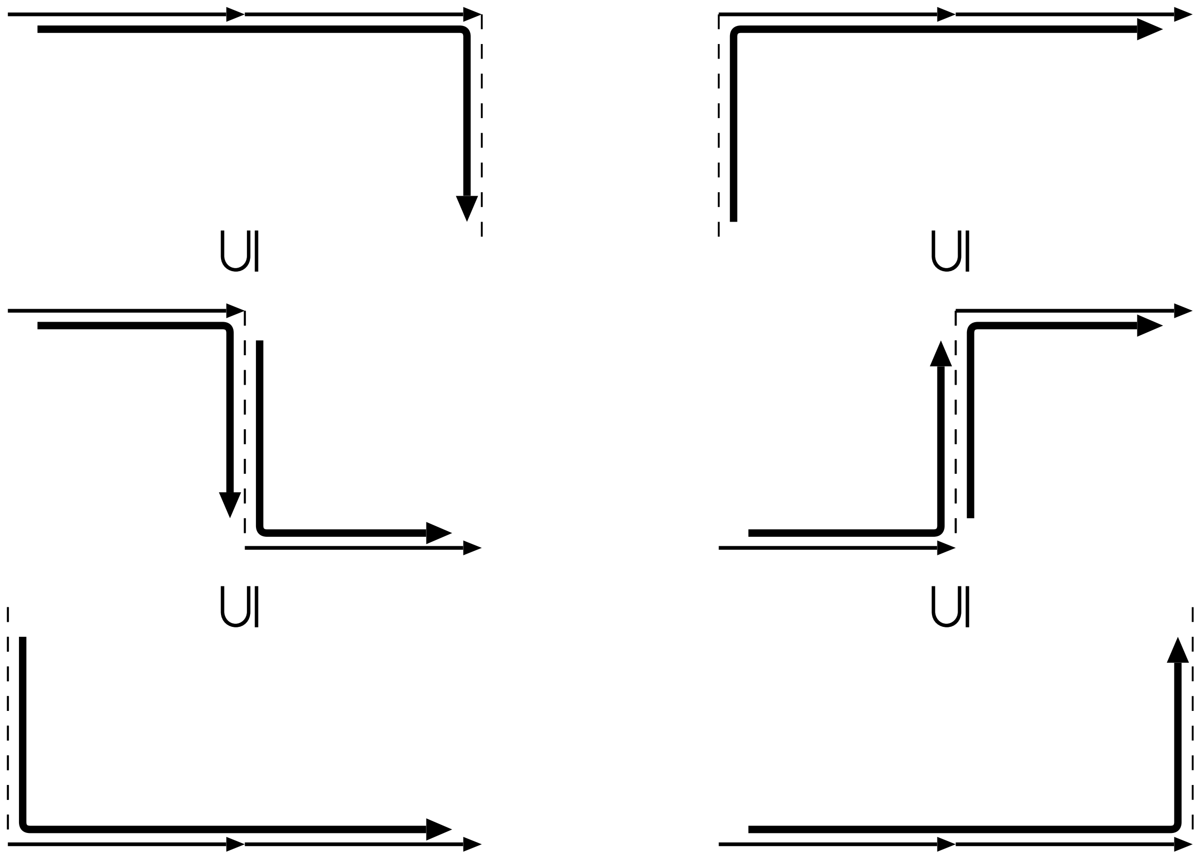
Technical note: The conditions for initialization and finalization are obtained by “identifying” either of the RHS/LHS pairs of corners in the diagrams above.

Soundness of simulation

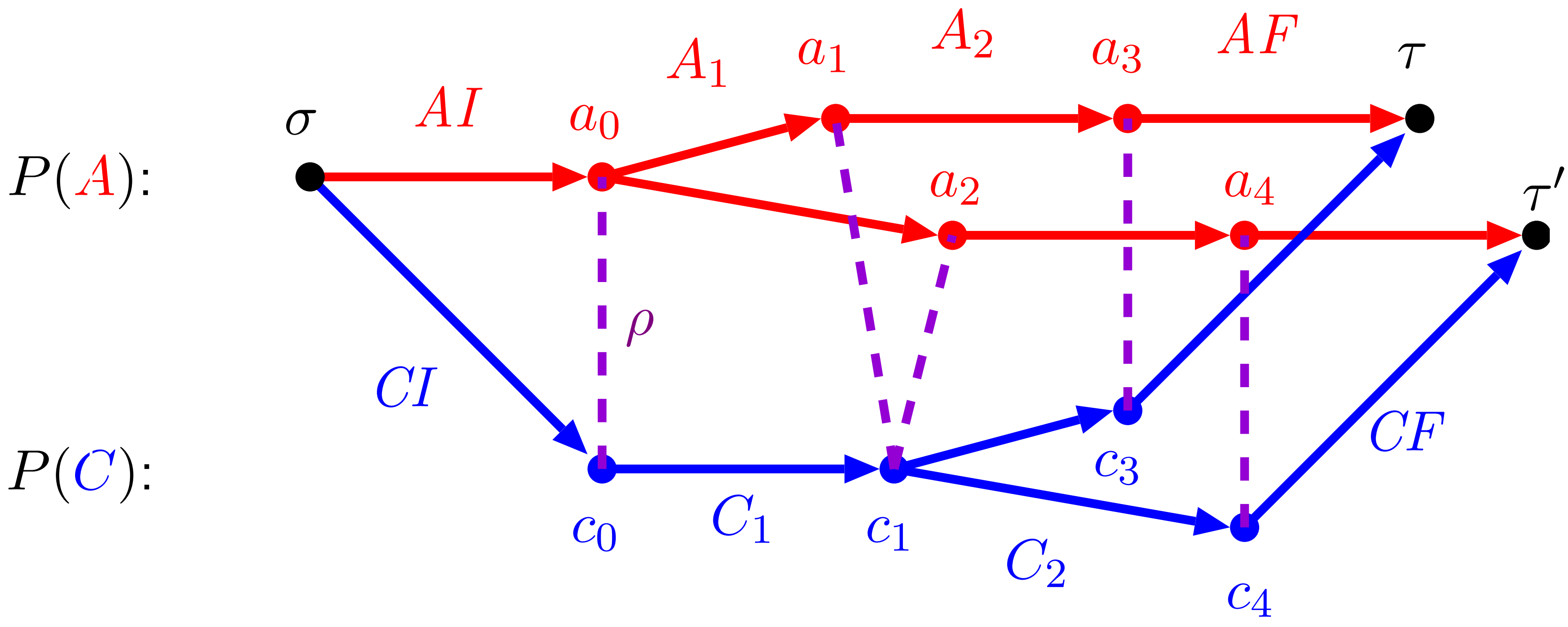
Both downward and upward simulation are *sound* techniques for proving data refinement. The induction steps for sequential composition look as follows.



Proofs



Incompleteness of downward simulation (1)



Incompleteness of downward simulation (2)

Assume ρ is a downward simulation relation between $(AI, (A_j)_{j \in \{1,2\}}, AF)$ and $(CI, (C_j)_{j \in \{1,2\}}, CF)$ where the relations in question are those depicted above.

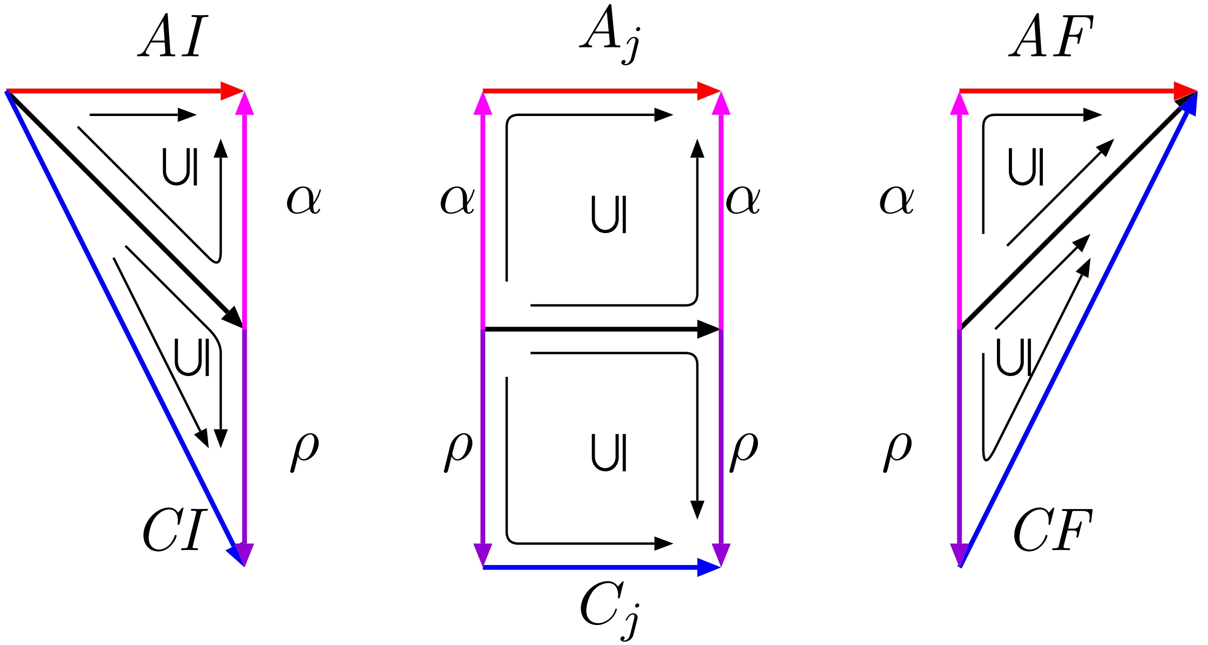
1. $CI \subseteq AI \,; \rho$, thus, $(a_0, c_0) \in \rho$.
2. $\rho \,; C_1 \subseteq A_1 \,; \rho$, thus, one of (a_1, c_1) and (a_2, c_1) is in ρ.
   W.l.o.g. assume that $(a_1, c_1) \in \rho$.
3. $\rho \,; C_2 \subseteq A_2 \,; \rho$, thus, $(a_3, c_4) \in \rho$.
4. $\rho \,; CF \subseteq AF$, which implies, that $(a_3, \tau') \in AF$,
   however, CF is only $\{(c_3, \tau), (c_4, \tau')\}$! **Contradiction!**

Completeness

The combination of downward and upward simulation is complete for proving refinement between data types.

Theorem 1 [HHS] If $\mathcal{C}$ refines $\mathcal{A}$ then there exist

- an intermediate data type $\mathcal{B}$,
- a downward simulation relation ρ between $\mathcal{B}$ and $\mathcal{C}$, and
- an upward simulation relation α between $\mathcal{B}$ and $\mathcal{A}$.



What’s out there?

Numerous (formal) methods exist for writing specifications and refining those to implementations:

- VDM (Raise, Z, B)
- Reynolds’ method
- Refinement Calculi of Back & von Wright, Gardiner & Morgan, Morris
- Hehner’s method
- Abadi & Lamport’s refinement mappings
- Lynch’s possibilities mappings

major development technique: stepwise refinement

All these methods are proved to be related in the Data Refinement book by Kai Engelhardt and me.

Key problem

- The soundness and completeness results of [HHS86] reduce the task of proving data refinement to:

  Proving that

  $$\begin{tikzcd}
  {} \arrow[r, "A"] \arrow[d, dashed, "\rho"'] \arrow[dr, phantom, "\rotatebox{90}{$\subseteq$}"] & {} \arrow[d, dashed, "\rho"] \\
  {} \arrow[r, "C"'] & {}
  \end{tikzcd}$$

  $\left.\begin{array}{r}\text{downwards}\\ \text{upwards}\end{array}\right\}$ **simulates**

  So we have to prove **inclusion between relations** ($\rho\,;C \subseteq A\,;\rho$ and $C\,;\rho^{-1} \subseteq \rho^{-1}\,;A$).

  I.e., we have a **relational characterization** of simulation.

- This relational characterization we want to compare with methods which use **assertional characterizations** of operations and simulation (Hoare logics, VDM, Reynolds, refinement calculi).

- **Key problem: How to relate these two characterizations?**

Assertional vs. relational characterizations of an operation

Assertional methods characterize operations by first-order logic assertions called pre- and postconditions. Questions:

1: Given an assertional characterization of an operation, which relation is determined by it?

2: Given a relational characterization of an operation: can this operation be expressed using pre- and postconditions?

Ad 2: Solved affirmatively in [Zwiers '89, LNCS 321] on the basis of recursion theory.

Ad 1: Solved using Galois connections as developed below.

Hoare formulae

Use Hoare formulae $\underset{\text{predicate}}{\{\varphi\}} \quad \underset{\text{operation}}{S} \quad \underset{\text{predicate}}{\{\psi\}}$ to specify operations, meaning:

$\{\varphi\}\, S\, \{\psi\}$ is valid (holds) iff

- *if* φ holds in initial state σ, *and if* S terminates for initial state σ in final state τ *then* ψ holds in τ.

Notation: $\models \{\varphi\}\, S\, \{\psi\}$ (validity)

Logical variables

- Specifying operations by Hoare formulae introduces the need for logical variables v, i.e., variables v whose values are not changed during program execution:

  $$\{x = v\}\ x := x + 1\ \{x = v + 1\},$$

  because, otherwise, no single axiom for $x := x + 1$.

- Leads to introduction of set $Logvar$ of logical variables disjoint from VAR, the set of program variables, and to logical variable states $\Gamma \overset{\mathsf{def}}{=} Logvar \to VAL$, $\gamma \in \Gamma$.

First connection

Using logical variable states, the meaning of $\models \{\varphi\}\, S\, \{\psi\}$ is:

$$\forall \sigma, \tau \in \Sigma. \forall \gamma \in \Gamma. (\gamma, \sigma) \in \mathcal{C}[\![\varphi]\!] \wedge (\sigma, \tau) \in \mathcal{P}[\![S]\!] \Rightarrow (\gamma, \tau) \in \mathcal{C}[\![\psi]\!],$$

with the meaning of assertions given by a relation between logical states and program states:

$$\mathcal{C}[\![\varphi]\!] \subseteq \Gamma \times \Sigma$$

and the meaning of operation S as a relation between program states:

$$\mathcal{P}[\![S]\!] \subseteq \Sigma \times \Sigma$$

This implies: $\models \{\varphi\}\, S\, \{\psi\} \Leftrightarrow\ \models \varphi \,;\, S \subseteq \psi$

using $r_1 \,;\, r_2 \overset{\text{def}}{=} \{(\sigma, \tau) \mid \exists \theta. (\sigma, \theta) \in r_1 \wedge (\theta, \tau) \in r_2\}$.

Second connection

- When operation op is specified by $\{\varphi\}\ op\ \{\psi\}$, we interpret op as the maximal relation r satisfying $\mathcal{C}[\![\varphi]\!]\ ;r \subseteq \mathcal{C}[\![\psi]\!]$.
- This max. relation is expressed by specification statement $\varphi \rightsquigarrow \psi$:
  $\mathcal{P}[\![\varphi \rightsquigarrow \psi]\!] \overset{\text{def}}{=} \{(\sigma, \tau) \mid \forall\gamma \in \Gamma.(\gamma, \sigma) \in \mathcal{C}[\![\varphi]\!] \Rightarrow (\gamma, \tau) \in \mathcal{C}[\![\psi]\!]\}$
- Since $\forall\sigma, \tau.\forall\gamma((\gamma, \sigma) \in \mathcal{C}[\![\varphi]\!] \wedge (\sigma, \tau) \in \mathcal{P}[\![S]\!] \Rightarrow (\gamma, \tau) \in \mathcal{C}[\![\psi]\!])$
  $\Leftrightarrow \forall\sigma, \tau.(\sigma, \tau) \in \mathcal{P}[\![S]\!] \Rightarrow (\forall\gamma.(\gamma, \sigma) \in \mathcal{C}[\![\varphi]\!] \Rightarrow (\gamma, \tau) \in \mathcal{C}[\![\psi]\!])$
  $\Leftrightarrow \qquad \mathcal{P}[\![S]\!] \subseteq \mathcal{P}[\![\varphi \rightsquigarrow \psi]\!]$,
  one obtains
  $$\{\varphi\}\, S\, \{\psi\} \Leftrightarrow\ \models S \subseteq \varphi \rightsquigarrow \psi.$$
- This clarifies why we interpret op as maximal relation:
  We do not want to restrict any refinement S of op unnecessarily.

Third connection

Let for $s \subseteq A \times C$ and $t \subseteq B \times C$

$$[t]s \overset{\text{def}}{=} \{(a,b) \in A \times B \,|\, \forall c \in C.(b,c) \in t \Rightarrow (a,c) \in s\}$$

then $\models \{\varphi\}\, S\, \{\psi\} \Leftrightarrow\ \models \varphi \subseteq [S]\psi$

Proof: $\forall \sigma \tau . \forall \gamma . (\gamma, \sigma) \in \mathcal{C}[\![\varphi]\!] \wedge (\sigma, \tau) \in \mathcal{P}[\![S]\!] \Rightarrow (\gamma, \tau) \in \mathcal{C}[\![\psi]\!]$

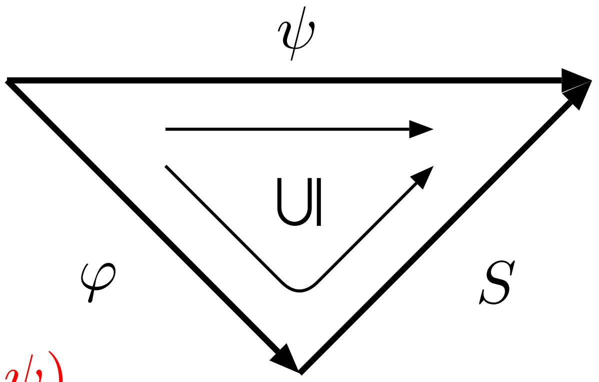
$$\Leftrightarrow \underbrace{\forall \gamma, \sigma . (\gamma, \sigma) \in \mathcal{C}[\![\varphi]\!]} \Rightarrow \underbrace{(\forall \tau . (\sigma, \tau) \in \mathcal{P}[\![S]\!] \Rightarrow (\gamma, \tau) \in \mathcal{C}[\![\psi]\!])}$$

$$\Leftrightarrow \forall \gamma, \sigma . (\gamma, \sigma) \in \mathcal{C}[\![\varphi]\!] \Rightarrow \qquad (\gamma, \sigma) \in \mathcal{C}[\![[S]\psi]\!]$$

$$\Leftrightarrow \mathcal{C}[\![\varphi]\!] \subseteq \mathcal{C}[\![[S]\psi]\!]$$

QED

Galois connection



Express maximal solutions for each of the relations on the LHS of $\varphi\,;S \subseteq \psi$ in terms of the remaining two relations.

$$S \subseteq \varphi \rightsquigarrow \psi \quad \Leftrightarrow \quad \varphi\,;S \subseteq \psi \quad (\Leftrightarrow \quad \varphi \subseteq [S]\,\psi)$$

It depends on the program semantics chosen whether total or partial correctness is expressed; the equivalence of these inclusions holds in both cases.

This Galois connection is our main technical tool in relating relational to assertional characterizations of operations.

Assertional characterization of simulation

Problem: How to characterize the maximal relation C ?-simulating $\varphi \rightsquigarrow \psi$ under abstraction relation ρ as a specification statement:

$$\begin{tikzcd}
{} \arrow[r, "\varphi \rightsquigarrow \psi"] \arrow[d, dashed, "\rho"'] \arrow[dr, phantom, "\subseteq"] & {} \arrow[d, dashed, "\rho"] \\
{} \arrow[r, "C"'] & {}
\end{tikzcd}$$

? $= L$ or downwards / L^{-1} or upwards

$\Rightarrow$ Once solved, ?-simulation is characterized, and therefore provable, within Hoare Logic

We solved this problem for both L and L^{-1}-simulation and for partial correctness and total correctness relational semantics [de Roever & Engelhardt, MFCS '96].

Simulation Theorems

Problem: How to characterize the maximal relation C downwards simulating $\varphi \rightsquigarrow \psi$ under abstraction relation ρ as a specification statement.

Solution for C:

(Partial correctness, relational semantics, downward simulation)

$$\exists \bar{a} \left(\rho \wedge (\bar{x}, \bar{a}) = (\bar{y_0}, \bar{b_0})\right) \rightsquigarrow \exists \bar{a} \left(\rho \wedge \forall \bar{x_0} \left(\varphi \left[{}^{(\bar{y_0}, \bar{b_0})}/_{(\bar{x}, \bar{a})}\right] \Rightarrow \psi\right)\right)$$

NB For the total correctness solution, add conjunct $\exists \bar{a} \left(\rho \wedge \exists \bar{x_0} \left(\varphi\right)\right)$ to the precondition. (This term expresses the domain of convergence of the total correctness solution.) This conjunct is essential in justifying the reduction of Reynolds' method for data refinement to downwards simulation for total correctness.

Proof sketch of the downward simulation theorem for partial correctness **(1)**

$$\begin{tikzcd}[column sep=huge, row sep=huge]
{} \arrow[r, "\varphi \rightsquigarrow \psi"] \arrow[d, dashed, "\beta"'] \arrow[dr, phantom, "\subseteq"] & {} \arrow[d, dashed, "\beta"] \\
{} \arrow[r, "S"'] & {}
\end{tikzcd}$$

S downward simulates $\varphi \rightsquigarrow \psi$ w.r.t. β

$$\begin{aligned}
&\iff \beta \,;\, S \subseteq (\varphi \rightsquigarrow \psi) \,;\, \beta \\
&\iff \qquad S \subseteq \beta \rightsquigarrow (\varphi \rightsquigarrow \psi) \,;\, \beta
\end{aligned}$$

Proof sketch of the downward simulation theorem for partial correctness **(2)**

From the solution

$$\begin{tikzcd}
\Sigma^{\bar{a},\bar{x}} \arrow[rr, "(\varphi \rightsquigarrow \psi)\ ;\ \beta"] \arrow[dr, swap, "\beta"] & \arrow[d, phantom, "\rotatebox{90}{$\subseteq$}"] & {} \\
& {} \arrow[ur, swap, "S"] &
\end{tikzcd}$$

provided by the Galois connection $\beta \rightsquigarrow (\varphi \rightsquigarrow \psi)\ ;\ \beta$ (which is not a specification statement but a relational term) we construct a specification statement $\varphi' \rightsquigarrow \psi'$

$$\begin{tikzcd}
\Gamma \arrow[rr, "\psi'"] \arrow[dr, swap, "\varphi'"] & \arrow[d, phantom, "\rotatebox{90}{$\subseteq$}"] & {} \\
& {} \arrow[ur, swap, "S"] &
\end{tikzcd}$$

with the same meaning for S by expressing binary relations, “$\rightsquigarrow$”, and “;” syntactically and replacing abstract program states in the upper left corner by concrete logical states.

Expressing binary relations and $\leadsto$ syntactically (1)

Given first-order logic predicates φ, ψ with free variables:

$$fv(\varphi) = \{\bar{x}, \bar{y}\} \quad \text{notation: } \varphi(\bar{x}\,;\bar{y}), \quad \bar{x} \cap \bar{y} = \emptyset$$

$$fv(\psi) = \{\bar{x}, \bar{z}\} \quad \text{notation: } \psi(\bar{x}\,;\bar{z}), \quad \bar{x} \cap \bar{z} = \emptyset$$

Then: $\mathcal{C}[\![\varphi]\!] \subseteq \Sigma^{\bar{x}} \times \Sigma^{\bar{y}}$, $\mathcal{C}[\![\psi]\!] \subseteq \Sigma^{\bar{x}} \times \Sigma^{\bar{z}}$, with $\Sigma^{\bar{u}} \overset{\text{def}}{=} [\bar{u} \longrightarrow \mathit{Val}]$

and $\mathcal{C}[\![\varphi]\!] \leadsto \mathcal{C}[\![\psi]\!] = \{(\sigma, \tau) \,|\, \forall\theta.(\theta, \sigma) \in \mathcal{C}[\![\varphi]\!] \Rightarrow (\theta, \tau) \in \mathcal{C}[\![\psi]\!]\}$

$$\subseteq \Sigma^{\bar{y}} \times \Sigma^{\bar{z}}$$

Since $\bar{y}$ and $\bar{z}$ in general not disjoint, indicate syntactically which variables are evaluated in initial state σ and which ones in final state τ, for $(\sigma, \tau) \in \mathcal{C}[\![\varphi]\!] \leadsto \mathcal{C}[\![\psi]\!]$.

Substitute primed versions v' for variables v evaluated in the initial state, with unprimed versions evaluated in the final state.

Expressing binary relations and $\rightsquigarrow$ syntactically (2)

Convention: primed variables v' evaluated in σ by $v'(\sigma, \tau) = \sigma(v)$, and unprimed ones v in τ by $v(\sigma, \tau) = \tau(v)$, and define

$$\varphi \rightsquigarrow \psi \overset{\text{def}}{=} (\forall \bar{x}.\varphi[\bar{y}'/\bar{y}] \rightarrow \psi)(\bar{y}\,;\bar{z}),$$

possibly renaming $\bar{x}$ in case $\bar{x} \cap \bar{y}' \neq \emptyset$.

Example 3 $(x = x_0 \rightsquigarrow x = x_0 + 1) = \forall x_0.(x' = x_0 \rightarrow x = x_0 + 1)$ characterizes $x := x + 1$.

Theorem: $\mathcal{C}[\![\varphi \rightsquigarrow \psi]\!] = \mathcal{C}[\![\varphi]\!] \rightsquigarrow \mathcal{C}[\![\psi]\!]$

Expressing “;” syntactically

Given first-order logic predicates $\varphi(\bar{x}\,;\bar{y})$ and $\psi(\bar{y}\,;\bar{z})$, “;” is usually defined by

$$(\exists \bar{y}.\varphi(\bar{x}\,;\bar{y}) \wedge \psi(\bar{y}\,;\bar{z}))(\bar{x}\,;\bar{z})$$

However this does NOT cater for our primed variable convention:

So one has $\varphi(\bar{x}'\,;\bar{y})$, $\bar{x}' \cap \bar{y} = \emptyset$

$\psi(\bar{y}'\,;\bar{z})$, $\bar{y}' \cap \bar{z} = \emptyset$

and defines

$$\varphi(\bar{x}'\,;\bar{y})\,;\psi(\bar{y}'\,;\bar{z}) \overset{\text{def}}{=} \exists \bar{u}.\varphi[\bar{u}/\bar{y}] \wedge \psi[\bar{u}/\bar{y}'] \quad \text{with } \bar{u} \cap \bar{x}' = \bar{u} \cap \bar{z} = \emptyset$$

Theorem: $\mathcal{C}[\![\varphi\,;\psi]\!] = \mathcal{C}[\![\varphi]\!]\,;\mathcal{C}[\![\psi]\!]$

Expressing representation relations syntactically

Binary representation relation β is expressed by a first-order predicate ρ relating values of abstract representation variables $\bar{a}$ to those of concrete representation variables $\bar{c}$, and lets the values of normal variables $\bar{x}$ – i.e., of non-representation variables – unchanged:

$$\beta(\bar{a}', \bar{x}' \,;\, \bar{c}, \bar{x}) \overset{\text{def}}{=} \rho(\bar{a}' \,;\, \bar{c}) \wedge \bar{x}' = \bar{x}, \text{ for appr. } \rho$$

Proof of downwards simulation theorem for partial correctness (1)

1. **Case $\varphi \leadsto \psi$:**

$$\text{Given: } \left.\begin{array}{r} \varphi(x_0\,;x,a) \\[1ex] \psi(x_0\,;x,a) \end{array}\right\} \Rightarrow (\varphi \leadsto \psi) = (\forall x_0.\varphi[x',a'/x,a] \to \psi)$$

2. **Case $(\varphi \leadsto \psi)\,;\beta$:**

$$\left.\begin{array}{r} (\varphi \leadsto \psi)(x',a'\,;x,a) \\[1ex] \underbrace{(\rho[a'/a] \land x' = x)(x',a'\,;x,c)}_{=\beta} \end{array}\right\} \Rightarrow$$

$$(\varphi \leadsto \psi)\,;(\rho[a'/a] \land x' = x) = \underbrace{\exists u,a.(\varphi \leadsto \psi)[u/x] \land \rho \land u = x}_{=\,(\exists a.(\varphi \leadsto \psi) \land \rho)(x',a'\,;x,c)}$$

Proof of downwards simulation theorem for partial correctness (2)

3. Case $\beta \leadsto (\varphi \leadsto \psi) \,;\, \beta$**:**

$$\underbrace{\rho[a'/a] \wedge x' = x}_{=\beta} \leadsto (\varphi \leadsto \psi) \,;\, (\underbrace{\rho[a'/a] \wedge x' = x}_{=\beta}) = \qquad \text{(by (2))}$$

$$\underbrace{\forall x'_0, a'_0 . (\rho[a'_0/a] \wedge x'_0 = x)[x', c'/x, c] \to (\exists a . \rho \wedge \forall x_0 . \varphi[x'_0, a'_0/x, a] \to \psi)}$$

$$= \rho[a'_0/a] \wedge x'_0 = x \leadsto \exists a . \rho \wedge \forall x_0 . \varphi[x'_0, a'_0/x, a] \to \psi$$

QED

I.e., $S \subseteq \beta \leadsto (\varphi \leadsto \psi) \,;\, \beta$ iff

$$\models \left\{ \rho[a'_0/a] \wedge x'_0 = x \right\} S \left\{ \exists a . \rho \wedge \forall x_0 . \varphi[x'_0, a'_0/x, a] \to \psi \right\}$$

Simplification possible in some cases

Theorem: For $\bar{x}$ list of program variables, $\bar{x}_0$ a list of logical variables occurring free in assertions φ and ψ, let $\bar{y}_0$ be a list of fresh logical variables of the same length as of $\bar{x}$. Then:

$$\varphi \leadsto \psi = \bar{x} = \bar{y}_0 \leadsto \forall \bar{x}_0(\varphi[\bar{y}_0/\bar{x}] \rightarrow \psi)$$

Theorem: For preconditions of form $\bar{x} = \bar{y}_0$ of if ρ^{-1} is a total function S downward simulates $\varphi \leadsto \psi$ under representation relation ρ iff

$$\models \Big\{\exists a(\rho \wedge \varphi)\Big\}\, S\, \Big\{\exists a(\rho \wedge \psi)\Big\}.$$

Semantic models

Unfortunately, the relational model for partial correctness is not appropriate for all of the methods we would like to discuss.

Instead we need **four** of them:

	relations	pred. transformers
partial corr.	Hoare (p.c.), Hehner	Gardiner
total corr.	VDM, Z, Reynolds, Hoare (t.c.), Abadi & Lamport, Lynch	Back & von Wright, Morgan

Syntax & semantics

$Prog$ is a reasonably broad language to express the essential features of the treated methods (from the data refinement point of view).

$$Prog \ni S ::= \varphi \leadsto \psi \mid X \mid S_1 \,;\, S_2 \mid S_1 \Box S_2 \mid \mu X.S$$

with relational semantics for partial correctness $\mathcal{P}[\![.]\!] : Prog \to 2^{\Sigma \times \Sigma}$ such that

$$\{\varphi\}\, S\, \{\psi\} \text{ is valid iff } \mathcal{P}[\![S]\!] \subseteq \mathcal{P}[\![\varphi \leadsto \psi]\!].$$

Example 4 $(x, y, s = x_0, y_0, s_0) \leadsto (x, y, s = x_0, y_0, s_0 \cup \{x_0\})$ expresses $s := s \cup \{x\}$ in Example 1.

4 semantics of programs

	relations	pred. transformers
partial corr.	$\mathcal{P}[\![S]\!] \subseteq \Sigma^2$	$\mathit{wlp}(S) : 2^\Sigma \overset{\text{mon}}{\rightarrow} 2^\Sigma$
total corr.	$\mathcal{P}_\perp[\![S]\!] \subseteq {\Sigma_\perp}^2$	$\mathit{wp}(S) : 2^\Sigma \overset{\text{mon}}{\rightarrow} 2^\Sigma$

Relating semantics (1)

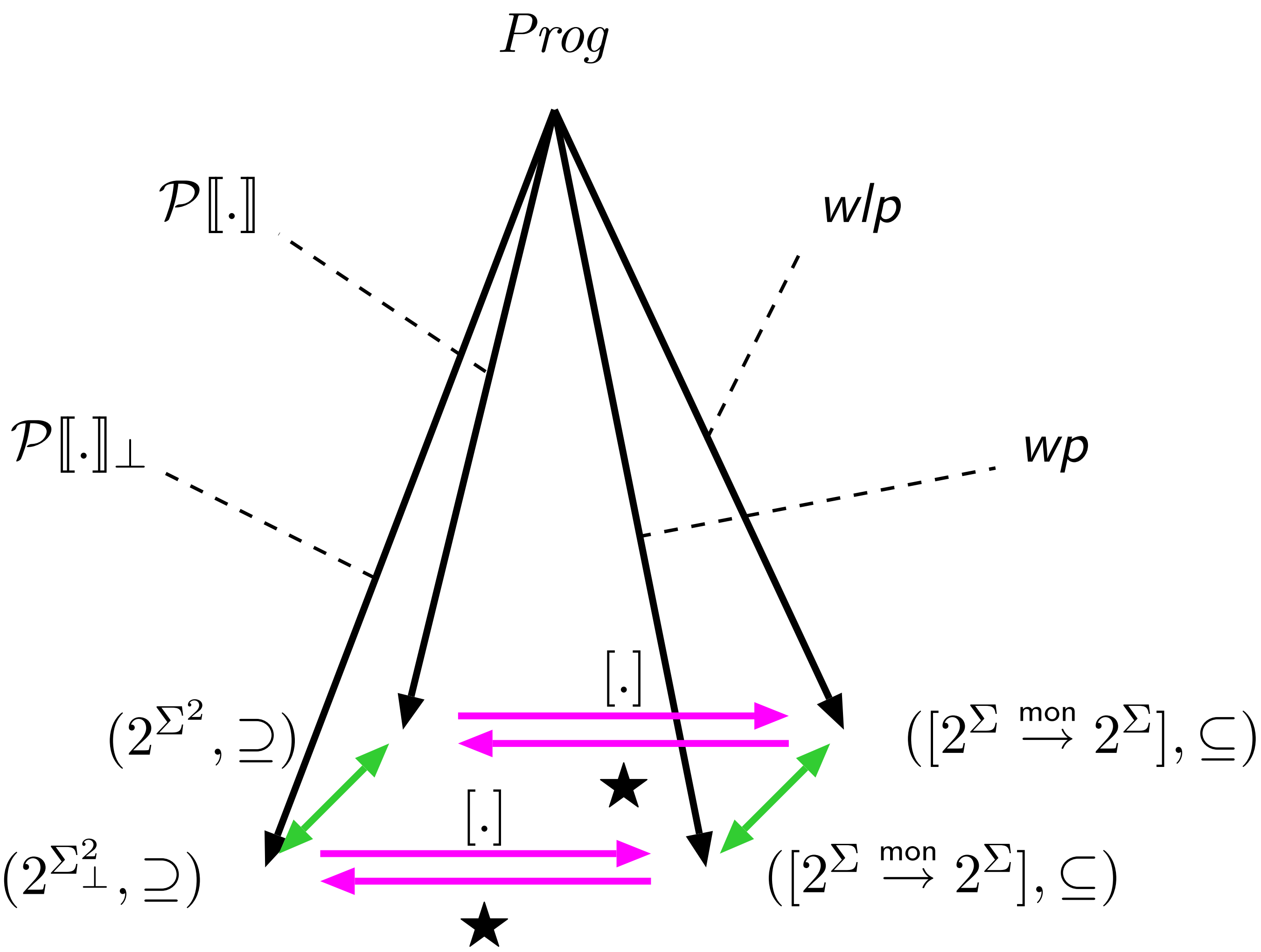
vertical connection: separation theorems

$$\begin{array}{ccccc}
\text{total corr.} & = & \text{partial corr.} & + & \text{termination} \\
[\varphi]S[\psi] & \Leftrightarrow & \{\varphi\}\,S\,\{\psi\} & \wedge & [\varphi]S[\mathsf{true}] \\
\mathit{wp}(S)\psi & \Leftrightarrow & \mathit{wlp}(S)\psi & \wedge & \mathit{wp}(S)\mathsf{true}
\end{array}$$

horizontal connection: Galois connection $(\bigstar, [.])$

$$\begin{array}{rcl}
\sigma \in [r]\,s & \Leftrightarrow & \forall \tau\,((\sigma, \tau) \in r \Rightarrow \tau \in s) \\
(\sigma, \tau) \in \bigstar P & \Leftrightarrow & \forall s\,(\sigma \in P(s) \Rightarrow \tau \in s)
\end{array}$$

Relating semantics (2)



$Prog$

$\mathcal{P}[\![.]\!]$

wlp

$\mathcal{P}[\![.]\!]_\perp$

wp

$[.]$

$(2^{\Sigma^2}, \supseteq)$

$([2^\Sigma \overset{\mathsf{mon}}{\longrightarrow} 2^\Sigma], \subseteq)$

$\bigstar$

$[.]$

$(2^{\Sigma^2_\perp}, \supseteq)$

$([2^\Sigma \overset{\mathsf{mon}}{\longrightarrow} 2^\Sigma], \subseteq)$

$\bigstar$

Proof system (1)

adaptation axiom

$$\vdash \{\pi\}\, \varphi \rightsquigarrow \psi \,\{\exists \bar{y_0}\, (\pi[{}^{\bar{y_0}}/_{\bar{x}}] \wedge \forall \bar{x_0}\, (\varphi[{}^{\bar{y_0}}/_{\bar{x}}] \Rightarrow \psi))\}$$

$\rightsquigarrow$-substitution rule

$$\frac{\{\varphi\}\, S_1\, \{\psi\}, \{\pi\}\, S_2[{}^{\varphi \rightsquigarrow \psi}/_X]\, \{\theta\}}{\{\pi\}\, S_2[{}^{S_1}/_X]\, \{\theta\}}$$

recursion rule

$$\frac{\{\pi\}\, S[{}^{\pi \rightsquigarrow \theta}/_X]\, \{\theta\}}{\{\pi\}\, \mu X.S\, \{\theta\}}$$

Proof system (2)

composition rule

$$\frac{\{\pi\}\, S_1\, \{\varphi\},\ \ \{\varphi\}\, S_2\, \{\rho\}}{\{\pi\}\, S_1 \,;\, S_2\, \{\rho\}}$$

choice rule

$$\frac{\{\pi\}\, S_1\, \{\rho\},\ \ \{\pi\}\, S_2\, \{\rho\}}{\{\pi\}\, S_1 \ \square\ S_2\, \{\rho\}}$$

consequence rule

$$\frac{\pi \Rightarrow \varphi,\ \ \{\varphi\}\, S\, \{\psi\},\ \ \psi \Rightarrow \rho}{\{\pi\}\, S\, \{\rho\}}$$

$=$ *sound* and (relatively) *complete* proof system (in the sense of Cook).

Reynolds' method

... we must transform our program to replace the abstract variable by a concrete variable representing its value. To do this, we will use the following general method:

R1. One or more concrete variables are introduced to store the representation of one or more abstract variables.

R2. A general invariant called the *representation invariant* is introduced, which describes the relationship between the abstract and concrete variables.

R3. Each assignment to an abstract variable (or more generally, each assignment that affects the representation invariant) is augmented with assignments to the concrete variables that re-establish the representation invariant (or achieve it, in case of an initialization).

R4. Each expression that contains an abstract variable but occurs outside of an assignment to an abstract variable is replaced by an expression that does not contain abstract variables but is guaranteed by the representation invariant to have the same value.

The last step will render the abstract variables **auxiliary**, so that their declarations and assignments can be **eliminated**. [Reynolds 1981]

Theorems

Theorem 2 Each data refinement step following Reynolds' recipe induces a case of total correctness downward simulation in the relational setting.

Theorem 3 Each data refinement step in VDM induces a case of total correctness downward simulation in the relational setting.

Example with Reynolds' method: steps R1 and R2

```
begin
    var $s$ : set of $\mathbb{N}$ ; $l : \mathbb{N}^*$;
    $s := \{5\}$;
    {geninv l: $elems(l) = s$}
    $S_1$;
    $s := s \cup \{x\}$;
    $S_2$;
    $y :=$ a member of $s$;
end
```

Example with Reynolds' method: step R3

```
begin
    var $s$ : set of $\mathbb{N}$ ; $l : \mathbb{N}^*$;
    $s := \{5\}$ ; $l := \langle 5 \rangle$;
    $\{$geninv I: elems$(l) = s\}$
    $S_1$;
    $\langle s := s \cup \{x\}$ ;    $l :=$ append$(l, x)\rangle$ ;
    $S_2$;
    $y :=$ a member of $s$;
end
```

Example with Reynolds' method: step R4

```
begin
    var $s$ : set of $\mathbb{N}$ ; $l$ : $\mathbb{N}^*$;
    $s := \{5\}$ ; $l := \langle 5 \rangle$;
    $\{$geninv I: $elems(l) = s\}$
    $S_1$;
    $\langle s := s \cup \{x\}$ ;    $l := append(l, x)\rangle$ ;
    $S_2$;
    $y := first(l)$;
end
```

Example with Reynolds' method: final step

```
begin
    var $l : \mathbb{N}^*$;
    $l := \langle 5 \rangle$;
    $S_1$;
    $l := \mathit{append}(l, x)$;
    $S_2$;
    $y := \mathit{first}(l)$;
end
```

Example in VDM (1)

We specify the abstract and concrete level operations of our example using VDM:

First the state variable is declared.

$$s : \textbf{set of } \mathbb{N} \quad \Big| \quad l : \mathbb{N}^*$$

Then its initial value is fixed.

$$s_0 = \{5\} \quad \Big| \quad l_0 = [5]$$

Example in VDM (2)

The operations are specified next.

ADDa $(x : \mathbb{N})$

ext wr s : **set of** $\mathbb{N}$

post $s = \overleftarrow{s} \cup \{x\}$

ADDc $(x : \mathbb{N})$

ext wr $l : \mathbb{N}^*$

post $l = \overleftarrow{l} \frown x$

GETa $(y : \mathbb{N})$

ext rd s : **set of** $\mathbb{N}$

pre $s \neq \emptyset$

post $y \in s$

GETc $(y : \mathbb{N})$

ext rd $l : \mathbb{N}^*$

pre $\mathsf{len}(l) > 0$

post $y = \mathsf{first}(l)$

VDM proof obligations (1)

The connection between the state spaces of the two levels under consideration is provided by *retrieve function*

$$\mathsf{elems} : \mathbb{N}^* \to \textbf{set of } \mathbb{N} \ .$$

The concrete data model ($\mathbb{N}^*$) shall be *adequate*, i.e., every abstract value has a corresponding concrete value:

$$l : \mathbb{N}^* \vdash \exists s : \textbf{set of } \mathbb{N} \, (\mathsf{elems} \ \ (l) = s)$$

All images of concrete initial states must be abstract initial states.

$$\vdash \mathsf{elems} \ ([5]) = \{5\}$$

VDM proof obligations (2)

The concrete precondition shall hold whenever the abstract precondition does. (*domain rule* for *GETa* and *GETc*)

$$l : \mathbb{N}^*,\ \mathsf{elems}\ (l) \neq \emptyset \vdash \mathsf{len}(l) > 0$$

The concrete operation should not break the abstract postcondition. (*result rule* for *ADDa* and *ADDc*)

$$\overleftarrow{l}, l : \mathbb{N}^*,\ \mathsf{elems}\ (l) \neq \emptyset,\ l = \overleftarrow{l} \frown x$$
$$\vdash \mathsf{elems}\ (l) = \mathsf{elems}\ (\overleftarrow{l}) \cup \{x\}$$

References

All references can be found in:

- Willem-Paul de Roever and Kai Engelhardt, **Data Refinement: Model-Oriented Proof Methods and their Comparison,** Cambridge Tracts in Theoretical Computer Science 47, Cambridge University Press, 1998.