Approximately Bisimilar Symbolic Models for Incrementally Stable Switched Systems

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Motivation

- Controller synthesis for switched systems:
  - Wide litterature on stability and stabilization
  - Not so rich for other objectives
    path planning, oscillation enforcement...

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    supervisory control, algorithmic game theory...
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    existence, computation...
  - This talk: for a class of incrementally stable switched systems

Outline of the Talk

1. Switched systems and incremental stability

2. Symbolic abstractions of switched systems:
   - Approximate bisimulation
   - Common Lyapunov function
   - Multiple Lyapunov functions

3. Symbolic models for the boost DC-DC converter

Switched Systems

A switched system is a quadruple:

$$\Sigma = (\mathbb{R}^n, P, \mathcal{P}, F),$$

where:

- $\mathbb{R}^n$ is the state space;
- $P = \{1, \ldots, m\}$ is the finite set of modes;
- $\mathcal{P} \subseteq \mathcal{S}(\mathbb{R}_0^+, P)$ is the set of switching signals;
  $\mathcal{S}(\mathbb{R}_0^+, P)$: *set of piecewise constant functions from* $\mathbb{R}_0^+$ *to* P
- $F = \{f_1, \ldots, f_m\}$ is a collection of vector fields indexed by P.

For $p \in P$, Σ_p denotes the continuous subsystem associated to f_p.

Trajectories of a Switched System

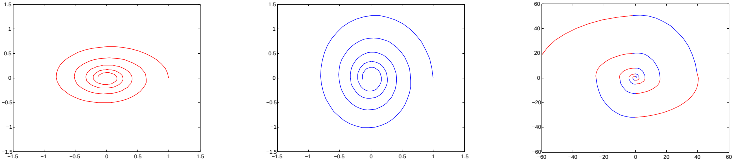
- A continuous, piecewise $\mathcal{C}^1$ function $\mathbf{x} : \mathbb{R}_0^+ \to \mathbb{R}^n$ is a trajectory of Σ if there exists $\mathbf{p} \in \mathcal{P}$ such that

$$\dot{\mathbf{x}}(t) = f_{\mathbf{p}(t)}(\mathbf{x}(t)) \text{ for almost all } t \in \mathbb{R}_0^+.$$

- $\mathbf{x}(t, x, \mathbf{p})$ denotes the point reached at time $t \in \mathbb{R}_0^+$ from the initial state x under the switching signal $\mathbf{p}$.
- $\mathbf{x}(t, x, p)$ denotes the point reached at time $t \in \mathbb{R}_0^+$ from the initial state x under the constant switching signal $\mathbf{p}(t) = p$ *i.e. trajectory of continuous subsystem* Σ_p.

Stability of Switched Systems

- Switching between stable subsystems may create unstable behaviors:

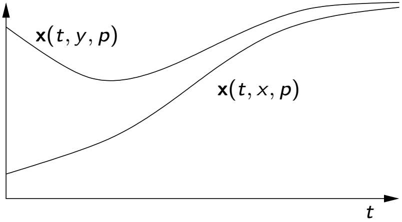


- Stability ensured via:
  - Common Lyapunov function
  - Multiple Lyapunov functions + dwell time

Incremental Stability

The subsystem Σ_p is incrementally globally asymptotically stable (δ-GAS) if there exists a $\mathcal{KL}$ function β_p such that for all $t \in \mathbb{R}_0^+$, for all $x, y \in \mathbb{R}^n$, the following condition is satisfied:

$$\|\mathbf{x}(t, x, p) - \mathbf{x}(t, y, p)\| \leq \beta_p(\|x - y\|, t)$$



δ-GAS Lyapunov Functions

A smooth function $V_p : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_0^+$ is a δ-GAS Lyapunov function for subsystem Σ_p if there exist $\mathcal{K}_\infty$ functions $\underline{\alpha}$, $\overline{\alpha}$ and $\kappa \in \mathbb{R}^+$ such that for all $x, y \in \mathbb{R}^n$:

$$\underline{\alpha}(\|x - y\|) \leq V_p(x, y) \leq \overline{\alpha}(\|x - y\|)$$

and

$$\frac{\partial V_p}{\partial x}(x, y) f_p(x) + \frac{\partial V_p}{\partial y}(x, y) f_p(y) \leq -\kappa V_p(x, y)$$

Theorem (Angeli 2002)

Σ_p is δ-GAS if and only if it admits a δ-GAS Lyapunov function.

Incremental Stability for Switched Systems

The switched system Σ is incrementally globally uniformly asymptotically stable (δ-GUAS) if there exists a $\mathcal{KL}$ function β such that for all $t \in \mathbb{R}_0^+$, for all $x, y \in \mathbb{R}^n$, for all switching signals $\mathbf{p} \in \mathcal{P}$, the following condition is satisfied:

$$\|\mathbf{x}(t, x, \mathbf{p}) - \mathbf{x}(t, y, \mathbf{p})\| \leq \beta(\|x - y\|, t)$$

The convergence rate β is independent of the switching signal $\mathbf{p}$.

Theorem

If there exists a common δ-GAS Lyapunov function for subsystems $\Sigma_1, \ldots, \Sigma_m$, then the switched system Σ is δ-GUAS.

Multiple δ-GAS Lyapunov Functions

$\mathcal{S}_{\tau_d}(\mathbb{R}_0^+, P)$ denotes the set of switching signals with dwell time τ_d.
The duration between two successive switching times is at least τ_d.

Theorem

Let $\Sigma_{\tau_d} = (\mathbb{R}^n, P, \mathcal{P}, F)$ with $\mathcal{P} \subseteq \mathcal{S}_{\tau_d}(\mathbb{R}_0^+, P)$. If for all $p \in P$, there exists a δ-GAS Lyapunov function V_p for subsystem $\Sigma_{\tau_d,p}$ and that in addition there exists $\mu \in \mathbb{R}^+$ such that:

$$\forall x, y \in \mathbb{R}^n,\ \forall p, p' \in P,\ V_p(x, y) \leq \mu V_{p'}(x, y).$$

If $\tau_d > \frac{\log \mu}{\kappa}$, then Σ_{τ_d} is δ-GUAS.

Supplementary Assumption

- In the following, we assume that there exists a $\mathcal{K}_\infty$ function γ such that, for all $p \in P$

$$\forall x, y, z \in \mathbb{R}^n,\ |V_p(x, y) - V_p(x, z)| \leq \gamma(\|y - z\|).$$

- Working on a compact subset $C \subseteq \mathbb{R}^n$:

$$|V_p(x, y) - V_p(x, z)| \leq \left(\max_{p \in P, x, y \in C} \left\| \frac{\partial V_p}{\partial y}(x, y) \right\| \right) \|y - z\|.$$

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Transition Systems

A transition system is a sextuple:

$$T = (Q, L, \longrightarrow, O, H, I),$$

where:

- a set of states Q;
- a set of labels L;
- a transition relation $\longrightarrow \subseteq Q \times L \times Q$;
- an output set O;
- an output function $H : Q \to O$;
- a set of initial states $I \subseteq Q$.

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T is said to be metric if the output set O is equipped with a metric d.

Approximate Bisimulation

$T_1 = (Q_1, L, \xrightarrow[1]{}, O, H_1, I_1)$, $T_2 = (Q_2, L, \xrightarrow[2]{}, O, H_2, I_2)$ are metric transition systems.

Let $\varepsilon \in \mathbb{R}_0^+$ be a given precision, a relation $R \subseteq Q_1 \times Q_2$ is an ε-approximate bisimulation relation between T_1 and T_2 if for all $(q_1, q_2) \in R$:

- $d(H_1(q_1), H_2(q_2)) \leq \varepsilon$;
- for all $q_1 \xrightarrow[1]{l} q_1'$, there exists $q_2 \xrightarrow[2]{l} q_2'$, such that $(q_1', q_2') \in R$;
- for all $q_2 \xrightarrow[2]{l} q_2'$, there exists $q_1 \xrightarrow[1]{l} q_1'$, such that $(q_1', q_2') \in R$.

Approximately Bisimilar Transition Systems

Transition systems T_1 and T_2 are said to be approximately bisimilar with precision ε (denoted $T_1 \sim_\varepsilon T_2$) if:

- for all $q_1 \in I_1$, there exists $q_2 \in I_2$, such that $(q_1, q_2) \in R$;
- for all $q_2 \in I_2$, there exists $q_1 \in I_1$, such that $(q_1, q_2) \in R$.

Switched Systems as Transition Systems

Consider a switched system $\Sigma = (\mathbb{R}^n, P, \mathcal{P}, F)$ with $\mathcal{P} = \mathcal{S}(\mathbb{R}_0^+, P)$ and a time sampling parameter $\tau_s \in \mathbb{R}^+$.

Let $T_{\tau_s}(\Sigma) = (Q_1, L_1, \underset{1}{\longrightarrow}, O_1, H_1, I_1)$ where:

- the set of states is $Q_1 = \mathbb{R}^n$;
- the set of labels is $L_1 = P$;
- the transition relation is given by

$$q \xrightarrow[1]{l} q' \text{ iff } \mathbf{x}(\tau_s, q, l) = q';$$

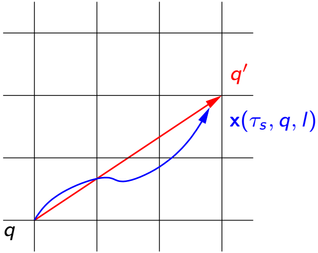
- the output set is $O_1 = \mathbb{R}^n$;
- the output function H_1 is the identity map over $\mathbb{R}^n$;
- the set of initial states is $I_1 = \mathbb{R}^n$.

Construction of the Symbolic Model

- We start by approximating the set of states $Q_1 = \mathbb{R}^n$ by:

$$[\mathbb{R}^n]_\eta = \left\{ q \in \mathbb{R}^n \;\middle|\; q_i = k_i \frac{2\eta}{\sqrt{n}},\ k_i \in \mathbb{Z},\ i = 1, ..., n \right\},$$

  where $\eta \in \mathbb{R}^+$ is a state space discretization parameter.
- Approximation of the transition relation:



Construction of the Symbolic Model

Let $T_{\tau_s,\eta}(\Sigma) = (Q_2, L_2, \underset{2}{\longrightarrow}, O_2, H_2, I_2)$ where:

- the set of states is $Q_2 = [\mathbb{R}^n]_\eta$;
- the set of labels remains the same $L_2 = L_1 = P$;
- the transition relation is given by

$$q \xrightarrow[2]{l} q' \text{ iff } \|\mathbf{x}(\tau_s, q, l) - q'\| \le \eta;$$

- the output set remains the same $O_2 = O_1 = \mathbb{R}^n$;
- the output function H_2 is the natural inclusion map: $H_2(q) = q \in \mathbb{R}^n$;
- the set of initial states is $I_2 = [\mathbb{R}^n]_\eta$.

Approximation Theorem

Theorem

Consider time and state space sampling parameters $\tau_s, \eta \in \mathbb{R}^+$ *and a desired precision* $\varepsilon \in \mathbb{R}^+$*. Let us assume that there exists* $V : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_0^+$ *which is a common* δ*-GAS Lyapunov function for subsystems* $\Sigma_1, \ldots, \Sigma_m$*. If*

$$\eta \leq \min\left\{\gamma^{-1}\left((1 - e^{-\kappa\tau_s})\underline{\alpha}(\varepsilon)\right), \overline{\alpha}^{-1}\left(\underline{\alpha}(\varepsilon)\right)\right\} \quad (1)$$

then, $T_{\tau_s}(\Sigma)$ *and* $T_{\tau_s,\eta}(\Sigma)$ *are approximately bisimilar with precision* ε*.*

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then, $T_{\tau_s}(\Sigma)$ and $T_{\tau_s,\eta}(\Sigma)$ are approximately bisimilar with precision ε.

Any precision can be achieved!

Sketch of the Proof

Idea: Show that $R = \{(q_1, q_2) \in Q_1 \times Q_2 | V(q_1, q_2) \leq \underline{\alpha}(\varepsilon)\}$ is an ε-approximate bisimulation relation:

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- For $(q_1, q_2) \in R$, $\|q_1 - q_2\| \leq \underline{\alpha}^{-1}(V(q_1, q_2)) \leq \varepsilon$.

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- For $(q_1, q_2) \in R$, $\|q_1 - q_2\| \leq \underline{\alpha}^{-1}(V(q_1, q_2)) \leq \varepsilon$.
- Let $q_1 \xrightarrow[1]{l} q_1'$, then $q_1' = \mathbf{x}(\tau_s, q_1, l)$, let $q_2 \xrightarrow[2]{l} q_2'$ then $\|\mathbf{x}(\tau_s, q_2, l) - q_2'\| \leq \eta$ and

$$\begin{aligned} V(q_1', q_2') &\leq V(q_1', \mathbf{x}(\tau_s, q_2, l)) + \gamma(\eta) \\ &\leq V(\mathbf{x}(\tau_s, q_1, l), \mathbf{x}(\tau_s, q_2, l)) + \gamma(\eta) \\ &\leq e^{-\kappa \tau_s} V(q_1, q_2) + \gamma(\eta) \\ &\leq e^{-\kappa \tau_s} \underline{\alpha}(\varepsilon) + \gamma(\eta) \leq \underline{\alpha}(\varepsilon) \end{aligned}$$

Sketch of the Proof (II)

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- Let $q_1 \in I_1 = \mathbb{R}^n$, there exists $q_2 \in I_2 = [\mathbb{R}^n]_\eta$ such that $\|q_1 - q_2\| \leq \eta$. Then,

$$V(q_1, q_2) \leq \overline{\alpha}(\|q_1 - q_2\|) \leq \overline{\alpha}(\eta) \leq \underline{\alpha}(\varepsilon).$$

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$T_{\tau_s}(\Sigma)$ and $T_{\tau_s,\eta}(\Sigma)$ are approximately bisimilar with precision ε.

Case of Multiple Lyapunov Functions

Approximation result holds if we impose a dwell time τ_d:

Theorem

Consider time and state space sampling parameters $\tau_s, \eta \in \mathbb{R}^+$, a desired precision $\varepsilon \in \mathbb{R}^+$ and a dwell time $\tau_d \in \mathbb{R}^+$. Let us assume that for all $p \in P$, there exists a δ-GAS Lyapunov function V_p for subsystem $\Sigma_{\tau_d,p}$. If $\tau_d > \frac{\log \mu}{\kappa}$ and

$$\eta \leq \min\left\{ \gamma^{-1}\left(\frac{\frac{1}{\mu} - e^{-\kappa\tau_d}}{1 - e^{-\kappa\tau_d}} (1 - e^{-\kappa\tau_s}) \underline{\alpha}(\varepsilon) \right), \overline{\alpha}^{-1}\left(\underline{\alpha}(\varepsilon)\right) \right\}$$

then, $T_{\tau_s}(\Sigma_{\tau_d})$ and $T_{\tau_s,\eta}(\Sigma_{\tau_d})$ are approximately bisimilar with precision ε.

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then, $T_{\tau_s}(\Sigma_{\tau_d})$ and $T_{\tau_s,\eta}(\Sigma_{\tau_d})$ are approximately bisimilar with precision ε.

Bound on the dwell time is the same that in the δ-GAS theorem!

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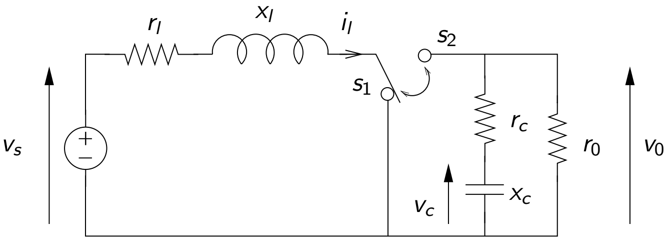
DC-DC Converter

- Power converter with switching control:



- State variable $x(t) = [i_l(t), v_c(t)]^T$.
- Control objective: regulate the output voltage
  Formulated as an invariance property.

DC-DC Converter

Dynamics of the system:

$$\dot{x}(t) = A_p x(t) + b, \quad p = 1, 2 \; .$$

where

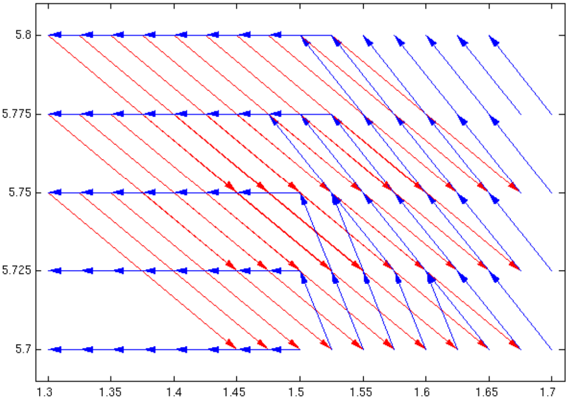
$$A_1 = \begin{bmatrix} -\frac{r_l}{x_l} & 0 \\ 0 & -\frac{1}{x_c}\frac{1}{r_0+r_c} \end{bmatrix}, \; A_2 = \begin{bmatrix} -\frac{1}{x_l}(r_l + \frac{r_0 r_c}{r_0+r_c}) & -\frac{1}{x_l}(\frac{r_0}{r_0+r_c}) \\ \frac{1}{x_c}\frac{r_0}{r_0+r_c} & -\frac{1}{x_c}\frac{1}{r_0+r_c} \end{bmatrix}, \; b = \begin{bmatrix} \frac{v_s}{x_l} \\ 0 \end{bmatrix}.$$

Existence of a common δ-GAS Lyapunov function of the form

$$V(x, y) = \sqrt{(x-y)^T M (x-y)}.$$

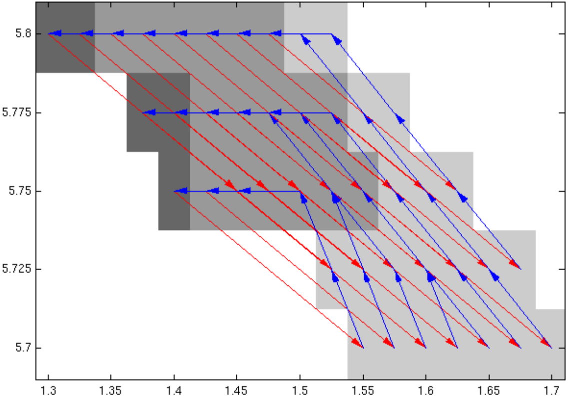
Symbolic Model of the DC-DC Converter

First (useless) abstraction: $\tau_s = 0.5$, $\eta = \frac{1}{40\sqrt{2}}$, $\varepsilon = 2.6$.



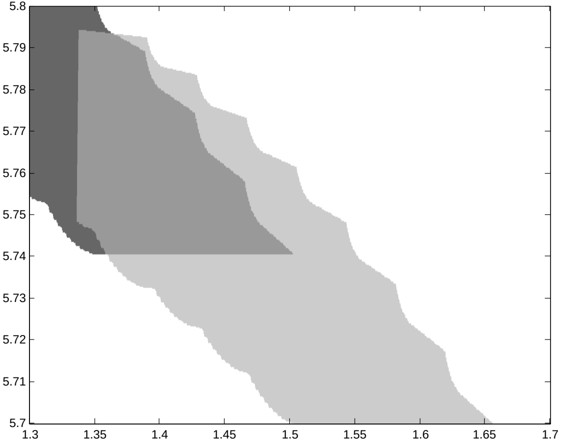
Control of the Symbolic Model

Supervisor for the symbolic model and the invariance property.



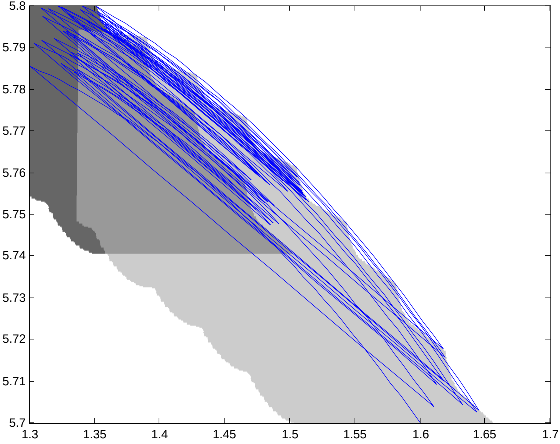
Control of the Symbolic Model (II)

Abstraction: $\tau_s = 0.5$, $\eta = \frac{1}{4000\sqrt{2}}$, $\varepsilon = 0.026$ (642001 states!).



Control of the DC-DC Converter

Corresponding trajectory of the switched system:



Conclusions

- Contributions:
  - Extension of GAS results for switched systems to δ-GAS.
  - Symbolic abstractions for a class of switched systems:
    1. Abstraction is effectively computable
    2. Precision of the abstraction can be chosen a priori
  - Application to the boost DC-DC converter
- Future work:
  - Multiscale symbolic models
  - On the fly computation of symbolic models (i.e. during control synthesis)
- References: A. Girard, G. Pola and P. Tabuada, *Approximately bisimilar symbolic models for incrementally stable switched systems*, to appear in HSCC'08.