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- Definition
- Differential equations
- Exponential Functions
- Examples
- Initial Value and Final Value Theorems

Definition

$$\mathcal{L}x(s) = \int_0^\infty x(t)e^{-st}$$

(under the existence condition)

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Because the Laplace transform enables transforming ordinary differential equations into algebraic equations (that we know how to manipulate usually)

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thanks to two properties:

1. **Linearity**:

$$\mathcal{L}(\alpha x + \beta y) = \alpha \mathcal{L} x + \beta \mathcal{L} y$$

2. The derivatives are transformed into **products**:

$$\mathcal{L}(x')(s) = s\mathcal{L}x(s) - x(0)$$

Proof

Integration by parts

$$\int_0^\infty x'(t)e^{-st} = [x(t)e^{-st}]_0^\infty - \int_0^\infty x(t)(-s)e^{-st}$$

if $\lim_{t\to\infty} x(t)e^{-st} = 0$, we have

$$\mathcal{L}(x')(s) = s\mathcal{L}x(s) - x(0)$$

Example: an ODE

First-order linear differential equation with constant coefficients:

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$$(s + a)\mathcal{L}y(s) = b\mathcal{L}x(s) + y(0)$$

Example: an ODE (cont'd)

$$(s + a)\mathcal{L}y(s) = b\mathcal{L}x(s) + y(0)$$

$$\boxed{\mathcal{L}y(s) = \frac{1}{s + a}(b\mathcal{L}x(s) + y(0))}$$

Example: an ODE (cont'd)

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$$\boxed{\mathcal{L}y(s) = \frac{1}{s+a}\left(b\mathcal{L}x(s) + y(0)\right)}$$

This differential equation has been "solved" and the solution is a **rational fraction**

Exponential Functions

The **exponential (gamma) functions** are transformed into **rational fractions**:

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$$\mathcal{L}\gamma_n(s) = \left(\frac{1}{s+\lambda}\right)^{n+1}$$

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By induction :

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$$\int_0^\infty e^{-\lambda t} e^{-st} = \int_0^\infty e^{-(s+\lambda)t} = \left[-\frac{e^{-(s+\lambda)t}}{s+\lambda} \right]_0^\infty = \frac{1}{s+\lambda}$$

provided that $\lim_{t \to \infty} e^{-(s+\lambda)t} = 0$

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Integration by parts

$$\begin{aligned} \int_0^\infty \frac{t^{n+1}}{(n+1)!} e^{-\lambda t} e^{-st} &= \left[-\frac{t^{n+1}}{(n+1)!} \frac{e^{-(s+\lambda)t}}{s+\lambda} \right]_0^\infty \\ &\quad - \int_0^\infty -\frac{t^n}{n!} \frac{e^{-(s+\lambda)t}}{s+\lambda} \\ &= \frac{1}{s+\lambda} \int_0^\infty \frac{t^n}{n!} e^{-(s+\lambda)t} \end{aligned}$$

Examples of signal

Step signal: $u(t) = \begin{cases} 1 \text{ si } t \geq 0 \\ 0 \text{ otherwise} \end{cases}$

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$$\mathcal{L}r(s) = \frac{1}{s^2}$$

Other signals

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$$\begin{aligned}
\mathcal{L}sin(s) &= \frac{1}{2i}\left(\frac{1}{s - i\omega} - \frac{1}{s + i\omega}\right) \\
&= \frac{1}{2i}\frac{s + i\omega - (s - i\omega)}{(s - i\omega)(s + i\omega)} \\
&= \frac{\omega}{s + \omega}
\end{aligned}$$

Example of System

First-order system:

$$\mathcal{L}y(s) = \frac{1}{s+a}(b\mathcal{L}x(s) + y(0))$$

Response to a step signal that starts with $y(0) = 0$:

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Response to a step signal that starts with $y(0) = 0$:

$$\mathcal{L}x(s) = \frac{1}{s}$$

$$\mathcal{L}y(s) = \frac{b}{s(s+a)}$$

Resolution

Partial fraction decomposition: $\dfrac{b}{s(s+a)} = \dfrac{A}{s} + \dfrac{B}{s+a}$

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$$\frac{b}{s+a} = A + \frac{Bs}{s+a}$$

$s = 0$

$$\frac{b}{0+a} = A$$

$$\frac{b(s+a)}{s(s+a)} = \frac{A(s+a)}{s} + \frac{B(s+a)}{s+a}$$

$$\frac{b}{s} = \frac{A(s+a)}{s} + B$$

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$\mathcal{L}y(s) = \frac{b}{a}\left(\frac{1}{s} - \frac{1}{s+a}\right)$. Hence, $y(t) = \frac{b}{a}(1 - e^{-at})$

Other Method

Using the Laplace approach, we need find the **poles** (roots of the polynomials)
However, we know how to do this for polynomials of degrees less than or equal to 5 (already difficult beyond degree 2)

Otherwise, **numerical integration**

Other Properties

Initial Value Theorem:

$$\boxed{\lim_{t \to 0} x(t) = \lim_{s \to \infty} s\mathcal{L}x(s)}$$

if the limits exist

Final Value Theorem:

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Applications

How to know the **final value** of the responses of a system to a step signal **without calculating the solution**?

Take the Laplace transform of the response (in the running example):

$$\mathcal{L}y(s) = \frac{b}{s(s+a)}$$

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How to know the **final value** of the responses of a system to a step signal **without calculating the solution**?

Take the Laplace transform of the response (in the running example):

$$\mathcal{L}y(s) = \frac{b}{s(s+a)}$$

It suffices to use the **Final Value Theorem**:

$$\lim_{s \to 0} s \frac{b}{s(s+a)} = \lim_{s \to 0} \frac{b}{(s+a)} = \frac{b}{a}$$