Correction of Some Exercises and Questions from Slides 1 (Introduction to Dynamical Systems) and Slides 2 (Linear Systems and Stability)
Course Feedback Control and Real-Time Systems

Thao Dang

1 Finite state system

What is maximum size of the transition relation (deterministic and nondeterministic)?
Answer: Suppose that the number of the states is N_s and the number of inputs is N_i.

- If the system is deterministic, from a state and with a given input, at most one successor state is possible and therefore the number of possible transitions from any state is at most N_i.
- If the system is deterministic, from a state and with a given input, at most N_s successor states are possible. Therefore, from any state, there are at most $N_i N_s$ transitions.

2 Babylonian method

Example: A program computing $\sqrt{x_0}$ using the babylonian method

$$x_{k+1} = \frac{1}{2}\left(x_k + \frac{x_0}{x_k}\right).$$

implemented using int, float, rationals, reals, ...

1. **Questions: state-space? initial state? inputs? outputs? time?**
   Answers: $\mathbb{R}$, x_0, no input, no output, time is $\mathbb{N}$ (natural numbers)
2. **Questions: transition relation? behaviors?**
   Answers: the transition relation is defined by $x_{k+1} = \frac{1}{2}\left(x_k + \frac{x_0}{x_k}\right)$. Behaviors are sequences x_k (with $k = 0, 1, 2, \ldots$) determined by this equation.

3. **Questions: deterministic? finite?**
Answers: deterministic (from a given x_k only one x_{k+1} is possible. Infinite (because the state space is infinite).

4. **Question: When is enumerative state-space exploration applicable?**
Answer: if the state space S is enumerable (that is if there exists a surjection $\mathbb{N} \rightarrow S$).

5. **Question: How to check if the sequence converges to $\sqrt{x_0}$?**
Answer: Recall that if a sequence converges monotonically and is bounded, then it has a limit.

   We first prove that the sequence computed by the Babylonian method satisfies this condition (Step 1), and we then prove that the limit is $\sqrt{x_0}$ (Step 2).

   For Step 1, we consider the case where $x_k > \sqrt{x_0}$. Then, $\frac{x_0}{x_k} < \frac{x_k^2}{x_k} = x_k$. Hence, $x_{k+1} = \frac{1}{2}\left(x_k + \frac{x_0}{x_k}\right) < \frac{1}{2}\left(x_k + x_k\right) = x_k$. In other words, $x_k > x_{k+1} > \sqrt{x_0}$. This indicates that the sequence x_k is monotonically decreasing and furthermore is bounded.

   Now we consider the case where $x_1 < \sqrt{x_0}$. We observe that

$$
\begin{aligned}
x_{k+1}^2 - x_0 \quad &= \tfrac{1}{4}\left(x_k + \tfrac{x_0}{x_k}\right)^2 - x_0 \\
&= \tfrac{1}{4}\left(x_k^2 + 2x_0 \tfrac{x_0^2}{x_k^2}\right) - x_0 \\
&= \tfrac{1}{4}\left(x_k^2 - 2x_0 + \tfrac{x_0^2}{x_k^2}\right) \\
&= \tfrac{1}{4}\left(x_k - \tfrac{x_0}{x_k}\right)^2 > 0
\end{aligned}
\tag{1}
$$

   This implies that $x_2 > \sqrt{x_0}$. In addition,

$$
\begin{aligned}
x_{k+1} - x_k \quad &= \tfrac{1}{2}(\tfrac{x_0}{x_k} - x_k) \\
&= \tfrac{x_0 - x_k^2}{2x_k}
\end{aligned}
\tag{2}
$$

   Hence, for $k > 2$, the sequence x_k is (by 1) bounded below by $\sqrt{x_0}$, and is monotonically decreasing (by (2) and $x_2 > \sqrt{x_0}$). Step 1 is thus completed.
   For Step 2 we will prove that the limit of the sequence x_k is $\sqrt{x_0}$. To this end, we remark that the limit $\gamma = lim_{k\rightarrow\infty} x_k$ satisfies $\gamma = \frac{1}{2}(\gamma + \frac{x_0}{\gamma})$. This means that $\gamma^2 = x_0$, which completes Step 2.

6. **Apply symbolic state-space exploration starting from $x_0 = 8$. Use integer intervals to describe sets of states. Overapproximate if necessary.**

Hint: for this question and the following questions, we enclose each number in the algorithm by an interval. Then, to perform the algorithm, we need to do interval arithmetic (that is each number in the calculation is replaced by an interval). Notice also that only integers are allowed, integer division is needed.

7. **Start from $x_0 = 9$. How can the precision be increased?**

8. **Does always rounding up or always rounding down cover all possibilities?**