Semantic Games for Synchrony

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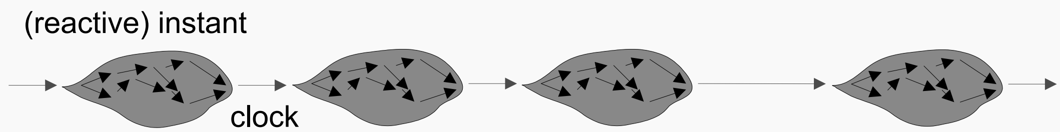
What is this talk about?

- To illustrate how the co-contravariant fixed point problem in synchronous languages can be handled by means of a game theory approach.
- To show that the Must/Cannot analysis of combinational Esterel corresponds to the computation of winning strategies of a particular game (the maze).
- Sequencing is modelled by a notion of stratified plays.

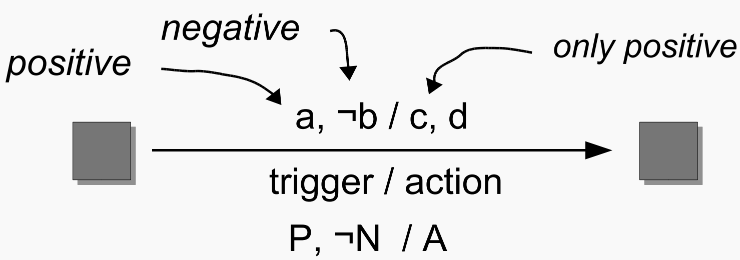
Synchronous Configurations: A Declarative View

Programming Synchronous Reactions



At every instant, signals may be present or absent, emitted or not emitted

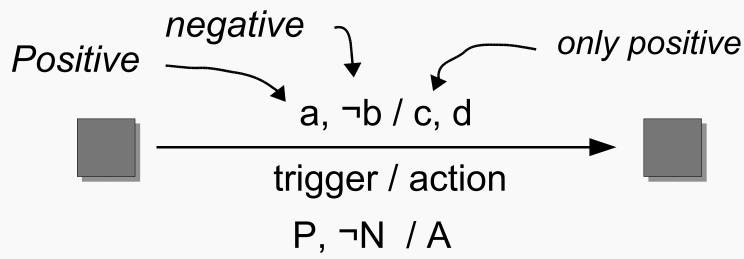
Transitions



"if a present and b absent, then emit c and d"

Programming Synchronous Reactions

Transitions

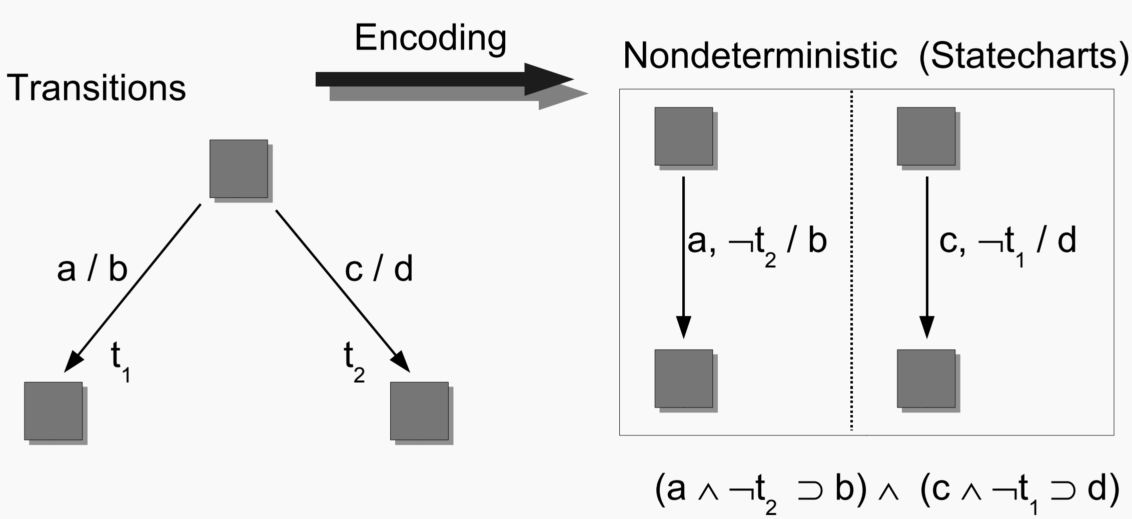


"if a present and b absent, then emit c and d"

Declarative
$a \wedge \neg b \supset c \wedge d$

How do we code configurations ?

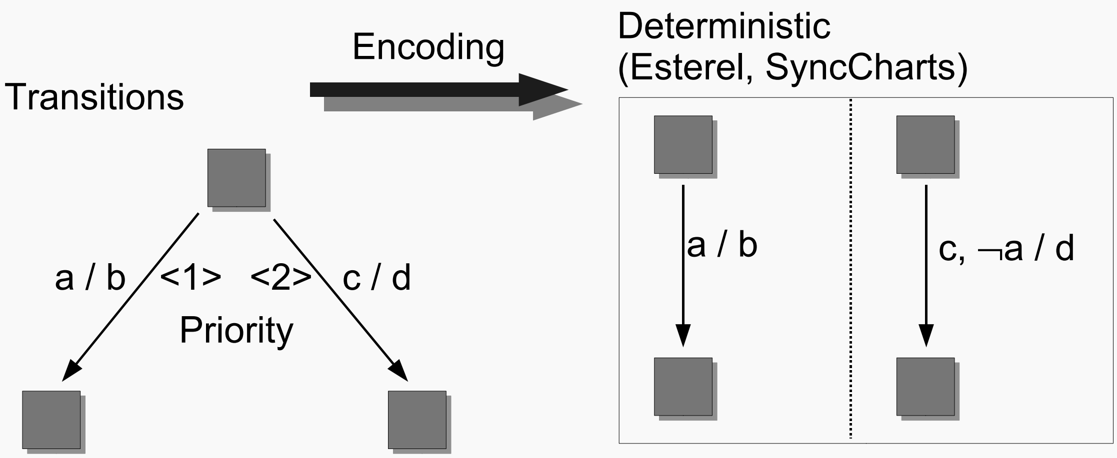
Reducible to parallel composition of transitions



$(a \wedge \neg t_2 \supset b) \wedge (c \wedge \neg t_1 \supset d)$

How do we code configurations ?

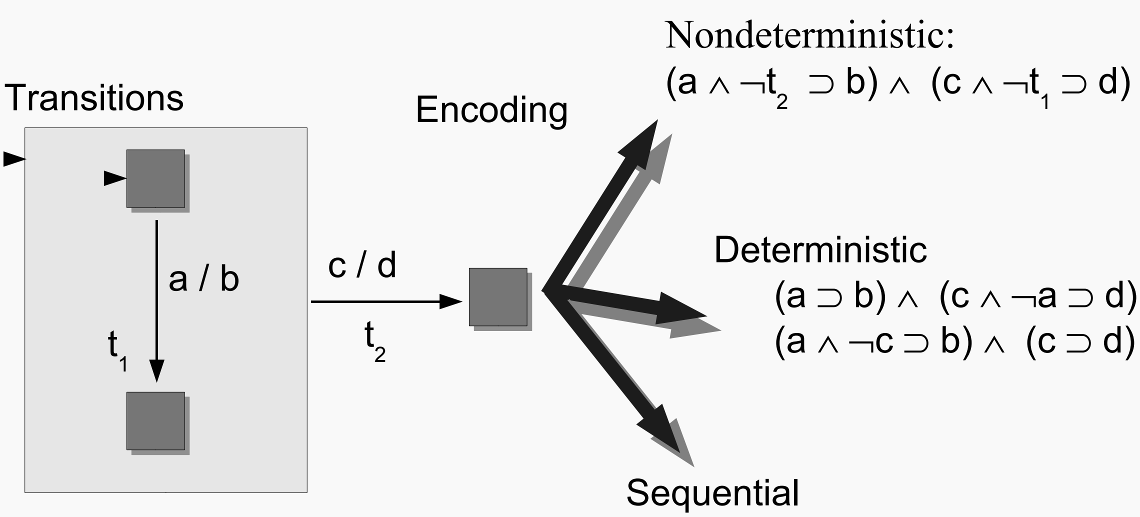
Reducible to parallel composition of transitions



$(a \supset b) \wedge (c \wedge \neg a \supset d)$

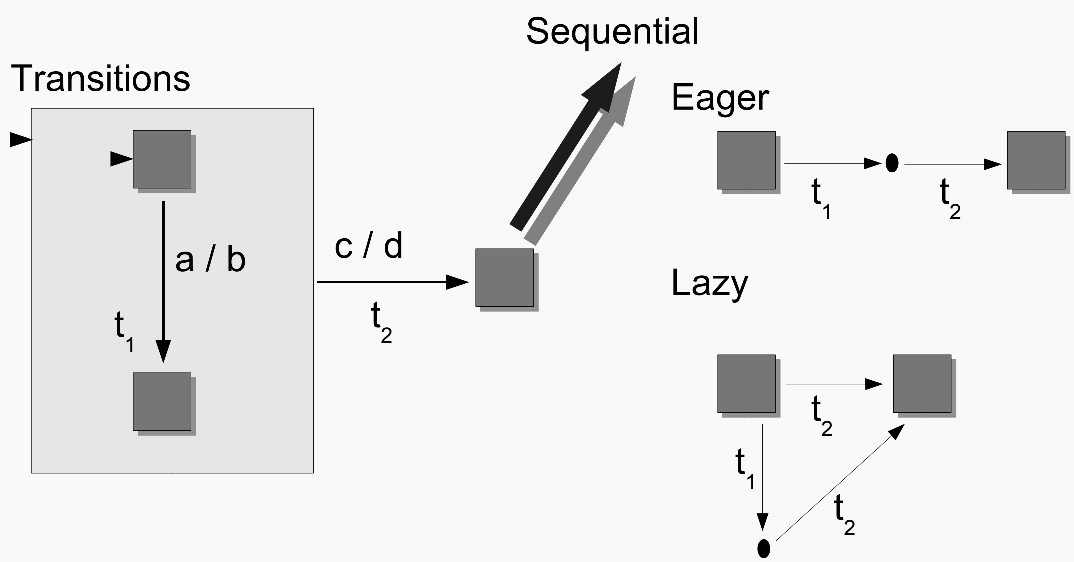
How do we execute configurations ?

Reducible to parallel composition of transitions



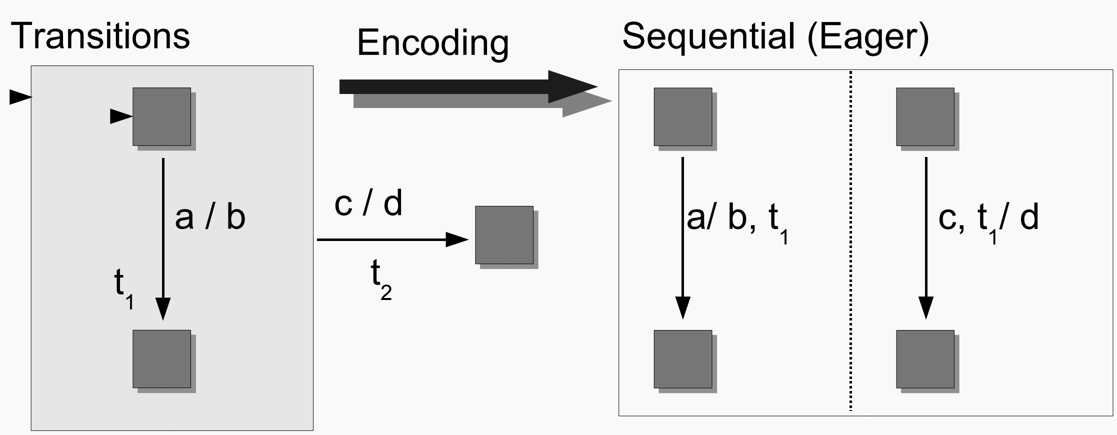
How do we code configurations ?

Reducible to parallel composition of transitions



How do we code configurations ?

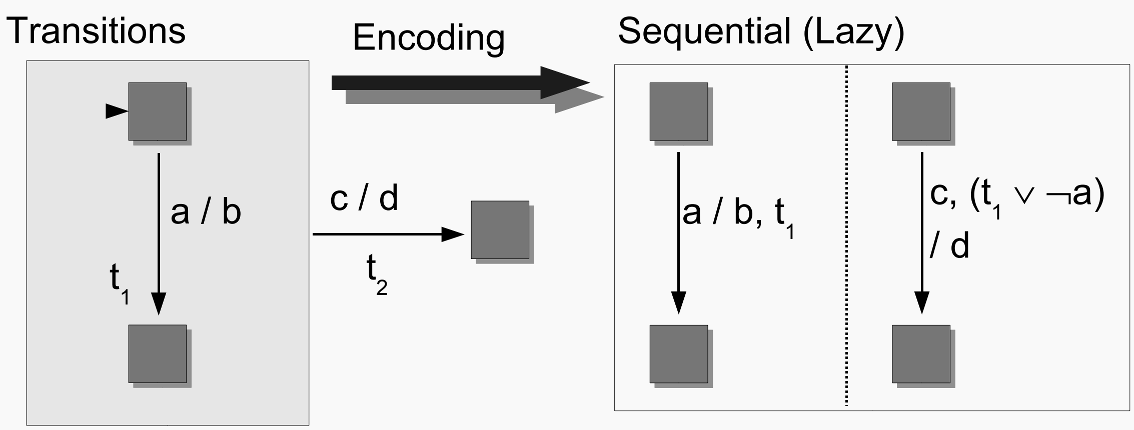
Reducible to parallel composition of transitions



$$(a \supset b \wedge t_1) \wedge (c \wedge t_1 \supset d)$$

How do we code configurations ?

Reducible to parallel composition of transitions



$$(a \supset b \wedge t_1) \wedge (c \wedge t_1 \supset d) \wedge (c \wedge \neg a / d)$$

How do we code configurations ?

environment input + configuration $\Rightarrow$ reaction

environment stimulus (event) $\Rightarrow$ **Configuration** $t_1 \| t_2 \| \ldots \| t_N$ $\Rightarrow$ reaction (event)

reducible to parallel composition of transitions

Choice and priorities can be represented using negations

How do we code configurations ?

configuration ⇒ reaction

Configuration

$t_1 \| t_2 \| \ldots \| t_N \| /a,e$ ⇒ reaction (event)

reducible to parallel composition of transitions

Choice and priorities can be represented using negations

Environment stimulus can be accounted for as part of the configuration

Configurations

Kernel syntax

(for "instantaneous" reactions in the combinational fragment)

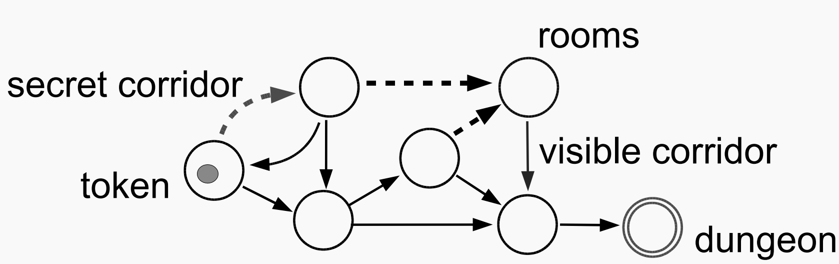
C ::= nothing	C ::= true
| emit s	| s
| present s then C	| s / C
| present s else C	| ¬s / C
| C || C	| C || C

present s then C else D   s / C \|\| ¬s / D

Have a break ... play a Maze

Finite 2-Player Games

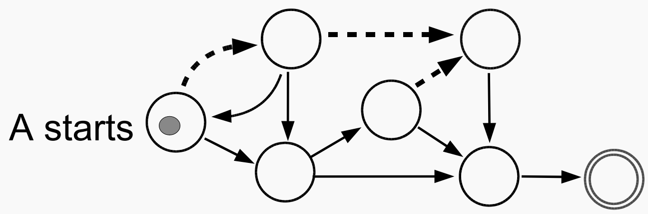
Finite game graph (maze)



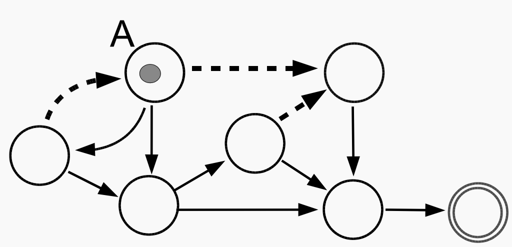
- Players *A*, *B* take alternate turns:
  visible corridor (→) = turn changes to opponent
  secret corridor (⇢) = player keeps his/her turn

- Winning Rule: “last player loses”

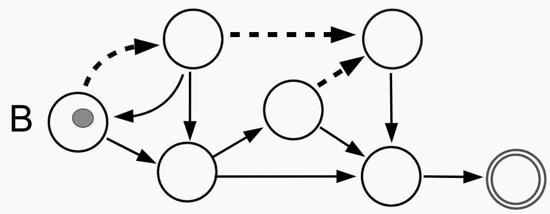
Finite 2-Player Games



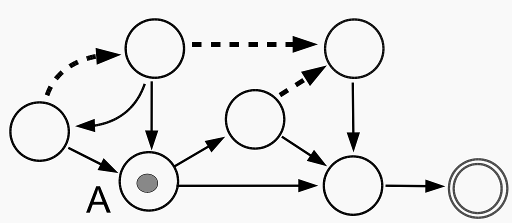
Finite 2-Player Games



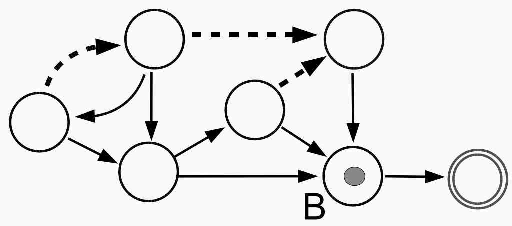
Finite 2-Player Games



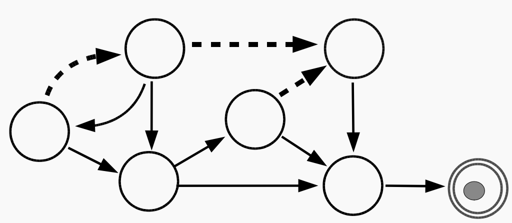
Finite 2-Player Games



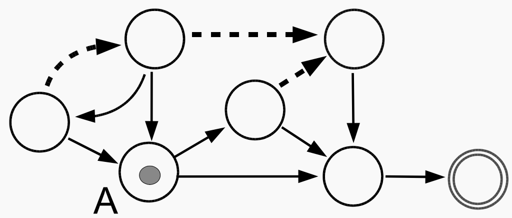
Finite 2-Player Games



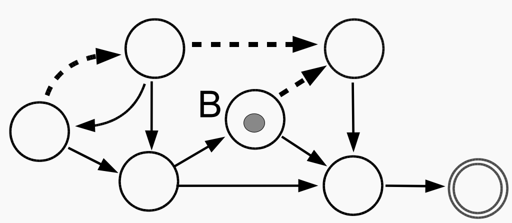
Finite 2-Player Games



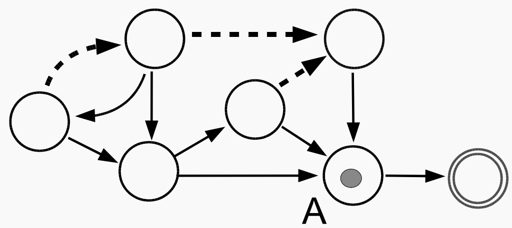
Finite 2-Player Games



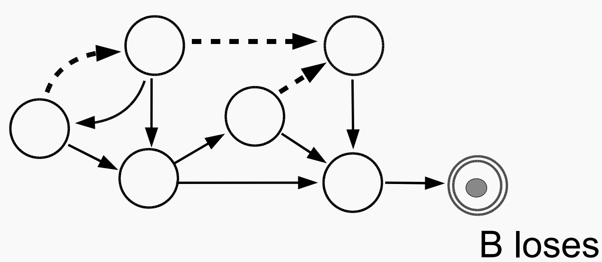
Finite 2-Player Games



Finite 2-Player Games



Finite 2-Player Games



Strategies

rooms visible secret corridors

Let M = (R, $\xrightarrow{\iota}$, $\xrightarrow{\tau}$) be a finite maze and $\xrightarrow{\gamma} := \xrightarrow{\iota} \cup \xrightarrow{\tau}$.

A strategy is a partial mapping $\alpha : R \rightarrow R$ such that
$\forall r$. if $\alpha(r)$ is defined then $r \xrightarrow{\gamma} \alpha(r)$.

A pair of strategies (α, β) and a start room r determines a unique play in M, denoted play(α, β, r), where
player A uses function α
player B uses function β
to determine his/her next move, as long as α and β are defined (maximality).

Winning Positions

Let length(α, β, r) be the length (possibly ∞).
Let last$(\alpha, \beta, r) \in \{A, B, \perp\}$ the last player in play(α, β, r).

A room $r \in R$ is called a

winning position (for the starting player A)
if $\exists\alpha.\forall\beta.$ last$(\alpha, \beta, r) = B$

losing position (for the starting player A)
if $\forall\alpha.\exists\beta.$ last$(\alpha, \beta, r) = A$

Winning Positions

Let length(α, β, r) be the length (possibly ∞).
Let last(α, β, r) $\in$ {*A*, *B*, $\perp$} the last player in play(α, β, r).

A room r $\in$ R is called an

n-winning position (for the starting player A)
if $\exists\alpha.\forall\beta$. length($\alpha$, β, r) $\leq$ n & last(α, β, r) = *B*

n-losing position (for the starting player A)
if $\forall\alpha.\exists\beta$. length($\alpha$, β, r) $\leq$ n & last(α, β, r) = *A*

n-safe position (for the starting player A)
if $\exists\alpha.\forall\beta$. length($\alpha$, β, r) $>$ n $\vee$ last(α, β, r) = *B*

Computing Winning Positions

Bounded Winning

R_{win}^{n} := {r$\in$R | r is n-winning } R_{safe}^{n} := {r$\in$R | r is n-safe }

Iterative Approximation:

R^{-1}_{win} := $\emptyset$
R^{-1}_{safe} := all

R^{n+1}_{win} := {r$\in$R | $\exists$s$\in$R.
$(r \xrightarrow{\tau} s \wedge s \in R^{n}_{win}) \vee (r \xrightarrow{\iota} s \wedge s \notin R^{n}_{safe})$}
R^{n+1}_{safe} := {r$\in$R | $\exists$s$\in$R.
$(r \xrightarrow{\tau} s \wedge s \in R^{n}_{safe}) \vee (r \xrightarrow{\iota} s \wedge s \notin R^{n}_{win})$}

Computing Winning Positions

The iteration constructs approximation sequences

$\emptyset \subseteq R^0_{win} \subseteq R^1_{win} \subseteq R^2_{win} \subseteq \ldots \subseteq R^n_{win} \subseteq \ldots$

$all \supseteq R^0_{safe} \supseteq R^1_{safe} \supseteq R^2_{safe} \supseteq \ldots \supseteq R^n_{safe} \supseteq \ldots$

in which $\emptyset \subseteq R^{n+1}_{win} \subseteq R^{n+1}_{safe} \subseteq all$.

Theorem (standard result)

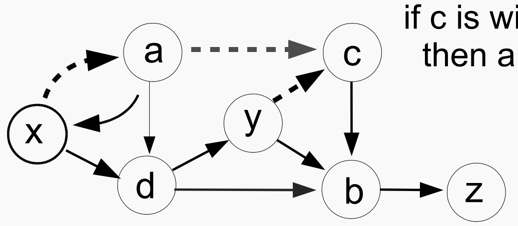
Let $R_{win} := \cup_n R^n_{win}$ and $R_{safe} := \cap_n R^n_{safe}$. Then,

- $r \in R_{win}$ iff r winning position
- $r \in all \setminus R_{safe}$ iff r losing position
- $r \in R_{safe} \setminus R_{win}$ iff r draw position.

So, what is the game in a synchronous reaction ?

Configurations as Mazes

Maze G_c



if c is winning position
then a is winning position

if b is losing position
then d is winning position

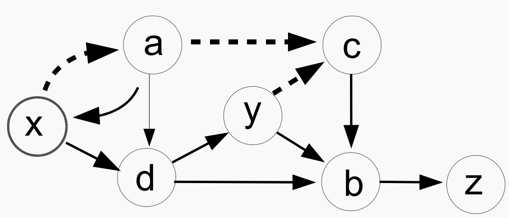
Configuration C

$c/a \parallel a/x \parallel \neg x/a \parallel \neg d/a \parallel \neg d/x \parallel \neg y/d \parallel c/y \parallel \neg b/c \parallel \neg b/y \parallel \neg b/d \parallel \neg z/b$

- rooms = signals, corridors = transitions
- winning position = signal present
- losing position = signal absent

Mazes as Configurations

Maze G_c



Configuration C

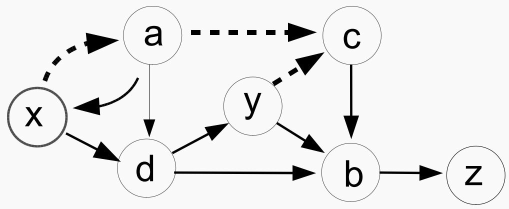
$c/a \parallel a/x \parallel \neg x/a \parallel \neg d/a \parallel \neg d/x \parallel \neg y/d \parallel c/y \parallel \neg b/c \parallel \neg b/y \parallel \neg b/d \parallel \neg z/b$

$R^{-1}_{win} = \emptyset$

$R^{-1}_{safe} = \{a, b, c, d, x, y, z\}$

The Esterel Game

Maze G_c



Configuration C

c/a ‖ a/x ‖ ¬x/a ‖ ¬d/a ‖ ¬d/x ‖ ¬y/d ‖ c/y ‖ ¬b/c ‖ ¬b/y ‖ ¬b/d ‖ ~~¬z~~/b

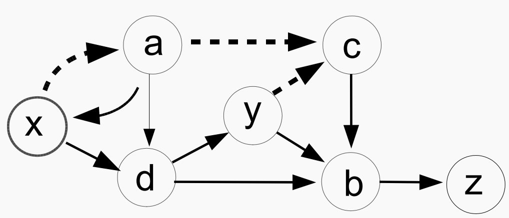
$R^0_{win} = \emptyset$

$R^0_{safe} = \{a, b, c, d, x, y\}$

present
z absent

The Esterel Game

Maze G_c



Configuration C

c/a ‖ a/x ‖ ¬x/a ‖ ¬d/a ‖ ¬d/x ‖ ¬y/d ‖ c/y ‖ ~~¬b/c~~ ‖ ~~¬b/y~~ ‖ ~~¬b/d~~ ‖ ~~¬z~~/b

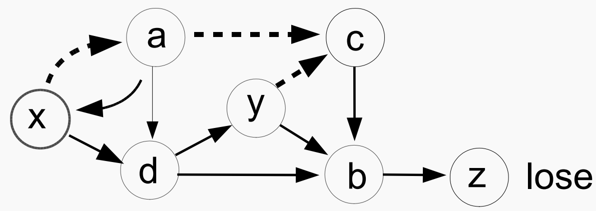
$R^1_{win} = \{ b \}$

$R^1_{safe} = \{a, b, c, d, x, y\}$

b present
z absent

The Esterel Game

Maze G_c



Configuration C

~~c/a~~ ‖ a/x ‖ ¬x/a ‖ ¬d/a ‖ ¬d/x ‖ ¬y/d ‖ ~~c/y~~ ‖ ~~¬b/c~~ ‖ ~~¬b/y~~ ‖ ~~¬b/d~~ ‖ ~~¬z~~/b

R^2_{win} = { b }

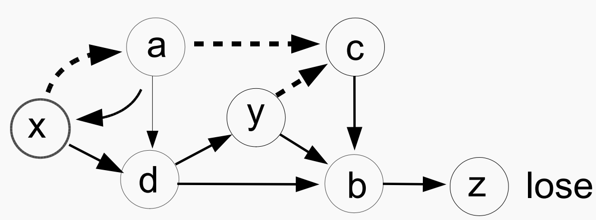
R^2_{safe} = {a, b, d, x, y }

b present

z,c absent

The Esterel Game

Maze G_c



Configuration C

~~c/a~~ ‖ a/x ‖ ¬x/a ‖ ¬d/a ‖ ¬d/x ‖ ~~¬y~~/d ‖ ~~c/y~~ ‖ ~~¬b/c~~ ‖ ~~¬b/y~~ ‖ ~~¬b/d~~ ‖ ~~¬z~~/b

R^3_{win} = { b }

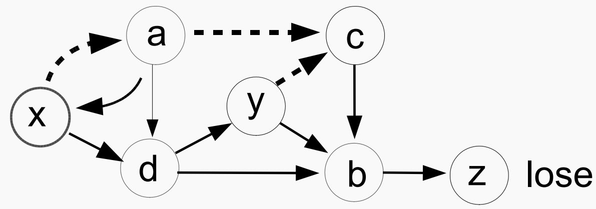
R^3_{safe} = {a, b, d, x }

b present

z,c,y absent

The Esterel Game

Maze G_c



Configuration C

~~c/a~~ ‖ a/x ‖ ¬x/a ‖ ~~¬d/a~~ ‖ ~~¬d/x~~ ‖ ~~¬y~~/d ‖ ~~c/y~~ ‖ ~~¬b/c~~ ‖ ~~¬b/y~~ ‖ ~~¬b/d~~ ‖ ~~¬z~~/b

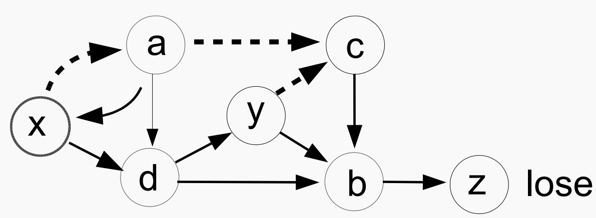
R^4_{win} = { b, d }

R^4_{safe} = {a, b, d, x }

b,d present
z,c,y absent

The Esterel Game

Maze G_c



Configuration C

~~c/a~~ ‖ a/x ‖ ¬x/a ‖ ~~¬d/a~~ ‖ ~~¬d/x~~ ‖ ~~¬y~~/d ‖ ~~c/y~~ ‖ ~~¬b/c~~ ‖ ~~¬b/y~~ ‖ ~~¬b/d~~ ‖ ~~¬z~~/b

R^5_{win} = { b, d } = R^4_{win}

R^5_{safe} = {a, b, d, x } = R^4_{safe}

Fixed point reached
b,d present
z,c,y absent

Coincidence Theorem

Theorem

Let C be a configuration and G_C be the game associated with C. Then, for all signals a ∈ Sigs:

- a is present in the Esterel reaction of C iff a is a winning position in G_C
- a is absent in the Esterel reaction of C iff a is a losing position in G_C
- a is non-constructive in the Esterel reaction of C iff a is a draw position