Fundamentals of Dynamical Systems

Course “Feedback Control and Real-Time Systems”

Master of Science in Informatics at Grenoble
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NOTICE

The slides do not cover all the contents of the Feedback Control part. Some exercises are solved on the white board.

Overview

Dynamical Systems

- Fundamentals of Dynamical Systems
- Specifying and Analyzing Properties

Fundamentals of Dynamical Systems

goal: a unified view of seemingly disparate systems

- using the same concepts
- adapting techniques where necessary
- combining different techniques when systems have heterogeneous components

... which they do in cyber-physical systems!

examples?

dynamical system

- precisely identified entity
  (we know what is part of the system and what isn't)
- defining behaviors over some notion of time
  (we know what "before" and "after" mean)
- with (possibly) observable outputs
- (possibly) influenced by a given set of inputs

examples for what is *not* a dynamical system?

Fundamentals of Dynamical Systems

behavior

- evolution of states over time
- (possibly) decorated with input or output

formalized as *executions, runs, words, traces, trajectories*,…

examples?

disturbance

- something that modifies the inputs or outputs of the system

random changes in the environment, electromagnetic interference, sensor noise, quantization error(!)

more examples?

Fundamentals of Dynamical Systems

deterministic system

- if the inputs are known, there is only one future behavior

nondeterministic system

- if the inputs are known, there is a known set of future behaviors
  (actual behavior may be different each time we run the system, but belong to the same set)

examples?

stochastic system

- if the inputs are known, the future behavior is known with a certain probability
  (it's the same behavior xyz% of the times we run the system)

examples?

Fundamentals of Dynamical Systems

state

- a set of independent physical quantities, the specification of which **completely** determines the future behavior of the system if the inputs are known

state-space

- the set of states of the system

example: motion of a car (with accelerator and brake pedals)

Fundamentals of Dynamical Systems

transition

- relates a state to a successor state
- may depend on time and inputs

transition relation

- defines for each state the possible **successor** states
- a subset of states $\times$ time $\times$ inputs $\times$ states

$$\text{state} \xrightarrow{\text{time,input}} \text{state}'$$

reachable states

- states in the closure of the transition relation
- starting from a given set of **initial states**

finite state system

- the state space and the inputs are finite sets

What is maximum size of the transition relation (deterministic/nondeterministic)?

state-space exploration (enumerative)

- starting from a given initial state, visit all reachable states, trying out all possible inputs
- = graph traversal, e.g., breadth-first search

always **terminates** if the state space is **finite**

example: check if the system can go to a given ”bad” state

infinite state system

- the state space is an infinite set (enumerable or not)

state-space exploration no longer terminates

Fundamentals of Dynamical Systems

symbolic state-space exploration

- like state-space exploration, but using sets of states
- terminate if successors $\subseteq$ visited states or bad states overlap visited states
- often uses **overapproximation** to operate on sets with simple descriptions (intervals…)

may terminate even if state space is **infinite**

Fundamentals of Dynamical Systems

Example: A program computing $\sqrt{x_0}$ using the babylonian method

$$x_{k+1} = \tfrac{1}{2}\left(x_k + \frac{x_0}{x_k}\right).$$

implemented using int,float,rationals,reals,…

- state-space? initial state? inputs? outputs? time?
- transition relation? behaviors?
- deterministic? finite?

Exercises:

Given an implementation using int,float,rationals,reals,…

1. When is enumerative state-space exploration applicable?
2. How to check if the sequence converges to $\sqrt{x_0}$?
3. Apply symbolic state-space exploration starting from $x_0 = 8$. Use integer intervals to describe sets of states. Overapproximate if necessary.
4. Start from $x_0 = 9$. How can the precision be increased?
5. Does always rounding up or always rounding down cover all possibilities?

Fundamentals of Dynamical Systems

Discrete-Time Dynamical System:

$$x_{k+1} = f\left(x_k, u_k\right).$$

- state-space? initial state? inputs? outputs? time?
- why ”discrete-time”?
- transition relation?
- deterministic? finite?

Examples: Finite state machine (digital computer)

Discrete-Time **Continuous** Dynamical System:

$$x_{k+1} = f\left(x_k, u_k\right).$$

- f is a continuous function of x and u:
- a small enough change in the input (or in time) generates an arbitrarily small change in the output

Examples: Digital controller (considering floating point as real numbers); sun position at noon every day

Fundamentals of Dynamical Systems

Discrete-Time **Continuous** Dynamical System:

$$x_{k+1} = f\left(x_k, u_k\right).$$

two main categories:

- f is **linear**: $x_{k+1} = Ax_k + Bu_k$
  either converging, diverging, or periodic
- f is **nonlinear**:
  possibly chaotic behavior

Linear Map

scalar case:

$$x_{k+1} = ax_k$$

for which values of a:

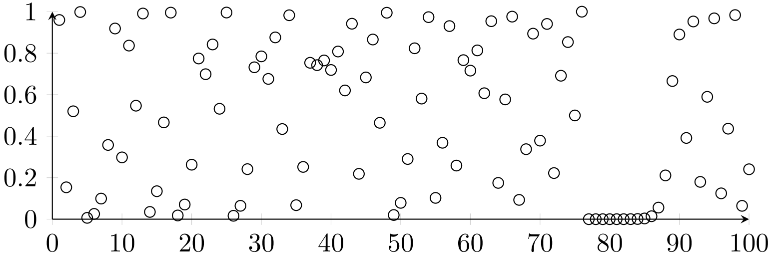
- converging,
- diverging,
- periodic?

Logistic Map

demographic model with reproduction and starvation

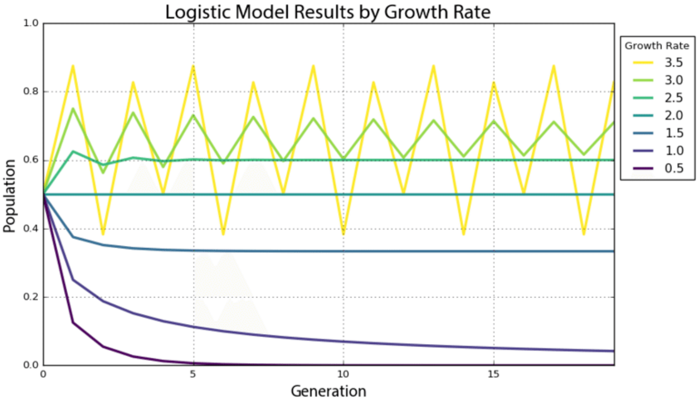
[R. May, 1976]

$$x_{k+1} = rx_k(1 - x_k)$$



$x_0 = 0.6$, $r = 4$

Logistic Map



Fundamentals of Dynamical Systems

Discrete-Time **Piecewise Continuous** Dynamical System:

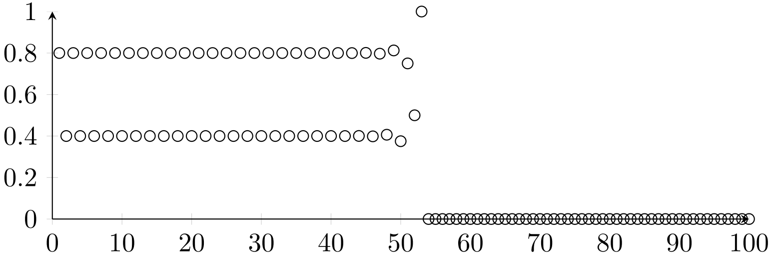
$$x_{k+1} = \begin{cases} f_1\left(x_k, u_k\right), & x_k \leq c_1 \\ \vdots & \\ f_i\left(x_k, u_k\right), & c_{i-1} < x_k \leq c_i \\ \vdots & \\ f_m\left(x_k, u_k\right), & x_k > c_m \end{cases}$$

- may exhibit complex behavior even for simple f_i

Example: continuous systems with saturation of signals

Tent Map

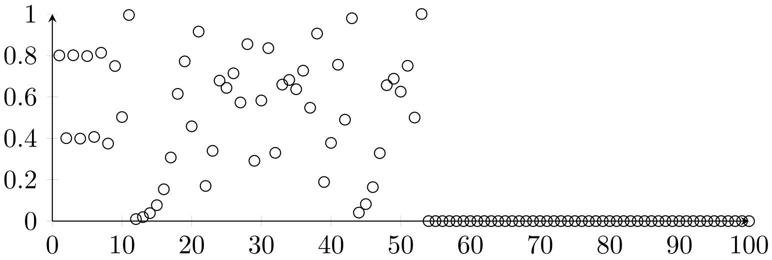
$$x_{k+1} = \begin{cases} \mu x_k, & x_k < \frac{1}{2}, \\ \mu(1 - x_k), & x_k \geq \frac{1}{2} \end{cases}$$



$x_0 = 0.6$, $\mu = 2$

Tent Map

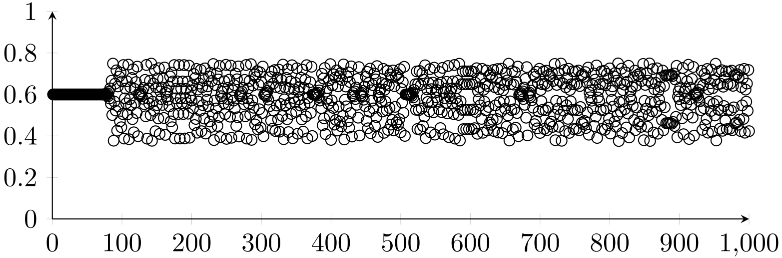
$$x_{k+1} = \begin{cases} \mu x_k, & x_k < \frac{1}{2}, \\ \mu(1 - x_k), & x_k \geq \frac{1}{2} \end{cases}$$



$x_0 = 0.6001$, $\mu = 2$

Tent Map

$$x_{k+1} = \begin{cases} \mu x_k, & x_k < \frac{1}{2}, \\ \mu(1 - x_k), & x_k \geq \frac{1}{2} \end{cases}$$



$x_0 = 0.6$, $\mu = 1.5$

Tent Map vs Logistic Map

tent map:

$$x_{k+1} = \begin{cases} \mu x_k, & x_k < \frac{1}{2}, \\ \mu(1 - x_k), & x_k \geq \frac{1}{2} \end{cases}$$

logistic map:

$$y_{k+1} = r y_k (1 - y_k)$$

for $\mu = 2$ and $r = 4$:

$$x_k = \frac{2}{\pi} \sin^{-1} \sqrt{y_k}$$

relation between *piecewise linear* and *nonlinear* system

Fundamentals of Dynamical Systems

Continuous-Time (Continuous) Dynamical System:

Typically given by a differential equation system:

$$\dot{x}(t) = f\left(x(t), u(t)\right).$$

- can be converted to discrete-time system by **sampling** at time points, e.g., $t = k\delta$

Example: Motion of a car

Discrete-Continuous Dynamical System (Hybrid System):

- mix of discrete and continuous dynamics
- discrete state changes are considered instantaneous
- discrete state determines continuous dynamics

Example: Motion of a car with gear shift

Levels of Abstraction

discrete (state) system (discrete dynamics)

continuous system (continuous dynamics)

- discrete or continuous time

discrete-continuous (hybrid) system

What is the ”right” model?

Example: Robot in a Maze

- robot turning at exactly 90 degrees, timing irrelevant:
  discrete system
  Can the robot leave the maze?
- maze door opens and closes at specific times: **timed system**
  Can the robot leave the maze while the door is open?
- robot not turning exactly 90 degrees:
  hybrid (discrete-continuous) system
  accumulation of deviations!
  Can the robot leave the maze while the door is open?

Overview

Dynamical Systems

- Fundamentals of Dynamical Systems
- Specifying and Analyzing Properties

Specifying and Analyzing Properties

boolean properties

- state property: state $\rightarrow \{\text{true}, \text{false}\}$
  e.g., predicate over the state variables
- behavior property: behavior $\rightarrow \{\text{true}, \text{false}\}$
  e.g., all states along the behavior satisfy the property
- system property: system $\rightarrow \{\text{true}, \text{false}\}$
  e.g., all behaviors from initial states satisfy property

these generalizations to behaviors over time are called
temporal logics

examples?

Specifying and Analyzing Properties

probabilistic properties

- behavior probability: behavior $\rightarrow [0, 1]$
  e.g., probability that the behavior is taken (from given initial state)
- probabilistic property: system $\rightarrow [0, 1]$
  e.g., probability that any behavior from the initial states satisfies the property

these generalizations to behaviors over time are called **probabilistic temporal logics**

examples?

Specifying and Analyzing Properties

safety: nothing bad ever happens

analysis techniques:

- inductive invariants,
- state-space exploration (enumerative, symbolic)

examples?

Specifying and Analyzing Properties

liveness: something good eventually happens

analysis techniques:

- temporal logics,
- model checking (generalization of state-space exploration),
- ranking functions

examples?

Specifying and Analyzing Properties

probabilistic safety and liveness: nothing bad/something good happens with a certain probability

analysis techniques:

- probabilistic temporal logics,
- model checking,
- fault-tree analysis

examples?

Specifying and Analyzing Properties

quantitative semantics of safety and liveness: distance to nothing bad/something good happening

- e.g., min distance of any behavior to violating the property

Example: $x(t) \leq c$ for all $t \geq 0$ (boolean safety)

- quantitative semantics $q = \min_{t \geq 0} c - x(t)$
  property satisfied iff $q \geq 0$.

measure of **robustness**

Specifying and Analyzing Properties

real-time scheduling: system achieves given tasks in given time frame

analysis techniques:

- model checking,
- worst-case execution time (WCET) analysis

quantitative property:

- computing worst-case execution time

examples?

Specifying and Analyzing Properties

stability: system will remain close to its steady state if disturbances (inputs) small enough

analysis techniques:

- linear algebra,
- Lyapunov functions (continuous ranking functions)

quantitative property:

- *stability (gain) margin*: amount that feedback can be increased while remaining stable

examples?