On Hybrid Control of Under-actuated Mechanical Systems*

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Abstract. In this work we present a novel control design methodology for under-actuated mechanical systems. As part of the design process we use the reachability analysis tool **d/dt** [ABDM99,D00] to see whether there is a switching sequence which can drive the system to a desired periodic orbit. Much of the work in the design of the control law is done manually using classical control techniques (unlike the fully-automatic approach advocated in [ABD+00]), and **d/dt** is used to complement these techniques. We hope this work will contribute to the proliferation of reachability-based techniques to the control engineer's tool box.

1 Introduction

The algorithmic approach to the analysis of hybrid systems, first put forward explicitly in [ACH+95], is inspired by a computer science approach to verification of automata. The system under consideration is viewed as a generator of trajectories and the problem of verification consists of checking whether there is an individual trajectory which violates some specification, e.g. reaches a bad state. Likewise, the controller synthesis problem is phrased as restricting systematically the set of all possible behaviors in order to satisfy a property. The algorithmic approach consists in making a *brute-force search* in the state-space, based only on the description of the system dynamics. Initially this approach has been applied to restricted classes of hybrid systems where the continuous dynamics has a constant derivative in every state, see e.g. [AD94] for timed automata, and [ACH+95,AMP95,HHW97] for hybrid automata. More recently attempts have been made to lift this approach to systems with non-trivial dynamics. In particular, some of the authors were involved in the development of **d/dt**, a tool for verification and controller synthesis for hybrid systems with linear continuous dynamics [ABDM99,D00]. The synthesis algorithm implemented

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in **d/dt** [ABD+00,D00] suggested a very idealistic scenario for switching-based control: the user defines the dynamics at the various modes, as well as the control objective, and the tool automatically generates the appropriate conditions for mode switching.

This approach attempts to obtain the *general-purpose* flavor of discrete verification tools and it is still very remote from control engineering practice. In the continuous world, every class of systems has its own special character as well as its corresponding mathematical tricks which are used extensively by engineers during the controller design process. Coordinate transformations, dimensionality reduction, simplifying assumptions or linearization cannot be captured by straightforward reachability analysis.

In this paper we show how reachability-based techniques can be combined with more "knowledge-based" methods in order to derive control strategies for a non-trivial class of dynamical systems, namely under-actuated mechanical systems. We propose a general methodology for designing controllers for such systems and demonstrate it on a double-pendulum example. The complexity of the system as given initially exceeds the current capabilities of reachability-based tools: its dynamics is non-linear and control is done using continuous actuation. Moreover, the system is of dimension n while the dimensionality of the available control is $m < n$. The proposed approach to control this system by switching is based on the following principles.

1. The state-space can be transformed and partitioned via a diffeomorphism ϕ into an m-dimensional part e_1 and an $(n-m)$-dimensional part e_2.
2. Using standard control techniques, e_1 can be controlled to zero. Given this control, the remaining part is a closed system which defines the dynamics of e_2 (called the Zero dynamics).
3. Each diffeomorphism induces a different control law for its zero dynamics and hence a particular "mode" for the dynamics of the the uncontrolled part of the system. We use a parameterized family of diffeomorphisms which becomes finite after discretizing the parameters.
4. The dynamics of e_2 at each mode can be linearized around its equilibrium point. It is possible to choose the parameters so that the linearized system has periodic orbits in every mode. It should be kept in mind that the validity of the linear model is restricted to the neighborhood of the equilibrium.
5. If our goal is to reach a specific periodic orbit, we can achieve it by a sequence of mode switchings. At each mode, however, a different quantity is controlled to zero. Hence, when we switch from controlling e_1 to controlling e_1', the latter should already be close to zero. This restricts the parts of the state-space of the e_2 system where switching is allowed and leads to modeling the system as a hybrid automaton where the transition guards reflect these constraints.

The role of **d/dt** is then to check whether, based on the hybrid automaton representation, it is possible to reach from one orbit to another by mode switching and how much time it takes.

2 Control of Under-actuated Mechanical Systems

2.1 Under-actuated Mechanical Systems

We consider the class of jointed mechanical systems without flexibilities, the dynamics of which is given by Lagrange equations:

$$M(q)\ddot{q} + N(q, \dot{q}) = W\Gamma \tag{1}$$

where M is the symmetric positive definite matrix defining the kinetic energy and N gathers generalized gravity, Coriolis and centrifugal forces; q is the $n-$dimensional vector of generalized (joint) coordinates; Γ includes all external generalized forces and W is a constant matrix.

If we now assume that the generalized forces are only actuation torques/forces (i.e the system is friction-free and no other potential-based actions occur), then the system is called *under-actuated* if rank $(W) < n$. Without loss of generality, we can consider that $W = \begin{pmatrix} I_m \\ 0_{n-m\times n} \end{pmatrix}$ with $m < n$ the number of actuators.

2.2 Zero Dynamics

Let us consider a diffeomorphism ϕ:

$$q \to \ \phi(q) = \begin{pmatrix} e_1(q) \\ e_2(q) \end{pmatrix} \tag{2}$$

where e_1 is m-dimensional. Then, the dynamics (1) projected on the constraint $e_1 = 0$ is called the *zero dynamics* associated with ϕ. It is given by:

$$P(q)(M(q)\ddot{q} + N(q, \dot{q})) = 0 \tag{3}$$

with $P = I_n - W(J_1 M^{-1} W)^{-1} J_1 M^{-1}$ the projection operator, in which $J_1 = \frac{\partial e_1}{\partial q}$. A control objective can therefore be to bring the system to this zero dynamics, specified by the goal task $e_1 = 0$, and to stabilize it. Since dim (e_1) = dim (Γ), all the available actuation forces/torques have to be used for that purpose. In fact, that can be done trough partial decoupling/feedback linearization: it can be easily seen that using the control

$$\Gamma = (J_1 M^{-1} W)^{-1}(u - \dot{J}_1 \dot{q} + J_1 M^{-1} N) \tag{4}$$

we obtain $\ddot{e}_1 = u$, assumed that $J_1 M^{-1} W$ is nonsingular. It then remains to specify an adequate input u which stabilizes e_1, asymptotically or in finite time, in order to drive the system to the zero dynamics. Once reached, its motion is then governed by eq. (3), which is free, since no more control is available. In many cases, this free motion is a periodic orbit. The idea now is to specify such a periodic orbit as a final goal, recalling that we can consider the choice of ϕ as a way to modify it. The problem addressed in the following is then to study the reachability of this behavior starting from given initial conditions, using a sequence $\phi_1, \phi_2 \ldots$, i.e successive jumps from an orbit to another one.

2.3 Handling the Periodic Orbits

Let us consider the case where $m = n-1$, i.e. the zero dynamics can be expressed using a single coordinate denoted by x_1. When the phase portrait of the system is a closed curve O, this periodic orbit, which characterizes the zero dynamics, can be uniquely specified by a pair (ϕ, X^0) where X^0 is a point on the orbit, for example the initial conditions. Let us assume (assumption A0) that the equation of O in the phase plane is of the form $V(x_1, \dot{x_1}) - \tilde{V} = 0$, the invariance being expressed by $\dot{V} = 0$. V is a so-called *Lyapunov* function. For a non-actuated conservative mechanical system, the natural V is the mechanical energy. Since it is not the case here, $\tilde{V}$ can only be called by analogy the "energy" level of the orbit.

Let us now consider the particular case where the set of ϕ_i consists of functions of given analytical form depending on a k-dimensional vector of real parameters p. Then p can be considered as an auxiliary control of the system. Giving some bounds to the parameters and the variables, so that they range over D_p and D_{X^0}, respectively, the set of all possible orbits for the system is

$$\mathbf{0} = \{O(p, X^0) : p \in D_p \; X^0 \in D_{X^0}\}.$$

When V is known, the set can also be parameterized by p and $\tilde{V}$.

The problem we address now is the following: let us define a desired behavior of the system as a goal orbit O^*; then, given an initial orbit $O_0 \neq O^*$, can we reach O^* by modifying p? We don't consider here related problems of automatic control: existence of the orbits, active stabilization, continuous control of p, which will be addressed in forthcoming papers. Instead, we focus our attention on a *discrete* approach, i.e. to the questions: is there a sequence of intersecting orbits allowing to reach O^* through jumps on the parameters and how long time will it take? Assuming here that these jumps are instantaneous and don't disturb the overall behavior (assumption A1), we can therefore forget the effect of the control (4) and consider for the analysis the related set of zero dynamics uniquely. We are therefore led back to a problem of reachability analysis of a hybrid system: each discrete state is an homogeneous differential equation associated with given values of the parameters; transitions are allowed when orbits of different modes are compatible with each other, i.e. when continuous state variables reach some particular values. We will illustrate the approach on the double pendulum example.

3 The Case of the Double Pendulum

The considered testbed is the double pendulum depicted in Figure 1. The reader is referred to [EGP99] for details on experimental issues. Terms in eq. (1) write for this system as:

$$M = \begin{pmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{pmatrix} \tag{5}$$

and:

$$N = \begin{pmatrix} N_1(q_1, q_2, \dot{q}_1, \dot{q}_2) \\ N_2(q_1, q_2, \dot{q}_1, \dot{q}_2) \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \end{pmatrix} + \begin{pmatrix} G_1 \\ G_2 \end{pmatrix} \tag{6}$$

with:

$$\begin{array}{l} m_{11} = m_1 l_1^2 + m_2(l_2^2 + L_1^2 + 2L_1 l_2 c2) \\ m_{12} = m_2(l_2^2 + L_1 l_2 c2) \\ m_{22} = m_2 l_2^2 \\ C_{11} = -m_2 L_1 l_2 s2 \dot{q}_2 \\ C_{12} = -m_2 L_1 l_2 s2(\dot{q}_1 + \dot{q}_2) \\ C_{21} = m_2 L_1 l_2 s2 \dot{q}_1 \\ C_{22} = 0 \\ G_1 = g((m_1 l_1 + m_2 L_1) s1 + m_2 l_2 s12) \\ G_2 = g m_2 l_2 s12 \end{array} \tag{7}$$

where $si := sin(q_i)$, $ci := cos(q_i)$, $sij := sin(q_i + q_j)$. We consider the case where only the hip is actuated. Therefore $W = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Let us now choose the diffeomorphism ϕ and the control Γ such that

$$e_1 = q_1 - a q_2 - b = 0 \; ; \; e_2 = q_2 \tag{8}$$

where a and b are two real parameters[1]. Therefore the zero dynamics we have to consider is simply:

$$\begin{cases} (m_{22} + a m_{12})\ddot{q}_2 + (a C_{21} + C_{22})\dot{q}_2 + G_2 = 0 \\ q_1 = a q_2 + b \end{cases} \tag{9}$$

where it assumed that $m_{22} + a m_{12} \neq 0$ (assumption A2, satisfied when $-\frac{l_2}{L_1+l_2} < a < \frac{l_2}{L_1-l_2}$). This system can be expressed in the single coordinate q_2. It is a second order nonlinear differential equation, for which the natural state vector is $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} q_2 - q_2^* \\ \dot{q}_2 \end{pmatrix}$. In order to perform reachability analysis, we have to linearize the system. Its equilibrium points $X^* = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ are solutions of $G_2(q_2^*) = 0$, i.e, for $a \neq -1$ (assumption A3):

$$q_2^* = -\frac{b + k\pi}{1 + a} \tag{10}$$

We consider in the following only the case $k = 0$. The equation of the system linearized around the center q_2^* is:

$$\dot{x} = Ax = \begin{pmatrix} 0 & 1 \\ -\alpha & 0 \end{pmatrix} x \tag{11}$$

[1] Note that expression (8) specifies the desired spatial trajectory of the tip of the double pendulum, while the "energy" level will set the amplitude and the time profile of its motion along this trajectory

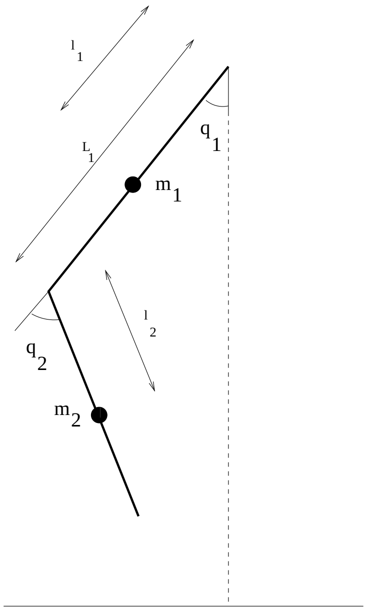

Fig. 1. A double pendulum.

where $\alpha = l_2 + \frac{a}{1+a} L_1 cos(\frac{b}{1+a})$. For ensuring the existence of periodic orbits, the eigenvalues of A have to be imaginary, which implies that α has to be strictly positive (assumption A4). The Lyapunov function associated with the system, i.e the energy level of an orbit is

$$V = \frac{1}{2}(\alpha x_1^2 + x_2^2) \tag{12}$$

For the purpose of reachability analysis it is more comfortable to work with the same system of coordinates in every state, hence we transform the linear dynamics of equation (11) into an affine dynamics over $y = (q_2, \dot{q_2})$:

$$\dot{y} = Ay + u = \begin{pmatrix} 0 & 1 \\ -\alpha & 0 \end{pmatrix} y + \begin{pmatrix} 0 \\ \alpha q_2^* \end{pmatrix} \tag{13}$$

Finally we have to remember that the system is submitted to physical bounds on the joints: $q_i \in [q_i^{min}, q_i^{max}]$. Introducing them in (8) leads to linear constraints on the parameters.

When we switch from ϕ to ϕ' there might be a transient period until the system settles in the new zero dynamics. In order to make assumption A1 (transitions are immediate) realistic we need to make sure that e'_1 and $\dot{e}'_1$ be already close to their zero. For q_1 this means

$$|e'_1| = |q_1 - a'q_2 - b'| < \epsilon_1 \tag{14}$$

Since $q_1 = aq_2 + b$ this reduces to

$$|(a - a')q_2 + (b - b')| < \epsilon_1 \tag{15}$$

For $\dot{q}_2$ we need:

$$|\dot{e}'_1| = |(a - a')\dot{q}_2| < \epsilon_2 \tag{16}$$

These conditions, which form rectangles in the phase-space of the zero dynamics, will be used as transition guards in the hybrid automaton model. Note that these conditions are symmetric, i.e. they are the same, in terms of q_2 and $\dot{q}_2$ for the transitions from (a', b') to (a, b). Of course, their global physical interpretation does depend on the source state of the transition.

The system is modeled as a hybrid automaton with 7 states, each representing a pair (a, b) of parameters (Figure 2). At each state the dynamics is of the form $\dot{x} = Ax + u$ where A and u for the various states are:

s_0	s_1	s_2	s_3	s_4	s_5	s_6
$\begin{matrix} 0 & 1 \\ -0.0479 & 0 \end{matrix}$	$\begin{matrix} 0 & 1 \\ -0.0878 & 0 \end{matrix}$	$\begin{matrix} 0 & 1 \\ -0.1167 & 0 \end{matrix}$	$\begin{matrix} 0 & 1 \\ -0.1982 & 0 \end{matrix}$	$\begin{matrix} 0 & 1 \\ -0.2326 & 0 \end{matrix}$	$\begin{matrix} 0 & 1 \\ -0.3143 & 0 \end{matrix}$	$\begin{matrix} 0 & 1 \\ -0.3555 & 0 \end{matrix}$
$\begin{matrix} 0 \\ 0.0011 \end{matrix}$	$\begin{matrix} 0 \\ 0.0000 \end{matrix}$	$\begin{matrix} 0 \\ -0.0012 \end{matrix}$	$\begin{matrix} 0 \\ 0.0000 \end{matrix}$	$\begin{matrix} 0 \\ -0.0039 \end{matrix}$	$\begin{matrix} 0 \\ -0.0090 \end{matrix}$	$\begin{matrix} 0 \\ -0.0140 \end{matrix}$

The transition guards are computed according to (15) and (16) with $\epsilon_1 = 0.05$ and $\epsilon_2 = 0.02$. In addition, we restrict the transitions to happen between pairs of "close" states, i.e. $|a - a'| \leq 0.15$ and $|b - b'| \leq 0.1$.

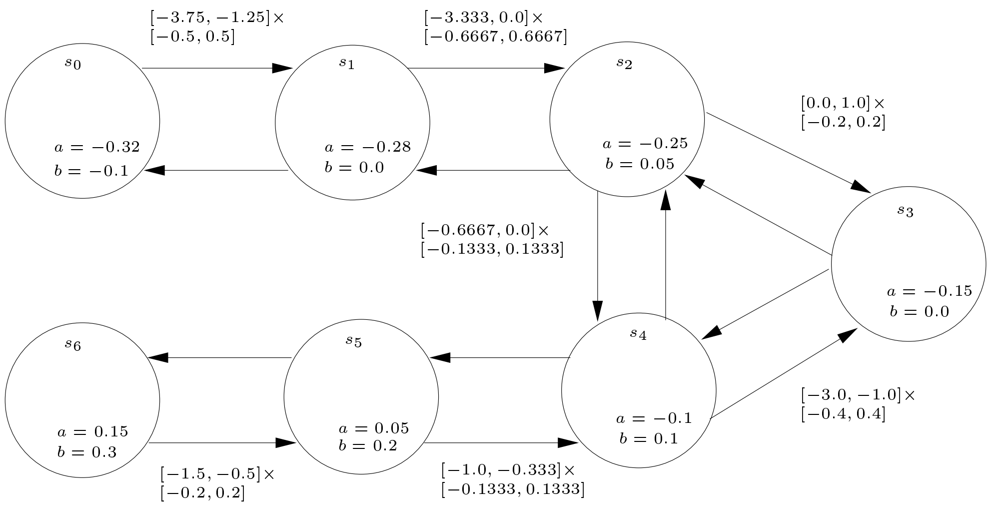

Fig. 2. The hybrid automaton for the double pendulum. The transition guards between pairs of states are written as products of intervals.

In order to facilitate the experimentation with **d/dt** we have augmented the input syntax to include parameters and formulae referring to them. For example, state s_0 and its outgoing transition is specified as:

```
state: 0;
matrixA:
 0.0  1.0,
 [-l2-(a0/(1+a0))*L1*cos(b0/(1+a0))]  0.0;
input: type convex_vert
 0.0 [(b0/(1+a0))*(-l2-(a0/(1+a0))*L1*cos(b0/(1+a0)))];
transition:
  label go01:
  if in guard:  type rectangle
    [-(-eps1+(b0-b1))/(a0-a1)]  [-(eps1+(b0-b1))/(a0-a1)],
    [eps2/(a0-a1)]  [-eps2/(a0-a1)];
  goto 1;
```

4 Results

The problem we solve with **d/dt** is the following: *given some initial low-energy orbit (more precisely, a connected set of orbits) is there a sequence of switchings that brings the system to its target, a higher-energy set of orbits?* This problem is essentially a controller synthesis problem for the *eventuality* specification, unlike the *safety* controller synthesis that we have treated in [ABD+00]. We are interested in reaching the desired orbit with the least number of mode switchings.

We illustrate informally the synthesis procedure that we employ in order to derive the switching controller. Consider an initial set of orbits characterized by the rectangle (in the $(q_2, \dot{q}_2)$ space) $P = [0.7 \times 0.9] \times [0.01, 0.02]$ at state s_3 and a goal orbit characterized by $F = [1.05, 1.3] \times [0.01, 0.02]$ at the same state. Starting from the inital set (s, P) we calculate, in a *breadth-first* manner, all its successors, i.e. continuous successors, and then, via intersection with the guards, the discrete successors. We continue until at some level k of the search tree, there is one or more paths having a leaf (s, Q) such that Q intersects F. The search graph of the first iteration is shown in Figure 3 and there are two intersections with the goal orbit after 4 transitions, along the paths s_3, s_2, s_3, s_2, s_3 and s_3, s_2, s_1, s_2, s_3. For every such path we do backward reachability analysis to find the predecessors of the goal orbit at every node and, in particular, the subset of P from which the goal can be reached by taking the k transitions that correspond to the path. This information is also used to derive the controller by restricting the guards. In our example we conclude that points satisfying $q_2 \in [0.7552, 0.9]$ can reach the goal orbit by following the sequence s_3, s_2, s_3, s_2, s_3 and those satisfying $q_2 \in [0.7152, 0.9]$ can do it following the sequence s_3, s_2, s_1, s_2, s_3. Note that from the interval $[0.7552, 0.9]$ both sequences can be taken.

If not all points in P are "covered" by the k-length sequences found in the first iteration, we restart the procedure from (s, P') where $P' \subseteq P$ is the subset of P

consisting of the points not covered yet. In our example P' consists of the points satisfying $q_2 \in [0.7, 0.7152]$. In the second iteration we find out that the goal orbit can be reached from any point in P' by either one of the three 6-transition sequences $s_3, s_2, s_3, s_2, s_3, s_2, s_3,\ \ s_3, s_2, s_3, s_2, s_1, s_2, s_3$ and $s_3, s_2, s_1, s_2, s_1, s_2, s_3$, and this concludes the computation. The fact that $\dot{q}_2$ does not matter here is particular to this example — with other sets of parameters the partition of the initial set did involve conditions on $\dot{q}_2$. The reachable states which correspond to the discovery of the sequence $s_3, s_2, s_1, s_2, s_1, s_2, s_3$ in the second iteration are depicted in Figure 4 and 5.

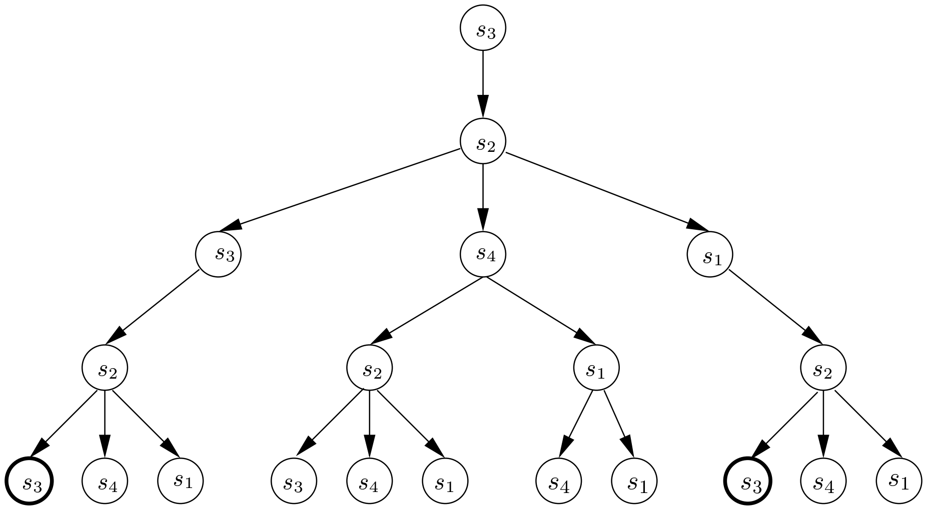

Fig. 3. The first iteration of the search tree. The goal orbits were first reached after 4 transitions along two paths of the tree.

5 Conclusion

We have investigated a new methodology for designing hybrid controllers which is partially-supported by our reachability analysis tool **d/dt**. Like [ABD+00] and [TLS00] this work explores the contribution of the hybrid automaton model to the alternative formulation and solution of problems in switching-based control. In this paper we have treated an interesting and open problem in robot control and provided a partial solution. To improve the performance of the algorithm, we plan to investigate other search procedures (backward computation and heuristic search) and validate our results via simulation.

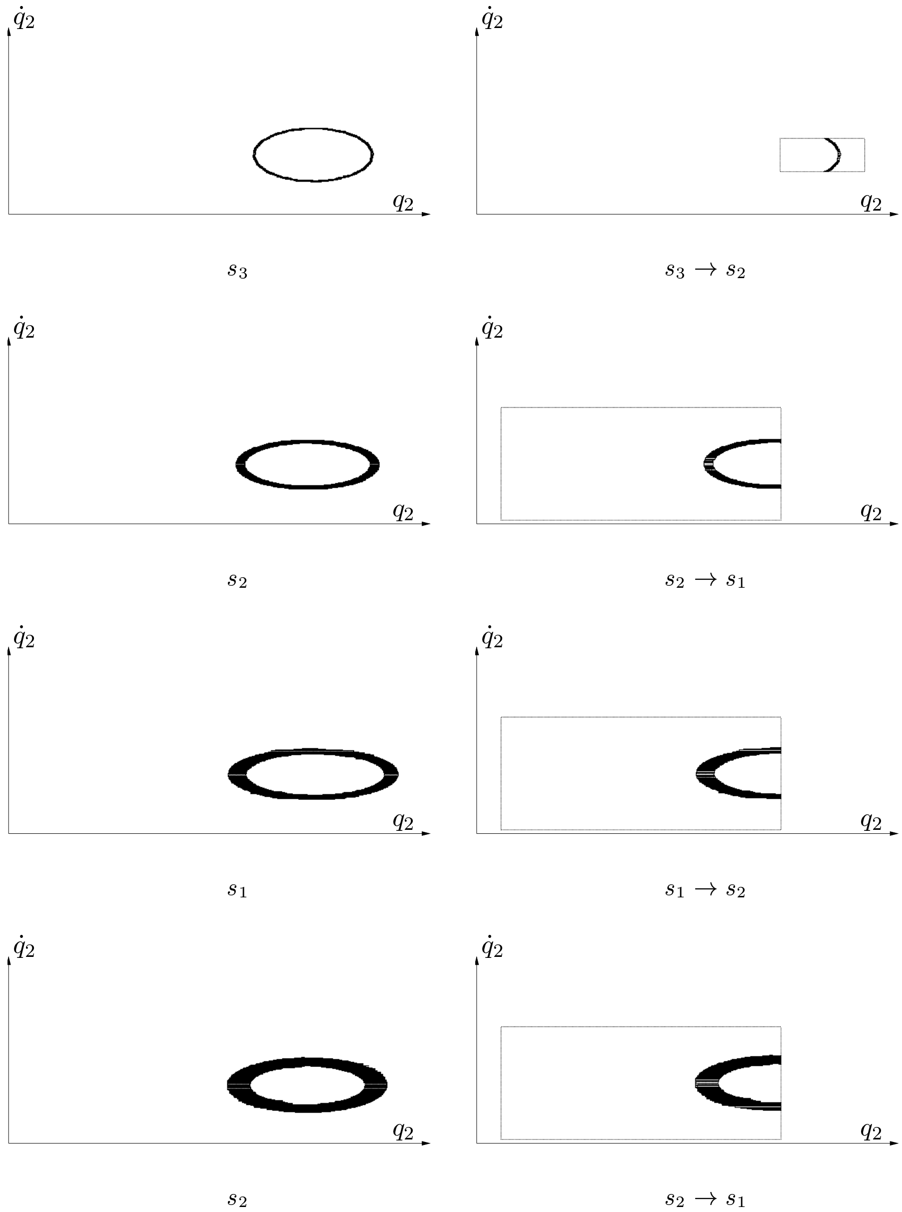

Fig. 4. Computation of reachable states for the sequence s_3, s_2, s_1, s_2. On the left we see the reachable set at mode s_i while at the right we show the intersecion with the guard from s_i to s_j.

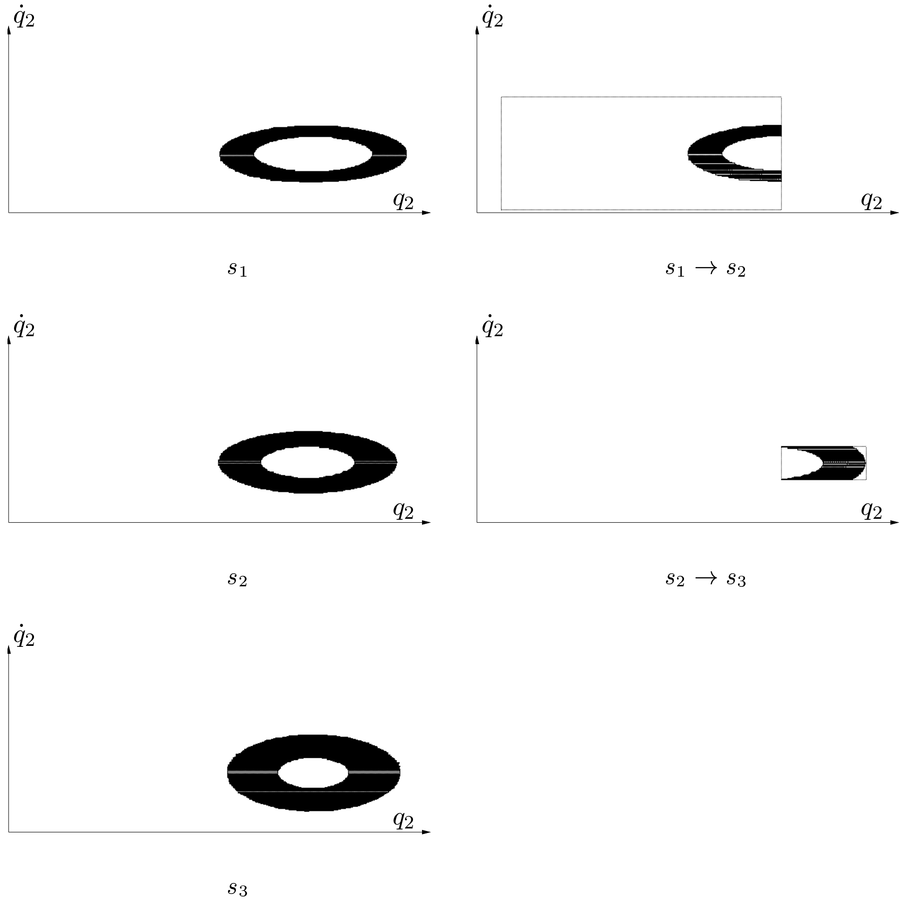

Fig. 5. Computation of reachable states for the sequence $s_3, s_2, s_1, s_2, s_1, s_2, s_3$ continued from Figure 4.

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